

Automata in groups and dynamics and induced systems of PDE in tropical geometry

Tsuyoshi Kato

1 Introduction

Automata appear in many branches of mathematics. From the view point of dynamical systems, they produce actions on spaces. Of particular interest for us is *theory of automata groups* which studies discrete group actions on trees.

In this paper we develop a new construction of dynamical scaling transform by use of *tropical geometry*, and apply automata groups to its framework in order to analyze global behaviour of both rational dynamical systems and some classes of systems of non linear partial differential equations.

Tropical geometry is a kind of scale transform between dynamical systems, which provides with a correspondence between automata and real rational dynamics. It allows us to study two dynamical systems at the same time, whose dynamical natures are very different from each other, by taking scaling limits of parameters.

There has been the extended development of partial differential equations by use of approximations by real rational dynamics. Combination of these two aspects creates a new connection between geometric group theory and partial differential equations. This is the main aspects which we will focus on in this paper. There are distinctive phenomena for each theory in automata, groups and dynamical systems, say calculating machine, freeness, ergodicity, etc. It would be of particular interests for us to translate these notions into and compare with each other.

A discrete group is called an automata group, if it is generated by actions on the rooted trees, whose rules are represented by an automaton. From

the view point of dynamical scaling limits, automata can be regarded as a frame of dynamical systems, which play the role of underlying mechanisms for rational dynamical systems and systems of partial differential equations.

In this paper we give a construction of dynamical scale transform from automata groups to partial differential equations. In particular we develop a method of dynamical scale transform from Mealy automata groups to first order hyperbolic systems of PDE.

If rational dynamical systems or systems of PDE contain automata groups as their underlying dynamics, then quite stably analytic properties of the groups can reflect to the dynamical structures of solutions to the systems, since fluctuation of the dynamics will be eliminated by the dynamical scale change to automata. This is a key point where geometric group theory enters, since it studies properties which are invariant with respect to quasi-isometry. So a basic and general question is, whether one can find some systems with global solutions which satisfy structural similarity with these properties, whenever we focus on some geometric properties of automata groups. As a particular property we will see that recursiveness in groups induces ‘quasi-recursiveness’ in the corresponding rational dynamics.

Theory of automata groups provide plenty of examples which possess quite characteristic properties. Here we focus on the finitely generated *infinite torsion groups* whose existence is a question called the *Burnside problem*. There have been many constructions of groups concerning it (see the references in [GNS] and [Z]). Of particular interest for us is by Aleshin-Grigorchuk as automata groups. In terms of dynamical systems, one can restate its property as infinite recursivity that any actions have finite orders, and the number of the minimal orders are infinite. In this paper we show that their automata produce the rational dynamical systems which satisfy *infinite quasi-recursivity*.

In this paper we include basic analysis of the hyperbolic systems of first order PDE of 2-variables, which arise from Mealy automata. In particular it includes existence, uniqueness, energy estimates with explicit estimates on the constants, and so on. It allows us to produce solutions to the PDE systems which can be applied to the asymptotic comparison theorem which we will describe below.

Even though the topics of this paper include two different fields, where one is discrete group theory and the other is PDE analysis, still the contents are written in a self-contained way.

Now let us introduce a class of dynamical systems which we treat. Let Q be a set called states and X be a space, which will be mostly in this paper a set of infinite sequences over a finite set. A *state dynamics* on X is the dynamical system given by elements of Q , such that every $q^0 \in Q$ determines an action on X . we call q^0 as the initial state.

An *automaton* is consisted by a finite rule which can create quite complicated state dynamics on the sequences of alphabets. Let S be another set called alphabets, and consider all the set of infinite sequences:

$$X_S = \{(s_0, s_1, \dots) : s_i \in S\}.$$

Let S^n be the product set of S by n times, and consider an automaton $\mathbf{A}$ which is given by a pair of functions:

$$\psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

where $\alpha, \beta \geq 0$. It gives rise to the state dynamics on X_S as follows. Let us choose any $q \in Q$ and $\bar{s} = (s_0, s_1, \dots) \in X_S$. Then the dynamics:

$$\mathbf{A}_q : X_S \rightarrow X_S, \quad \mathbf{A}_q(\bar{s}) = (s'_0, s'_1, \dots)$$

is determined inductively by:

$$s'_i = \psi(q_i, s_i, \dots, s_{i+\alpha}), \quad q_{i+1} = \phi(q_i, s_i, \dots, s_{i+\beta}) \quad (q_0 = q)$$

which we call the state dynamics. Besides the dynamics over X_S , the change of the state sets play important roles in the hidden dynamics. We call $\mathbf{A}$ as the initial automaton, if we specify the initial states.

Any sequences $\bar{q}^j = (q^0, \dots, q^j) \in X_Q^{j+1}$ give dynamics by compositions:

$$\mathbf{A}_{\bar{q}^j} = \mathbf{A}_{q^j} \circ \dots \circ \mathbf{A}_{q^0} : X_S \rightarrow X_S.$$

It can happen that different automata give the same state dynamics. In such case, the dynamics $\mathbf{A}_q$ are the same, but dynamics of change of the state sets can be very different. Such two automata are called *equivalent*.

The above type of the state dynamics contains two important cases of discrete dynamics which appear in tropical geometry:

(A) Integrable systems of cell automaton: One arises from the scaling limits of the integrable systems. Let us just present one of the typical case,

the ultra-discrete Lotka Volterra cell automaton, whose dynamics is given by the rule ([HT], [TTMS]):

$$s'_i = \varphi(s'_{i-1}, s_i, s_{i+1}) = s'_{i-1} + \max(1, s_i) - \max(1, s_{i+1})$$

Let us rewrite this by a state dynamics. For $S = Q$ with the initial state $q_0 = q$, consider the state dynamics with $\psi : Q \times S \rightarrow S, \phi : Q \times S^3 \rightarrow Q$ by:

$$\psi(q, s) = q, \quad \phi(q, s_0, s_1, s_2) = q + \max(1, s_1) - \max(1, s_2).$$

One can assign $s'_i = q_i$ with $s'_0 = q$, which describes the above automaton.

There are many other cases of the integrable systems given by cell automata of the above types (see [K4]). Recently we have discovered that some discrete integrable systems arise quite naturally at limit of Mealy automata in some sense, which will be appeared in another article.

(B) Mealy automaton: Suppose both $\alpha = \beta = 0$ and so:

$$\psi : Q \times S \rightarrow S, \quad \phi : Q \times S \rightarrow Q.$$

If we identify X_S with the rooted regular tree, then the state dynamics $\mathbf{A}_q$ give the semi-group actions on the tree, since the actions can be restricted level-setwisely. They are invertible actions if and only if $\psi : (q, \cdot) : S \cong S$ are isomorphic as sets for all $q \in Q$. In the invertible case, the group generated by these states is called the *automata group* given by the automaton (ψ, ϕ) . We will mostly treat regular trees whose degree coincide with the number of the alphabets.

General state dynamics will not give actions on the trees, but still it may be possible to hold isomorphisms $\mathbf{A}_q : X_S \cong X_S$, which are identified with the actions on the boundary of the trees.

In 4.A.3, we give an example of a non Mealy automaton which gives the isomorphic actions on the boundary of the tree, but does not induce the actions on the tree itself.

Automata groups contain several discrete groups which are quite characteristic in geometric group theory. There are so many interesting examples, but it is beyond the scope of this paper to give complete references. Let us just list some of known results:

(1) Automata groups with intermediate growth, more generally with fluctuations ([Gri2],[Br]),

- (2) Automata groups which are infinite and torsion ([Al], [Gri1]),
- (3) Automata groups with non-uniformly exponential growth ([W]),
- (4) A criterion of amenability ([BKN]),
- (5) Classification of automata groups with small numbers of states ([GNS]),
- (6) Construction of lampf-lighter type automata ([GZ]).

One may also find the various papers concerning automata groups both on the homepage of Grigorchuk and Zuk.

Our basic idea of research direction is to study how such geometric or analytic properties reflect to the structures of the rational dynamics and PDE systems, if they contain such groups as the frame of the dynamical systems. As an application, we will verify existence of *infinite quasi-recursivity* for some rational dynamical systems.

Below let us include summary of the contents of this paper.

Tropical geometry: A relative $(\max, +)$ -function φ is a piecewise linear function equipped with its presentation of the form:

$$\varphi(\bar{x}) = \max(\alpha_1 + \bar{a}_1\bar{x}, \dots, \alpha_m + \bar{a}_m\bar{x}) - \max(\beta_1 + \bar{b}_1\bar{x}, \dots, \beta_l + \bar{b}_l\bar{x}).$$

Tropical geometry associates the parametrized rational function (see [Mi]), which we call a *relatively elementary function* :

$$f_t(\bar{z}) \equiv \frac{k_t(\bar{z})}{h_t(\bar{z})} = \frac{\sum_{k=1}^m t^{\alpha_k} \bar{z}^{\bar{a}_k}}{\sum_{k=1}^l t^{\beta_k} \bar{z}^{\bar{b}_k}}.$$

They admit one to one correspondence of their presentations to φ , and take positive real numbers if the inputs are also positive.

A crucial property is that f_t converge to φ by letting $t \rightarrow \infty$ in some sense. Among various ways of scaling limits, tropical geometry behaves quite nicely, which allows us to obtain several uniform estimates by comparisons of both dynamical behaviours at the same time.

It is quite characteristic of $(\max, +)$ -functions that different presentations can give the same functions. For example:

$$\varphi(y, x) = \max(x, -x) - y, \quad \psi(x, y) = \max(\varphi(x, y), -y)$$

are the same functions but have different presentations. The corresponding rational functions are mutually $f_t(w, z) = w^{-1}(z + z^{-1})$ and $g_t(w, z) = w^{-1}(z + z^{-1} + 1)$, which are different even as functions. This motivates us to introduce a notion of *tropical equivalence* between such f_t and g_t .

Let us take finite sets $S, Q \subset \mathbb{Z}$, and consider an automaton $\mathbf{A}$ given by:

$$\varphi : Q \times S^{\alpha+1} \rightarrow S, \quad \psi : Q \times S^{\beta+1} \rightarrow Q.$$

In general these can be extended over the real numbers by piecewise linear functions which admit the presentations by the relative $(\max, +)$ functions:

$$\begin{aligned} \psi(\bar{r}) &= \max(\alpha_1 + \bar{a}_1 \bar{r}, \dots, \alpha_{\alpha+2} + \bar{a}_{\alpha+2} \bar{r}) - \max(\beta_1 + \bar{b}_1 \bar{r}, \dots, \beta_{\alpha+2} + \bar{b}_{\alpha+2} \bar{r}), \\ \phi(\bar{l}) &= \max(\gamma_1 + \bar{c}_1 \bar{l}, \dots, \gamma_{\beta+2} + \bar{c}_{\beta+2} \bar{l}) - \max(\delta_1 + \bar{d}_1 \bar{l}, \dots, \delta_{\beta+2} + \bar{d}_{\beta+2} \bar{l}). \end{aligned}$$

State systems of the rational dynamics: *The state dynamics* with respect to the pair (ψ, ϕ) are given by the discrete dynamics inductively defined by the iterations:

$$x_i^{j+1} = \psi(y_i^j, x_i^j, \dots, x_{i+\alpha}^j), \quad y_{i+1}^j = \phi(y_i^j, x_i^j, \dots, x_{i+\beta}^j)$$

where $x_i^0 = x_i$ and $y_0^j = y^j$ are the initial values for $i, j \geq 0$.

Let us visualize this dynamics as follows. $y = y^0$ determines the map:

$$\mathbf{A}_y : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}, \quad \mathbf{A}_y(x_0, x_1, \dots) = (x_0^1, x_1^1, x_2^1, \dots).$$

So by composition, finite sequences $(y^0, y^1, \dots, y^l)$ give the maps:

$$\begin{aligned} \mathbf{A}_{(y^0, y^1, \dots, y^l)} &= \mathbf{A}_{y^l} \circ \dots \circ \mathbf{A}_{y^0} : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}, \\ \mathbf{A}_{(y^0, y^1, \dots, y^j)}(x_0, x_1, \dots) &= (x_0^{j+1}, x_1^{j+1}, x_2^{j+1}, \dots). \end{aligned}$$

If $\alpha = \beta = 0$, then the Meay automaton gives the automata groups, and the state dynamics above exactly extend the group actions over the trees.

Let us consider the corresponding parametrized rational functions f_t and g_t with respect to φ and ψ respectively.

The state system of the rational dynamics is given by the corresponding rational dynamics:

$$z_i^{j+1} = f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \quad w_{i+1}^j = g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j)$$

where the initial values are given by $z_i^0 = z_i, w_0^j = w^j > 0$.

Now let (ψ^1, ϕ^1) and (ψ^2, ϕ^2) be pairs of $(\max, +)$ -functions, and (f_t^1, g_t^1) and (f_t^2, g_t^2) be the corresponding relatively elementary functions. Assume that both pairs are tropically equivalent:

$$(f_t^1, g_t^1) \sim (f_t^2, g_t^2).$$

For $l = 1, 2$, let $\{w^j(l)\}_{j \geq 0}$ and $\{z_i(l)\}_{i \geq 0}$ be the initial sequences of positive numbers, and denote the solutions by $(z_i^j(l), w_i^j(l))$ to the state systems of the rational dynamics:

$$\begin{aligned} z_i^{j+1}(l) &= f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ w_{i+1}^j(l) &= g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)). \end{aligned}$$

with the initial values $z_i^0(l) = z_i(l)$ and $w_0^j(l) = w^j(l)$.

Let $P_i(c) = \frac{c^{i+1}-1}{c-1}$. Our basic tool of the analysis for the rational dynamical systems are given by the following:

Proposition 1.1. *There are constants $M \geq 1$ and $c \geq 0$ independently of the initial values so that the asymptotic uniform-estimates hold:*

$$\begin{aligned} &\max\left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}, \frac{z_i^{j+1}(2)}{z_i^{j+1}(1)}, \frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}, \frac{w_{i+1}^j(2)}{w_{i+1}^j(1)}\right) \\ &\leq M^{2P_{i+j}(\gamma+1)(c)} [\{(z_i(l), w^j(l))\}_{l=1}^2]^{\tilde{c}^{i+1+j}(\gamma+1)} \end{aligned}$$

where $\gamma = \max(\alpha, \beta)$, $\tilde{c} = \max(c, 1)$, and the initial rates are given by:

$$[\{(z_i(l), w^j(l))\}_{l=1}^2] = \sup_{i,j} \max\left(\frac{z_i(1)}{z_i(2)}, \frac{z_i(2)}{z_i(1)}, \frac{w^j(1)}{w^j(2)}, \frac{w^j(2)}{w^j(1)}\right).$$

In particular if $c < 1$ and if the initial values are the same, then one obtains the uniform estimates by some constants M^C . Both M and c can be known immediately once the presentations of the defining functions are given. In fact c is the Lipschitz constant of the pair (ψ, ϕ) .

For our later purpose of the application to PDE analysis, we verify the uniform estimates when the orbits fluctuate under controll of torpically equivalent functions as below.

Let $(\{z_i^j(l)\}_{i,j}, \{w_i^j(l)\}_{i,j})$ be as above.

Theorem 1.2. *Suppose $(f_t^1, g_t^1) \sim (f_t^2, g_t^2)$ are pairwise tropically equivalent.*

If another sequences $\{w_i^j\}_{i,j}$ and $\{z_i^j\}_{i,j}$ satisfy the dynamical inequalities:

$$\begin{aligned} f_t^1(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) &\leq z_i^{j+1} \leq f_t^2(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \\ g_t^1(w_i^j, z_i^j, \dots, z_{i+\beta}^j) &\leq w_{i+1}^j \leq g_t^2(w_i^j, z_i^j, \dots, z_{i+\beta}^j) \end{aligned}$$

then the uniform estimates hold for $l = 1, 2$:

$$\left(\frac{z_i^j(l)}{z_i^{j+1}}\right)^{\pm 1}, \quad \left(\frac{w_i^j(l)}{w_{i+1}^j}\right)^{\pm 1} \leq M^{4P_{i+j(\gamma+1)}(c)} \left[\sup_{i,j} \max\left\{\left(\frac{z_i^0(l)}{z_i^0}\right)^{\pm 1}, \left(\frac{w_0^j(l)}{w_0^j}\right)^{\pm 1}\right\}\right]^{z^{i+1+j(\gamma+1)}}$$

We will apply these estimates for $f_t^1 = \frac{1}{N}f_t$ and $f_t^2 = Nf_t$ to the analysis of PDE for some integer $N \geq 1$.

Notice that we have removed the condition of monotonicity of functions which were assumed in [K3].

Automata groups and tropical geometry: Let us take finite sets $S, Q \subset \mathbb{Z}$, and consider an automaton $\mathbf{A}$ given by the pair $\varphi : Q \times S^{\alpha+1} \rightarrow S$ and $\psi : Q \times S^{\beta+1} \rightarrow Q$ with their extensions by relative $(\max, +)$ functions (ψ, ϕ) .

We say that the extensions are *stable* over (Q, S) , if there are some $0 < \delta < 1$ and $0 \leq \mu < 1$ so that the Lipschitz constants of the pair (ψ, ϕ) is bounded by μ on δ neighbourhoods of $Q, S \subset \mathbb{R}$ respectively.

We verify that for Mealy automata, stable extensions always exist. It would be possible to construct stable extensions for general automata, by performing little bit more complicated constructions.

If two automata are equivalent, the corresponding state systems of the rational dynamics show mutual structural similarity on the large scale as:

Theorem 1.3. *Suppose $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalent, and choose stable extensions of them.*

Then for any $C \geq 1$, there exists $t_0 > 1$ so that for all $t \geq t_0$ and any initial values $\bar{z}(l)$ and $\bar{w}(l)$ with the bounds:

$$C^{-1}k_i \leq z_i(l) \leq Ck_i, \quad C^{-1}q^j \leq w^j(l) \leq Cq^j$$

for $\bar{k} \in X_S$ and $\bar{q} \in X_Q$, then the uniform estimates hold:

$$\max\left\{\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)}\right\} < C^4.$$

This enables us to transform various group theoretic properties over the rational dynamics. Below we focus on the case of groups of infinite torsion, but the method itself can be applied much broader situations, which is based on the above theorem.

Infinite quasi-recursive rational dynamics: Let us consider the automata groups of infinite torsion, which give solutions to the Burnside problem. Let us transform such phenomena to the corresponding state systems of the rational dynamics, which we call the *rational Burnside problem*.

Firstly we introduce some notations. For subsets $X \subset \mathbb{R}$, let us denote by $X^{\mathbb{N}}$ all the set of infinite sequences by elements in X . Let us consider two dynamics:

$$\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2 : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$$

and denote their orbits by:

$$(\bar{\mathbf{A}}_k)^j(\bar{z}) = (z_0^j(k), z_1^j(k), \dots)$$

for $k = 1, 2$. Let us say that:

(1) $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are *C-close* over $X \subset \mathbb{R}$, if for any $\bar{z} \in X^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$, the corresponding orbits satisfy the uniform estimates hold for all $i, j = 0, 1, 2, \dots$:

$$\max\left(\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)}\right) \leq C.$$

(2) $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are *C'-separated* over X , if there is some $\bar{z} \in X^{\mathbb{N}}$ so that the bounds hold for some i :

$$\max\left(\frac{z_i^1(1)}{z_i^1(2)}, \frac{z_i^1(2)}{z_i^1(1)}\right) \geq C'$$

Let (f_t, g_t) be a pair of relatively elementary functions, and consider the state system of the rational dynamics. One can restate it in the compatible way to the dynamics by automata. For $w \in \mathbb{R}$, let:

$$\bar{\mathbf{A}}_w : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}, \quad (z_0, z_1, \dots) \rightarrow (z'_0, z'_1, \dots)$$

be given by the rule:

$$z'_i = f(w_i, z_i, \dots, z_{i+\alpha}), \quad w_{i+1} = g(w_i, z_i, \dots, z_{i+\beta})$$

where $w_0 = w$. For finite sequences $\bar{w} = (w^0, \dots, w^l) \in \mathbb{R}^{l+1}$, we denote:

$$\bar{\mathbf{A}}_{\bar{w}} \equiv \bar{\mathbf{A}}_{w^l} \circ \dots \circ \bar{\mathbf{A}}_{w^0} : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}.$$

Let us say that the state system of the parametrized rational dynamics by (f_t, g_t) is *infinite quasi-recursive* if there exist a constant $C > 0$ and subsets:

$$X_t, Y_t \subset \mathbb{R}$$

so that for any $C' \geq 1$, the followings hold for all $t \geq t_0 = t_0(C')$:

- (1) For any words $\bar{w} \in Y_t^{l+1}$, there exist some $p = p(\bar{w}) \in \mathbb{N}$ such that $(\bar{\mathbf{A}}_{\bar{w}})^p$ and the identity are C -close over X_t .
- (2) There are infinitely many words $\bar{w}^j \in Y_t^*$ so that $\bar{\mathbf{A}}_{\bar{w}^j}$ and $\bar{\mathbf{A}}_{\bar{w}^{j'}}$ are mutually C' -separated over X_t for any $j \neq j'$.

Let $S = \{s_0, s_1\} \subset \mathbb{Z}$ and $R = \{q^0, q^1\} \subset \mathbb{Z}$ be any embeddings, and denote $N_C(t^S) = \{z \in (0, \infty) : C^{-1}t^s \leq z \leq Ct^s : s \in S\}$.

We have the existence of parametrized infinitely quasi-recursive dynamics:

Theorem 1.4. *There exists a pair of relatively elementary functions (f_t, g_t) so that for any $C > 0$, the state dynamics is infinitely quasi-recursive over:*

$$(X_t, Y_t) = (N_C(t^S), N_C(t^R)).$$

Approximations of systems of PDE by rational dynamics: It would be of interest to study how globally analytic properties of automata group actions on trees effect on the associated dynamics of the PDE systems.

Let us state a general procedure to induce PDE systems from automata. Let $0 < \epsilon \leq 1$ be constants. Let us consider a function $u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ of class $C^{\mu+1}$, and take the Taylor expansions up to order $\mu + 1$:

$$u(x + i\epsilon, s + j\epsilon) = u + i\epsilon u_x + j\epsilon u_s + \frac{(i\epsilon)^2}{2} u_{2x} + \cdots + \frac{(j\epsilon)^{(\mu+1)}}{(\mu+1)!} u_{(\mu+1)s}(\xi_{ij}).$$

Let us describe a general way to approximate solutions to the systems of PDE by the state systems of rational dynamics.

Let $f_t = \frac{a_t}{b_t}$ and $g_t = \frac{c_t}{d_t}$ be relatively elementary functions, and consider the state systems of the rational dynamics:

$$z_i^{j+1} = f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \quad w_{i+1}^j = g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j).$$

Let us introduce the change of variables by:

$$i = \frac{x}{\epsilon}, \quad j = \frac{s}{\epsilon}, \quad u(x, s) = z_i^j, \quad v(x, s) = w_i^j$$

Then we take the difference, and insert the Taylor expansions:

$$\begin{aligned} z_i^{j+1} - f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) &= u(x, s + \epsilon) - f_t(v(x, s), u(x, s), \dots, u(x + \alpha\epsilon, s)) \\ &= \frac{P_1(\epsilon, t, u, v, u_s, u_x, \dots, u_{\mu x}) + R_1(\epsilon, t, u, v, \dots, u_{(\mu+1)x}(\xi))}{b_t(v(x, s), u(x, s), \dots, u(x + \alpha\epsilon, s))} \\ &\equiv \mathbf{L}_1(\epsilon, t, u, v, u_s, \dots, u_{\mu x}) + \epsilon^{\mu+1} \mathbf{E}_1(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi_{ij})\}_{\bar{a}, i, j}). \end{aligned}$$

where P_1 and R_1 are polynomials, and each monomial in R_1 contains derivatives of u of order $\mu + 1$.

Similarly we have the expansions:

$$\begin{aligned} w_{i+1}^j - g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j) &= v(x + \epsilon, s) - g_t(v(x, s), u(x, s), \dots, u(x + \beta\epsilon, s)) \\ &= \frac{P_2(\epsilon, t, u, v, u_x, v_x, \dots, u_{\mu x}) + R_2(\epsilon, t, u, v, \dots, u_{(\mu+1)x}(\xi'))}{d_t(v(x, s), u(x, s), \dots, u(x + \beta\epsilon, s))} \\ &\equiv \mathbf{L}_2(\epsilon, t, u, v, u_x, v_x, \dots, u_{\mu x}) \\ &\quad + \epsilon^{\mu+1} \mathbf{E}_2(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi'_{ij})\}_{\bar{a}, i, j}, \{v_{\bar{a}}(\xi'_{ij})\}_{\bar{a}, i, j}) \end{aligned}$$

where each monomial in R_1 contains derivatives of u or v of order $\mu + 1$.

We say that $\mathbf{L}_i$ and $\mathbf{E}_i$ are the *leading* and *error* terms respectively, and call the parametrized systems of PDE of order μ :

$$P_1(\epsilon, t, u, v, u_s, \dots, u_{\mu s}, u_{\mu x}) = 0, \quad P_2(\epsilon, t, u, v, v_x, \dots, u_{\mu x}, v_{\mu x}) = 0$$

the *induced systems of partial differential equations* with respect to (ψ, ϕ) .

Let us take four relatively elementary functions f^1, f^2 and g^1, g^2 so that $f^1 \sim f^2$ and $g^1 \sim g^2$ are tropically equivalent mutually. For $l = 1, 2$, let:

$$P_1^l(\epsilon, t, u, v, u_s, \dots, u_{\mu s}, u_{\mu x}) = 0, \quad P_2^l(\epsilon, t, u, v, v_x, \dots, u_{\mu x}, v_{\mu x}) = 0$$

be the induced systems of PDEs of order μ .

For functions of $C^{\mu+1}$ class, let us introduce the *higher distorsion* which are the relatively uniform norms of their $\mu + 1$ -differentiations (3.B):

$$K(u, v) \equiv \sup_{(x, s) \in [0, \infty)^2} \max \left[\frac{\|(u, v)\|_{\mu, \alpha}^1}{u(x, s + \epsilon)}, \frac{\|(u, v)\|_{\mu, \beta}^2}{v(x + \epsilon, s)} \right].$$

Theorem 1.5. *For $l = 1, 2$, let C be the bigger one of their error constants.*

Let $(u^l, v^l) : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ be positive solutions to the above systems respectively, so that the estimates:

$$0 \leq CK(u^l, v^l) \leq (1 - \delta)\epsilon^{-1}$$

are satisfied for some positive $\delta > 0$.

Then they satisfy the asymptotic estimates for all $(x, s) \in [0, \infty) \times [0, \infty)$:

$$\begin{aligned} & \left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s), \quad \left(\frac{v^1}{v^2}\right)^{\pm 1}(x, s) \\ & \leq (N_0 M)^{6P_{\epsilon^{-1}(x+s(\gamma+1))}^{(c)}} \left[(u^1, v^1) : (u^2, v^2)\right]_{\epsilon} \tilde{c}^{\epsilon^{-1}(x+s(\gamma+1))+1} \end{aligned}$$

where N_0 is any integer with $N_0 \geq \max(\delta^{-1}, 2 - \delta)$.

We say that a solution to the above PDE systems is *admissible*, if the higher distortions satisfy the above estimates.

In practice, in order to apply the above result, there are two types of questions where:

- (1) existence of positive solutions, (2) admissibility of such solutions.

Among all the induced PDE systems, the hyperbolic Mealy systems which we treat below are the one which arise from automata groups. For such class, existence follows always and we construct the admissible systems of PDE quite concretely.

Hyperbolic systems of PDE in tropical geometry: In the case of Mealy dynamics given by:

$$x(i, j + 1) = \psi(y(i, j), x(i, j)), \quad y(i + 1, j) = \phi(y(i, j), x(i, j))$$

the induced first order systems of the equations:

$$\epsilon u_s = f_t(v, u) - u, \quad \epsilon v_s = g_t(v, u) - v$$

are called the *hyperbolic Mealy systems*.

In this paper we develop basic analysis of the hyperbolic Mealy systems. In particular we verify existence of positive solutions, uniqueness and explicit energy estimates. Construction of admissible solutions to the hyperbolic Mealy systems involve the interplay of estimates between piecewise linear and differentiable dynamics.

Let us treat a case when the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$ (5.C). Let us put the initial domain:

$$I_0 = [0, \infty) \times \{0\} \cup \{0\} \times [0, \infty).$$

Theorem 1.6. *Suppose that the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$, and give the positive initial values:*

$$u, v : I_0 \rightarrow [r + q, R - q].$$

Then: (1) there exists a positive solution

$$u, v : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$$

with the uniform bounds $r + q \leq u(x, s), v(x, s) \leq R - q$.

(2) The solution is unique.

We also induce the energy estimates which are well known for the hyperbolic systems, but here we also estimate the constants explicitly.

In 5.D, we construct admissible systems of PDE from piecewisely linear functions. For this purpose we introduce some modification of automata.

Definition 1.1. $(\bar{\psi}, \bar{\phi})$ is an ϵ -refinement of the pair (ψ, ϕ) , if there is a positive number N so that for any $\bar{q} \in X_Q$ and $\bar{k} \in X_S$, there are paths $y : \{0, 1, \dots\} \rightarrow \mathbb{R}$ and $x : \{0, 1, \dots\} \rightarrow \mathbb{R}$ with:

$$\begin{aligned} y(jN) &= q^j, & |y(j+1) - y(j)| &\leq \epsilon, \\ x(iN) &= k_i, & |x(i+1) - x(i)| &\leq \epsilon \end{aligned}$$

for all $i, j \in \{0, 1, \dots\}$, such that the corresponding orbits:

$$(\{x_i^j\}, \{y_i^j\}), \quad (\{s_i^j\}, \{q_i^j\})$$

with respect to $(\bar{\psi}, \bar{\phi})$ and (ψ, ϕ) respectively, satisfy the equalities:

$$x_{iN}^{jN} = k_i^j, \quad y_{iN}^{jN} = q_i^j.$$

(ψ, ϕ) is refinable, if there is an ϵ -refinement for any small $\epsilon > 0$.

Let $\mathbf{A}$ be a Mealy automaton with 2 alphabets, equipped with a representative (ψ, ϕ) by relatively $(\max, +)$ -functions. For any $(s_0, s_1, \dots) \in X_2$ and $(q^0, q^1, \dots) \in X_{m+1}$, let:

$$\mathbf{A}_{(q^0, \dots, q^{l-1})}(s_0, s_1, \dots) = ((s_0^l, s_1^l, \dots) \in X_2$$

be the orbits of the automata group actions.

For its refinement $(\bar{\psi}, \bar{\phi})$ and their tropical correspondences $(\bar{f}_t, \bar{g}_t)$, let us consider the hyperbolic Mealy systems:

$$u_s = \bar{f}_t(v, u) - u, \quad v_x = \bar{g}_t(v, u) - v.$$

Theorem 1.7. For any $C > 0$ and any $t \geq t(C) > 1$, there are refinements $(\bar{\psi}, \bar{\phi})$ of (ψ, ϕ) with the pairs of tropical correspondences $(\bar{f}_t, \bar{g}_t)$ so that:

- (1) $(\bar{f}_t, \bar{g}_t)$ admits admissible solutions,
- (2) for any another pairs (f_t, g_t) toropically equivalent to $(\bar{f}_t, \bar{g}_t)$, any admissible solutions to the equations:

$$u_s = f_t(v, u) - u, \quad v_x = g_t(v, u) - v$$

whose initial values satisfy the inclusions for all $k = 0, 1, 2, \dots$:

$$d(u(Nk, 0), S), \quad d(v(0, Nk), Q) \leq C$$

then they satisfy the asymptotic estimates for some $M, c \geq 1$:

$$\left(\frac{u(Ni, Nj)}{s_i^j}\right)^{\pm 1} \leq M^{P_{N(i+j)+1}(c)}$$

2 Tropical geometry

2.A Tropical transform: A *relative* $(\max, +)$ -function φ is a piecewise linear function equipped with its presentation of the form:

$$\varphi(\bar{x}) = \max(\alpha_1 + \bar{a}_1 \bar{x}, \dots, \alpha_m + \bar{a}_m \bar{x}) - \max(\beta_1 + \bar{b}_1 \bar{x}, \dots, \beta_l + \bar{b}_l \bar{x})$$

where $\bar{a}_l \bar{x} = \sum_{i=1}^n a_l^i x_i$, $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\bar{a}_l = (a_l^1, \dots, a_l^n)$, $\bar{b}_l \in \mathbb{Q}^n$ and $\alpha_i, \beta_i \in \mathbb{R}$.

We say that the multiple integer $M \equiv ml$ is the *number of the components*.

φ is Lipschitz since it is piecewise linear. This is the most important property for our analytic estimates later.

Throughout this paper, we equip the metric on $\mathbb{R}^n$ by:

$$|(x_1, \dots, x_n)| = \sup_{1 \leq i \leq n} |x_i|.$$

Corresponding to φ , tropical geometry associates the parametrized rational function given by:

$$f_t(\bar{z}) \equiv \frac{k_t(\bar{z})}{h_t(\bar{z})} = \frac{\sum_{k=1}^m t^{\alpha_k} \bar{z}^{\bar{a}_k}}{\sum_{k=1}^l t^{\beta_k} \bar{z}^{\bar{b}_k}}$$

where $\bar{z}^{\bar{a}} = \prod_{i=1}^n z_i^{a_i}$, $\bar{z} = (z_1, \dots, z_n) \in \mathbb{R}_{>0}^n$ ([LM], [V], [Mi]).

We say that f_t is a *relatively elementary function*, and both terms $h_t(\bar{z}) = \sum_{k=1}^l t^{\beta_k} \bar{z}^{\bar{b}_k}$ and $k_t(\bar{z}) = \sum_{k=1}^m t^{\alpha_k} \bar{z}^{\bar{a}_k}$ are just elementary functions.

These two functions φ and f_t admit one to one correspondence between their presentations. In fact they are connected passing through some intermediate functions φ_t , which Maslov introduced as dequantization of the real line $\mathbb{R}$.

Let us briefly explain the aspects of scaling limit in tropical geometry. For $t > 1$, there is a family of semi-rings R_t which are all the real number $\mathbb{R}$ as sets. The multiplications and the additions are respectively given by:

$$x \oplus_t y = \log_t(t^x + t^y), \quad x \otimes_t y = x + y.$$

As $t \rightarrow \infty$ one obtains the equality:

$$x \oplus_\infty y = \max(x, y).$$

By use of these semi-ring structure, one has R_t -polynomials of the form:

$$\varphi_t(\bar{x}) = (\alpha_1 + \bar{a}_1 \bar{x}) \oplus_t \cdots \oplus_t (\alpha_m + \bar{a}_m \bar{x}) - (\beta_1 + \bar{b}_1 \bar{x}) \oplus_t \cdots \oplus_t (\beta_l + \bar{b}_l \bar{x})$$

2.A.2 Basic properties: So far we have seen three different types of functions, φ , f_t and φ_t . Let us list some basic properties they satisfy.

(A) φ and φ_t are connected as:

$$\lim_{t \rightarrow \infty} \varphi_t = \varphi$$

and the limit satisfies the relative $(\max, +)$ equation $\varphi_\infty(\bar{x}) = \varphi(\bar{x})$ as in 2.A. In fact we have the uniform estimates as below. Let M be the number of the components for φ .

Lemma 2.1 (K2). *The uniform estimates hold:*

$$\sup_{\bar{x} \in \mathbb{R}^n} |\varphi_t(\bar{x}) - \varphi(\bar{x})| \leq \log_t M.$$

Proof: For convenience we include the proof for the simple case. We verify the estimate $|x \oplus_t y - \max(x, y)| \leq \log_t 2$. Assume $x = \max(x, y)$. Then:

$$x \oplus_t y = \log_t(t^x + t^y) = \log_t(t^x(1 + t^{y-x})) = x + \log_t(1 + t^{y-x}).$$

Since $y - x \leq 0$ are non positive, the estimates $\log_t(1 + t^{y-x}) \leq \log_t 2$ hold. The general case easily follows from this. This completes the proof.

Remark 2.1: At a glance tropical geometry seems a special kind among various scale transforms. However we would point out the following fact, which suggests that tropical geometry possesses some universality from the arithmetic view points:

Lemma 2.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map with the property:*

$$f(x + y) = f(x)f(y), \quad x, y \in \mathbb{R}.$$

Then there is some $t > 0$ so that one of $f(x) = t^x$ or $f(x) = 0$ holds.

Proof: Firstly $f(x) \geq 0$ hold, since $f(x) = f(\frac{x}{2})^2$.

$f(a) = (f(1))^a$ hold for any rational $a = \frac{m}{n} \in \mathbb{Q}_{>0}$. This follows from the equalities $f(1) = f(n\frac{1}{n}) = f(\frac{1}{n})^n$ and $f(a) = f(\frac{1}{n})^m = f(1)^{\frac{m}{n}} = f(1)^a$. By continuity, the same formula holds for any $a \in \mathbb{R}$.

$f \equiv 0$ hold iff $f(b) = 0$ holds for some $b \in \mathbb{R}$, since the equalities $f(a) = f(a - b)f(b)$ hold.

$f(0) = 0$ or $f(0) = 1$ must hold by the equality $f(0) = f(0)f(0)$. So $f \equiv 0$ hold iff $f(0) = 0$.

Conversely suppose $f(0) = 1$ holds. Then $f(a) > 0$ hold for any $a \in \mathbb{R}$, and so $f(a) = t^a$ hold with $t \equiv f(1) > 0$. This completes the proof.

(B) Three types of the functions equipped with their presentations:

$$\varphi, \quad \varphi_t, \quad f_t$$

have one to one correspondences with each other. Namely one can obtain the presentations of all these functions at the same time, once the coefficients $\alpha_i, \bar{a}_i, \beta_j, \bar{b}_j$ are determined.

Notice that as functions, the presentations of relative $(\max, +)$ functions φ are not uniquely determined in general, unlike to the case of rational functions.

For example $\varphi(x, y) = \max(-y, y) - x$ and $\psi(x, y) = \max(\varphi(x, y), -x)$ are the same functions, but have the different presentations. Correspondingly the rational functions have the presentations as:

$$f(z, w) = z^{-1}(w^{-1} + w), \quad g(z, w) = z^{-1}(w^{-1} + w + 1)$$

which are mutually different even as functions.

This leads us to the following notion. Let φ^1 and φ^2 be two relative $(\max, +)$ -functions with n variables. Then φ^2 is *equivalent* to φ^1 ($\varphi^1 \sim \varphi^2$), if they are the same as functions, so $\varphi^1(x_1, \dots, x_n) = \varphi^2(x_1, \dots, x_n)$ hold for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ but possibly they may have different presentations.

Definition 2.1 (K2). *Let f_t^1 and f_t^2 be two relatively elementary functions. f_t^1 and f_t^2 are mutually tropically equivalent, if the corresponding $(\max, +)$ -functions φ^1 and φ^2 are equivalent.*

For example if f_t corresponds to φ , then $3f_t$ corresponds to $\max(\varphi, \varphi, \varphi)$, and so on. So f_t and Nf_t are tropically equivalent for all $N = 1, 2, \dots$. More generally f_t and af_t are tropically equivalent for any $a \in \mathbb{Q}_{>0}$.

For $N, M \geq 1$ and f_t , let us put:

$$\tilde{f}_t(z_0, \dots, z_{n-1}) \equiv f_t\left(\frac{M}{N}z_0, \dots, \frac{M}{N}z_{n-1}\right).$$

Lemma 2.3. *f_t and $\tilde{f}_t$ are tropically equivalent.*

Proof: The function $z \rightarrow \frac{M}{N}z$ corresponds tropically to:

$$x \rightarrow \mu(x) \equiv \max(x, \dots, x) - \max(0, \dots, 0) = x.$$

Let φ be the corresponding relative $(\max, +)$ -function to f_t . Then the another relative $(\max, +)$ -function:

$$\tilde{\varphi}(x_0, \dots, x_{n-1}) \equiv \varphi(\mu(x_0), \dots, \mu(x_{n-1})) = \varphi(x_0, \dots, x_{n-1})$$

corresponds to $\tilde{f}$. This completes the proof.

(C) Let us relate φ_t with f_t . Let $\text{Log}_t : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}^n$ be given by:

$$(x_0, \dots, x_{n-1}) = \text{Log}_t(z_0, \dots, z_{n-1}) \equiv (\log_t z_0, \dots, \log_t z_{n-1}).$$

Proposition 2.4 (LM,V). *$f_t \equiv (\log_t)^{-1} \circ \varphi_t \circ \text{Log}_t : \mathbb{R}_{>0}^n \rightarrow (0, \infty)$ is a parametrized rational function $f_t(\bar{z}) \equiv \frac{k_t(\bar{z})}{h_t(\bar{z})} = \frac{\sum_{k=1}^m t^{\alpha_k} \bar{z}^{\bar{a}_k}}{\sum_{k=1}^l t^{\beta_k} \bar{z}^{\bar{b}_k}}$.*

One can check this equality by direct calculations. Even though verification is quite easy, it plays an important role in tropical geometry.

In particular $f_t > 0$ take positive values on $\mathbb{R}_{>0}^n = (0, \infty)^n$. Notice that $f_t(\bar{0}) = 0$ may occur.

Let $(\varphi, \varphi_t, f_t)$ be the triplet as above, and M be the number of the components. Let us induce C^0 comparisons:

Lemma 2.5. *Suppose φ is uniformly bounded from both above and below as $a \leq \varphi(\bar{x}) \leq b$. Then f_t admits uniform bounds:*

$$t^a M^{-1} \leq f_t(\bar{z}) \leq t^b M.$$

Proof: For any $\bar{z} \in \mathbb{R}_{>0}^n$, let us denote $\bar{x} = \text{Log}_t(\bar{z})$. By use of proposition 2.4, we have the equalities:

$$f_t(\bar{z}) = t^{\varphi_t(\text{Log}_t(\bar{z}))} = t^{\varphi_t(\bar{x})} = t^{\varphi(\bar{x})} t^{\varphi_t(\bar{x}) - \varphi(\bar{x})}.$$

Then By lemma 2.1 we have the estimates:

$$t^a M^{-1} \leq f_t(\bar{z}) \leq t^b M.$$

This completes the proof.

Next let us induce C^1 comparisons. Let ψ and ϕ be two $(\max, +)$ -functions so that the equality $\varphi = \psi - \phi$ holds.

Lemma 2.6. *Suppose (1) φ is uniformly bounded from both above and below, and (2) ϕ is bounded from below. Then any derivatives of f_t are uniformly bounded from above, after change by troipically equivalent one, if necessarily.*

Proof: The bounds $a \leq \varphi = \psi - \phi \leq b$ hold by the assumption.

Let us write $f_t = \frac{h_t}{k_t}$, where both h_t and k_t are polynomials parametrized by $t > 1$, which correspond to ψ and ϕ respectively.

By the assumption, there is some $N \geq 0$ so that the equality $\phi' = \max(-N, \phi)$ holds as functions. Then we may assume positivity $k_t(0) > 0$, by change of ϕ by ϕ' if necessarily, which corresponds to $k_t + t^{-N}$.

The uniform bounds of the range of f_t implies that degrees of h_t and k_t coincide each other. Let us consider the derivative $f'_t = \frac{h'_t}{k_t} - \frac{h_t k'_t}{k_t^2}$. The both terms have the property that the degree of the denominators are strictly larger than that of the numerators.

These imply that the derivatives of f_t are also uniformly bounded from above. This completes the proof.

Notice that the higher derivateives can be considered similarly. In section 5, we induce more detailed estimates.

2.B State dynamics: In 2.B we study analysis of dynamical systems by relatively elementary functions. Let:

$$\varphi : \mathbb{R}^{\alpha+2} \rightarrow \mathbb{R}, \quad \psi : \mathbb{R}^{\beta+2} \rightarrow \mathbb{R}$$

be two piecewise-linear functions which admit their presentations by the relative $(\max, +)$ functions:

$$\begin{aligned} \psi(\bar{r}) &= \max(\alpha_1 + \bar{a}_1 \bar{r}, \dots, \alpha_{\alpha+2} + \bar{a}_{\alpha+2} \bar{r}) - \max(\beta_1 + \bar{b}_1 \bar{r}, \dots, \beta_{\alpha+2} + \bar{b}_{\alpha+2} \bar{r}), \\ \phi(\bar{l}) &= \max(\gamma_1 + \bar{c}_1 \bar{l}, \dots, \gamma_{\beta+2} + \bar{c}_{\beta+2} \bar{l}) - \max(\delta_1 + \bar{d}_1 \bar{l}, \dots, \delta_{\beta+2} + \bar{d}_{\beta+2} \bar{l}) \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\bar{a}, \bar{b} \in \mathbb{R}^{\alpha+2}$ and $\bar{c}, \bar{d} \in \mathbb{R}^{\beta+2}$ are all constants.

Later on we denote by $M = \max(M_\psi, M_\phi)$ and $c = \max(c_\psi, c_\phi)$ as the bigger ones of the numbers of the components and the Lipschitz constants for ψ and ϕ respectively.

Let us take initial sequences in $\mathbb{R}$:

$$\{x_i\}_{i \geq 0}, \quad \{y^j\}_{j \geq 0}.$$

Definition 2.2. *The state dynamics with respect to the pair (ψ, ϕ) are given by the discrete dynamics inductively defined by the iterations:*

$$\begin{aligned} x_i^{j+1} &= \psi(y_i^j, x_i^j, \dots, x_{i+\alpha}^j), \\ y_{i+1}^j &= \phi(y_i^j, x_i^j, \dots, x_{i+\beta}^j) \end{aligned}$$

where the initial values are given by $x_i^0 = x_i$ and $y_0^j = y^j$ for $i, j = 0, 1, 2, \dots$

2.B.2 Comparisons between iterated dynamics: Let (ψ_t, ϕ_t) be the tropical correspondences of the pair (ψ, ϕ) . One can also consider another state dynamics by use of (ψ_t, ϕ_t) instead of the pair. In fact we have uniform estimates of their orbits, which we describe below.

Recall the dynamics of automata in the introduction. There are dynamics of the states y_i^j behind the actions $\mathbf{A}_{(y^0, y^1, \dots, y^j)}$. Firstly let us start analyzing the orbits $\{y_0^j\}_j$, since their treatment is relatively simple compared with $\{x_i^j\}_{i,j}$.

Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a relative $(\max, +)$ -function of n variables, with the number of the components M and the Lipschitz constant c . Let φ_t be the corresponding function.

Now we consider the dynamics of the states, and choose any initial data $(x_0, x_1, \dots) \in \mathbb{R}^{\mathbb{N}}$ and $y_0 \in \mathbb{R}$. Then one considers two discrete dynamics defined inductively by the iterations for $i \geq 0$:

$$\begin{aligned} y_{i+1} &= \varphi(y_i, x_i, \dots, x_{i+\beta}), \\ y'_{i+1} &= \varphi_t(y'_i, x_i, \dots, x_{i+\beta}) \end{aligned}$$

where $y'_0 = y_0$. We denote $\bar{x}_i = (x_i, \dots, x_{i+\beta})$ for simplicity of the notation.

Let us put the polynomials of degree i :

$$P_i(c) = \frac{c^{i+1} - 1}{c - 1}.$$

Notice the equality:

$$cP_i(c) + 1 = P_{i+1}(c).$$

Lemma 2.7. *The uniform estimates:*

$$|y_i - y'_i| \leq P_{i-1}(c) \log_t M$$

hold for all $i \geq 0$.

Proof: Firstly the estimates $|y'_1 - y_1| = |\varphi_t(y_0, \bar{x}_0) - \varphi(y_0, \bar{x}_0)| \leq \log_t M$ hold by lemma 2.1.

Next we have the estimates:

$$\begin{aligned} |y_2 - y'_2| &= |\varphi(y_1, \bar{x}_1) - \varphi_t(y'_1, \bar{x}_1)| \\ &\leq |\varphi(y_1, \bar{x}_1) - \varphi(y'_1, \bar{x}_1)| + |\varphi(y'_1, \bar{x}_1) - \varphi_t(y'_1, \bar{x}_1)| \\ &\leq c|y_1 - y'_1| + \log_t M \leq (c + 1) \log_t M \end{aligned}$$

Similarly we have the following estimates:

$$\begin{aligned} |y_3 - y'_3| &= |\varphi(y_2, \bar{x}_2) - \varphi_t(y'_2, \bar{x}_2)| \\ &\leq |\varphi(y_2, \bar{x}_2) - \varphi(y'_2, \bar{x}_2)| + |\varphi(y'_2, \bar{x}_2) - \varphi_t(y'_2, \bar{x}_2)| \leq [c(c + 1) + 1] \log_t M \end{aligned}$$

By iterating the same estimates, one obtains the conclusion.

This completes the proof.

2.C Basic estimates for orbits: The estimates for the dynamics of $\{x_i^j\}_{i,j}$ involve more complicated analysis. As preliminaries, we verify some general estimates which will be used later.

For four sequences $\{p_i\}_i, \{q^j\}_j, \{x_i\}_i, \{y^j\}_j$ by real numbers, let us introduce the numbers:

$$|\{p_i, x_i\}_i; \{q^j, y^j\}_j| \equiv \sup_{i,j} \{|p_i - x_i|, |y^j - q^j|\}.$$

Let us take four sequences:

$$\{p_i^j\}_{i,j \geq 0}, \{q_i^j\}_{i,j \geq 0}, \{x_i^j\}_{i,j \geq 0}, \{y_i^j\}_{i,j \geq 0}.$$

For the Lipschitz constants c , let us put:

$$\tilde{c} = \max(c, 1).$$

The following type of the estimates are applied when we consider Mealy automata. The general cases are treated after this version.

Lemma 2.8. Suppose these sequences satisfy the following estimates:

$$|p_i^{j+1} - x_i^{j+1}|, |q_{i+1}^j - y_{i+1}^j| \leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|) + T$$

for some $T \in \mathbb{R}$ and $c \geq 0$.

Then they satisfy the estimates for all $i, j \geq 0$:

$$|p_i^{j+1} - x_i^{j+1}|, |q_{i+1}^j - y_{i+1}^j| \leq P_{i+j}(c)T + \tilde{c}^{i+j}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.$$

In particular:

(1) If they satisfy the same initial conditions:

$$p_i^0 = x_i^0, \quad q_0^j = y_0^j \quad (i, j \geq 0)$$

then the uniform estimates hold:

$$|p_i^{j+1} - x_i^{j+1}|, |q_{i+1}^j - y_{i+1}^j| \leq P_{i+j}(c)T.$$

(2) If $c < 1$ holds, then $|p_i^{j+1} - x_i^{j+1}|$ and $|q_{i+1}^j - y_{i+1}^j|$ are both uniformly bounded.

Proof of lemma 2.10: We verify the conclusion by induction on $i + j \geq 1$. For $i + j = 1$, the estimates:

$$\begin{aligned} |p_0^1 - x_0^1|, |q_1^0 - y_1^0| &\leq c \max(|q_0^0 - y_0^0|, |p_0^0 - x_0^0|) + T \\ &\leq T + c|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

hold. So the conclusion follows for $i + j = 1$.

Suppose the conclusion holds for all (i, j) with $i + j \leq N$, and take any (i, j) with $i + j = N \geq 1$.

Firstly assume both $i, j \neq 0$. Then the estimates hold:

$$\begin{aligned} |p_i^{j+1} - x_i^{j+1}|, |q_{i+1}^j - y_{i+1}^j| &\leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|) + T \\ &\leq c(P_{i+j-1}(c)T + \tilde{c}^{i+j-1}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\ &= P_{i+j}(c)T + \tilde{c}^{i+j}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

Next suppose $j = 0$ and $i = N \geq 1$. Then:

$$\begin{aligned} |p_i^1 - x_i^1|, |q_{i+1}^0 - y_{i+1}^0| &\leq c \max(|q_i^0 - y_i^0|, |p_i^0 - x_i^0|) + T \\ &\leq c \max(P_{i-1}(c)T + \tilde{c}^{i-1}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|, |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\ &\leq P_{i+j}(c)T + \tilde{c}^i|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j| \end{aligned}$$

since the estimates hold:

$$|p_i^0 - x_i^0| \leq |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j| \leq \tilde{c}^i |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|$$

We can treat the case $i = 0$ by the same way. Thus we have verified the claim for $i + j \leq N + 1$. This finishes the induction step.

This completes the proof.

Let us take four sequences $\{p_i\}_i, \{q^j\}_j, \{x_i\}_i, \{y^j\}_j$ as before. Let $\alpha, \beta \geq 0$ be constants and put:

$$\gamma = \max(\alpha, \beta).$$

Now we verify the general version which can be applied to estimate the orbits of the state dynamics. The argument is more complicated.

Proposition 2.9. *Suppose these satisfy the following estimates:*

$$\begin{aligned} |p_i^{j+1} - x_i^{j+1}| &\leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|, \dots, |p_{i+\alpha}^j - x_{i+\alpha}^j|) + T \\ |q_{i+1}^j - y_{i+1}^j| &\leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|, \dots, |p_{i+\beta}^j - x_{i+\beta}^j|) + T \end{aligned}$$

Then they satisfy the estimates:

$$\begin{aligned} |p_i^{j+1} - x_i^{j+1}|, |q_{i+1}^j - y_{i+1}^j| \\ \leq P_{i+j(\gamma+1)}(c)T + \tilde{c}^{i+1+j(\gamma+1)} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

Proof: We split the proof into several steps.

Step 1: Firstly we claim that the estimates below hold by induction on i :

$$|q_i^0 - y_i^0| \leq P_{i-1}(c)T + \tilde{c}^i |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.$$

For $i = 0$, the estimates $|q_0^0 - y_0^0| \leq |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|$ hold by definition.

Suppose the claim hold up to $i \geq 0$. Then:

$$\begin{aligned} |q_{i+1}^0 - y_{i+1}^0| &\leq c \max(|q_i^0 - y_i^0|, |p_i^0 - x_i^0|, \dots, |p_{i+\beta}^0 - y_{i+\beta}^0|) + T \\ &\leq c(P_{i-1}(c)T + \tilde{c}^i |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\ &\leq P_i(c)T + \tilde{c}^{i+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

Thus they hold up to $i + 1$. This verifies the claim.

Then we have the estimates:

$$\begin{aligned}
|p_i^1 - x_i^1| &\leq c \max(|q_i^0 - y_i^0|, |p_i^0 - x_i^0|, \dots, |p_{i+\alpha}^0 - y_{i+\alpha}^0|) + T \\
&\leq c(P_{i-1}(c)T + \tilde{c}^i |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\
&\leq P_i(c)T + \tilde{c}^{i+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.
\end{aligned}$$

Step 2: Next we claim that the estimates hold:

$$|q_i^1 - y_i^1| \leq P_{i+\beta}(c)T + \tilde{c}^{i+\beta+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.$$

We proceed by induction on i . For $i = 0$, the estimates:

$$|q_0^1 - y_0^1| \leq |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j| \leq P_\beta(c)T + \tilde{c}^{\beta+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|$$

hold by definition, since $\tilde{c} \geq 1$ holds.

Suppose the above estimates hold up to $i \geq 0$. Then:

$$\begin{aligned}
|q_{i+1}^1 - y_{i+1}^1| &\leq c \max(|q_i^1 - y_i^1|, |p_i^1 - x_i^1|, \dots, |p_{i+\beta}^1 - x_{i+\beta}^1|) + T \\
&\leq c \max(|q_i^1 - y_i^1|, P_{i+\beta}(c)T + \tilde{c}^{i+\beta+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\
&\leq c(P_{i+\beta}(c)T + \tilde{c}^{i+\beta+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\
&= P_{i+\beta+1}(c)T + \tilde{c}^{i+\beta+2} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|
\end{aligned}$$

where we used step 1 at the second inequalities. So the above estimates also hold for $i + 1$, and we have verified the claim.

Then we have the estimates:

$$\begin{aligned}
|p_i^2 - x_i^2| &\leq c \max(|q_i^1 - y_i^1|, |p_i^1 - x_i^1|, \dots, |p_{i+\alpha}^1 - x_{i+\alpha}^1|) + T \\
&\leq c \max(P_{i+\beta}(c)T + \tilde{c}^{i+\beta+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|, \\
&\quad P_{i+\alpha}(c)T + \tilde{c}^{i+\alpha+1} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\
&= P_{i+\gamma+1}(c)T + \tilde{c}^{i+\gamma+2} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.
\end{aligned}$$

Step 3: Let us verify the estimates for the general case by induction on j .

In step 1, 2, we have verified the conclusions for $j \leq 1$ and all $i \geq 0$.

Suppose the conclusions hold up to $j - 1 \geq 1$ and all $i \geq 0$. Firstly let us consider the pair (y_i^j, q_i^j) . We claim that the estimates hold for all $i \geq 0$:

$$|q_i^j - y_i^j| \leq P_{i-1+j(\gamma+1)}(c)T + \tilde{c}^{i+j(\gamma+1)} |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|.$$

The estimates $|q_0^j - y_0^j| \leq |\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|$ hold by definition. Let us proceed by induction on i . Suppose the above estimates hold up to $i \geq 0$. Then:

$$\begin{aligned} |q_{i+1}^j - y_{i+1}^j| &\leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|, \dots, |p_{i+\beta}^j - x_{i+\beta}^j|) + T \\ &\leq c \max(P_{i-1+j(\gamma+1)}(c)T + \tilde{c}^{i+(j+1)(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|, \\ &\quad P_{i+\beta+(j-1)(\gamma+1)}(c)T + \tilde{c}^{i+\beta+1+(j-1)(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\ &= P_{i+j(\gamma+1)}(c)T + \tilde{c}^{i+1+j(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

So the estimates also hold for $i + 1$, and we have verified the claim.

Then we claim that the estimates:

$$|p_i^{j+1} - x_i^{j+1}| \leq P_{i+j(\gamma+1)}(c)T + \tilde{c}^{i+1+j(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|$$

hold. This follows from the following:

$$\begin{aligned} |p_i^{j+1} - x_i^{j+1}| &\leq c \max(|q_i^j - y_i^j|, |p_i^j - x_i^j|, \dots, |p_{i+\alpha}^j - x_{i+\alpha}^j|) + T \\ &\leq c \max(P_{i-1+j(\gamma+1)}(c)T + \tilde{c}^{i+j(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|, \\ &\quad P_{i+\alpha+(j-1)(\gamma+1)}(c)T + \tilde{c}^{i+\alpha+1+(j-1)(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|) + T \\ &= P_{i+j(\gamma+1)}(c)T + \tilde{c}^{i+1+j(\gamma+1)}|\{p_i^0, x_i^0\}_i; \{q_0^j, y_0^j\}_j|. \end{aligned}$$

Thus we have verified the claim under the induction hypothesis up to $j - 1$.

This completes the proof.

2.C.2 Uniform estimates between the orbits for Mealy automata:

Let ψ and ϕ be a pair of relative $(\max, +)$ -functions of two variables. Let c and M be the maximums of the Lipschitz constants and the numbers of their components respectively.

Let ψ_t and ϕ_t be the tropical correspondences to ψ and ϕ respectively, and consider the systems of the equations:

$$\begin{aligned} x(i, j+1) &= \psi(y(i, j), x(i, j)), \\ y(i+1, j) &= \phi(y(i, j), x(i, j)), \end{aligned}$$

$$\begin{aligned} x'(i, j+1) &= \psi_t(y'(i, j), x'(i, j)), \\ y'(i+1, j) &= \phi_t(y'(i, j), x'(i, j)). \end{aligned}$$

with the same initial values $x'(i, 0) = x(i, 0) = x_i$ and $y'(0, j) = y(0, j) = y^j$.

Lemma 2.10. *The uniform estimates hold:*

$$|x(i, j+1) - x'(i, j+1)|, |y(i+1, j) - y'(i+1, j)| \leq P_{i+j}(c) \log_t M.$$

Proof: By lemma 2.1, both the estimates hold:

$$\begin{aligned} |x(i, j+1) - x'(i, j+1)| &= |\psi(y(i, j), x(i, j)) - \psi_t(y'(i, j), x'(i, j))| \\ &\leq |\psi(y(i, j), x(i, j)) - \psi(y'(i, j), x'(i, j))| \\ &\quad + |\psi(y'(i, j), x'(i, j)) - \psi_t(y'(i, j), x'(i, j))| \\ &\leq c \max(|y(i, j) - y'(i, j)|, |x(i, j) - x'(i, j)|) + \log_t M, \end{aligned}$$

$$\begin{aligned} |y(i+1, j) - y'(i+1, j)| &= |\phi(y(i, j), x(i, j)) - \phi_t(y'(i, j), x'(i, j))| \\ &\leq |\phi(y(i, j), x(i, j)) - \phi(y'(i, j), x'(i, j))| \\ &\quad + |\phi(y'(i, j), x'(i, j)) - \phi_t(y'(i, j), x'(i, j))| \\ &\leq c \max(|y(i, j) - y'(i, j)|, |x(i, j) - x'(i, j)|) + \log_t M. \end{aligned}$$

By applying lemma 2.10 for $p_i^j = x'(i, j)$, $x(i, j) = x_i^j$, $q_i^j = y'(i, j)$ and $y_i^j = y(i, j)$ with $T = \log_t M$, one obtains the desired result.

This completes the proof.

2.C.3 Uniform estimates for the state dynamics: Now let us consider the general case. Let ψ and ϕ be a pair of relative $(\max, +)$ -functions as in 2.B.

Let us choose the initial data $\{x_i\}_{i \geq 0}$ and $\{y^j\}_{j \geq 0}$ respectively, and consider two state dynamics given by the systems of the equations:

$$\begin{aligned} x(i, j+1) &= \psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)) \\ y(i+1, j) &= \phi(y(i, j), x(i, j), \dots, x(i+\beta, j)), \end{aligned}$$

$$\begin{aligned} x'(i, j+1) &= \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j)) \\ y'(i+1, j) &= \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta, j)), \end{aligned}$$

with the same initial values:

$$x(i, 0) = x'(i, 0) = x_i, \quad y(0, j) = y'(0, j) = y^j.$$

We verify the following:

Proposition 2.11. *The uniform estimates hold:*

$$|x(i, j+1) - x'(i, j+1)|, |y(i+1, j) - y'(i+1, j)| \leq P_{i+j(\gamma+1)}(c) \log_t M.$$

Proof: The proof is parallel to lemma 2.12. By lemma 2.1, one has the estimates:

$$\begin{aligned} & |x(i, j+1) - x'(i, j+1)| \\ &= |\psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)) - \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))| \\ &\leq |\psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)) - \psi(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))| \\ &\quad + |\psi(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j)) - \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))| \\ &\leq c \max(|y(i, j) - y'(i, j)|, |x(i, j) - x'(i, j)|, \\ &\quad \dots, |x(i+\alpha, j) - x'(i+\alpha, j)|) + \log_t M, \end{aligned}$$

$$\begin{aligned} & |y(i+1, j) - y'(i+1, j)| \\ &= |\phi(y(i, j), x(i, j), \dots, x(i+\beta, j)) - \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta, j))| \\ &\leq |\phi(y(i, j), x(i, j), \dots, x(i+\beta, j)) - \phi(y'(i, j), x'(i, j), \dots, x'(i+\beta, j))| \\ &\quad + |\phi(y'(i, j), x'(i, j), \dots, x'(i+\beta, j)) - \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta, j))| \\ &\leq c \max(|y(i, j) - y'(i, j)|, |x(i, j) - x'(i, j)|, \\ &\quad \dots, |x(i+\beta, j) - x'(i+\beta, j)|) + \log_t M. \end{aligned}$$

By applying proposition 2.11 for $p_i^j = x'(i, j)$, $x(i, j) = x_i^j$, $q_i^j = y'(i, j)$ and $y_i^j = y(i, j)$ with $T = \log_t M$, one obtains the result.

This completes the proof.

2.C.4 Initial value dependence: Let ϕ , ψ and M, c be as in 2.B, and consider the state dynamics:

$$\begin{aligned} x(i, j+1) &= \psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)), \\ y(i+1, j) &= \phi(y(i, j), x(i, j), \dots, x(i+\beta, j)). \end{aligned}$$

Here we consider how the initial values influence on the long time behavior of the dynamics. For $l = 1, 2$, let $\{x_i(l)\}_{j \geq 0}$ and $\{y^j(l)\}_{j \geq 0}$ be two initial data, and denote the corresponding solutions by $\{(x_l(i, j), y_l(i, j))\}_{l=1,2}$ with $x_l(i, 0) = x_i(l)$ and $y_l(0, j) = y^j(l)$. Recall $|\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|$ in 2.C.

Lemma 2.12. *The estimates hold:*

$$\begin{aligned} |x_1(i, j+1) - x_2(i, j+1)|, |y_1(i+1, j) - y_2(i+1, j)| \\ \leq \tilde{c}^{i+1+j(\gamma+1)} |\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|. \end{aligned}$$

Proof: Let us consider the estimates:

$$\begin{aligned} |x_1(i, j+1) - x_2(i, j+1)| = \\ |\psi(y_1(i, j), x_1(i, j), \dots, x_1(i+\alpha, j)) - \psi(y_2(i, j), x_2(i, j), \dots, x_2(i+\alpha, j))| \\ \leq c \max(|y_1(i, j) - y_2(i, j)|, \dots, |x_1(i+\alpha, j) - x_2(i+\alpha, j)|) \end{aligned}$$

$$\begin{aligned} |y_1(i+1, j) - y_2(i+1, j)| = \\ |\psi(y_1(i, j), x_1(i, j), \dots, x_1(i+\beta, j)) - \psi(y_2(i, j), x_2(i, j), \dots, x_2(i+\beta, j))| \\ \leq c \max(|y_1(i, j) - y_2(i, j)|, \dots, |x_1(i+\beta, j) - x_2(i+\beta, j)|). \end{aligned}$$

Then by applying proposition 2.11 for $p_i^j = x_1(i, j)$, $x_i^j = x_2(i, j)$ and $q_i^j = y_1(i, j)$, $y_i^j = y_2(i, j)$ with $T = 0$, one obtains the desired estimates.

This completes the proof.

2.D Rational dynamics: Let $\varphi : \mathbb{R}^{\alpha+2} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^{\beta+2} \rightarrow \mathbb{R}$ be relative $(\max, +)$ functions as in 2.B, with the constants M and c as before.

Passing through the scale transform in tropical geometry, one obtains two parametrized rational functions f_t and g_t with respect to φ and ψ respectively:

$$f_t(\bar{z}) = \frac{t^{\alpha_1} \bar{z}^{\bar{a}_1} + \dots + t^{\alpha_{\alpha+2}} \bar{z}^{\bar{a}_{\alpha+2}}}{t^{\beta_1} \bar{z}^{\bar{b}_1} + \dots + t^{\beta_{\alpha+2}} \bar{z}^{\bar{b}_{\alpha+2}}}, \quad g_t(\bar{w}) = \frac{t^{\gamma_1} \bar{w}^{c_1} + \dots + t^{\gamma_{\beta+2}} \bar{w}^{c_{\beta+2}}}{t^{\delta_1} \bar{w}^{d_1} + \dots + t^{\delta_{\beta+2}} \bar{w}^{d_{\beta+2}}}.$$

Let us take initial values:

$$0 < z_i < \infty, \quad 0 < w^j < \infty$$

for $i, j = 0, 1, 2, \dots$

Definition 2.3. *The dynamical system given by:*

$$\begin{aligned} z(i, j+1) &= f_t(w(i, j), z(i, j), \dots, z(i+\alpha, j)), \\ w(i+1, j) &= g_t(w(i, j), z(i, j), \dots, z(i+\beta, j)) \end{aligned}$$

is called the state system of the rational dynamics, where the initial values are given by $z(i, 0) = z_i$ and $w(0, j) = w^j$.

Let ψ_t and ϕ_t be the tropical correspondences to ψ and ϕ respectively. Let us consider the state dynamics:

$$\begin{aligned} x'(i, j+1) &= \psi_t(y'(i, j), x'(i, j), \dots, x(i+\alpha, j)) \\ y'(i+1, j) &= \phi_t(y'(i, j), x'(i, j), \dots, x(i+\beta, j)) \end{aligned}$$

with the initial values $x'(i, 0) = \log_t z_i$ and $y'(0, j) = \log_t w^j$.

Lemma 2.13. *The equalities hold:*

$$x'(i, j) = \log_t z(i, j), \quad y'(i, j) = \log_t w(i, j).$$

Proof: This follows from Proposition 2.4. This completes the proof.

2.D.2 Analysis on the equivalent dynamics for simple case: Compare the contents here with 2.B.2.

Let φ^1 and φ^2 be relative $(\max, +)$ -functions, which are mutually equivalent with their Lipschitz constant c . Let M be the bigger one of the numbers of the components.

Let g_t^1 and g_t^2 be the corresponding functions which are mutually tropically equivalent (see definition 2.1).

Let us take $w_0 \in (0, \infty)$ and an initial sequence $\{z_0, z_1, \dots\} \in \mathbb{R}_{>0}^{\mathbb{N}}$ by positive numbers. Then one considers the rational dynamics defined inductively by the iterations:

$$w_{i+1}^l = g_t^l(w_i^l, z_i, \dots, z_{i+\beta}) \quad (l = 1, 2)$$

with $w_0^l = w_0$. Notice that these orbits take positive values.

Lemma 2.14. *The uniform estimates hold for all $i \geq 0$:*

$$\frac{w_i^1}{w_i^2}, \quad \frac{w_i^2}{w_i^1} \leq M^{2P_{i-1}(c)}$$

Proof: Let us put $x_i = \log_t z_i$ and $y_0 = \log_t w_0$, and consider the discrete dynamics defined inductively by the iterations:

$$y_{i+1}^l = \varphi^l(y_i, x_i, \dots, x_{i+\beta}), \quad (y_{i+1}^l)' = \varphi_t^l(y_i', x_i, \dots, x_{i+\beta})$$

with $y_0^l = (y_0^l)' = y_0$. The equalities:

$$y_i^1 = y_i^2$$

hold for all $i \geq 0$, since φ^1 and φ^2 are mutually equivalent.

By proposition 2.4, the equalities hold:

$$(y^l)_i' = \log_t w_i^l \quad (l = 1, 2)$$

The following estimates follow from lemma 2.9:

$$\begin{aligned} \log_t \left\{ \max \left(\frac{w_i^1}{w_i^2}, \frac{w_i^2}{w_i^1} \right) \right\} &= |\log_t w_i^1 - \log_t w_i^2| = |(y^1)_i' - (y^2)_i'| \\ &\leq |(y^1)_i' - y_i^1| + |y_i^1 - y_i^2| + |y_i^2 - (y^2)_i'| \\ &\leq 2P_{i-1}(c) \log_t M = \log_t M^{2P_{i-1}(c)}. \end{aligned}$$

Since $\log_t$ are increasing, these estimates imply the desired one.

This completes the proof.

Later on we will use the notations:

$$\left(\frac{w}{w'} \right)^{\pm 1} \equiv \max \left(\frac{w}{w'}, \frac{w'}{w} \right).$$

2.D.3 Analysis on the equivalent dynamics: For $l = 1, 2$, let $\{w^j(l)\}_{j \geq 0}$ and $\{z_i(l)\}_{i \geq 0}$ be sequences by positive numbers.

Definition 2.4. *The initial rate is given by the positive number:*

$$[\{(z_i(l), w^j(l))\}_{l=1}^2] = \sup_{i,j} \max \left(\frac{z_i(1)}{z_i(2)}, \frac{z_i(2)}{z_i(1)}, \frac{w^j(1)}{w^j(2)}, \frac{w^j(2)}{w^j(1)} \right) \geq 1.$$

Let us put $\log_t z_i(l) = x_i(l)$ and $\log_t w^j(l) = y^j(l)$. Then the equality holds:

$$\log_t [\{(z_i(l), w^j(l))\}_{l=1}^2] = |\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|.$$

For the later notation, see 2.C.

Let (ψ^1, ϕ^1) and (ψ^2, ϕ^2) be paris of $(\max, +)$ -functions so that $\psi^1 \sim \psi^2$ and $\phi^1 \sim \phi^2$ are mutually equivalent. We say that (ψ^1, ϕ^1) and (ψ^2, ϕ^2) are *pairwisely equivalent*.

Let c_φ and M_φ be the Lipschitz constants and the numbers of the components for φ respectively, and put:

$$c = \max(c_{\psi^1}, c_{\psi^2}, c_{\phi^1}, c_{\phi^2}), \quad M = \max(M_{\psi^1}, M_{\psi^2}, M_{\phi^1}, M_{\phi^2}).$$

Let (f_t^1, g_t^1) and (f_t^2, g_t^2) be the relatively elementary functions with respect to (ψ^1, ϕ^1) and (ψ^2, ϕ^2) .

Let $\{w^j(l)\}_{j \geq 0}$ and $\{z_i(l)\}_{i \geq 0}$ be the initial sequences by positive numbers, and denote the solutions by $(z_i^j(l), w_i^j(l))$ to the state systems of the rational dynamics:

$$\begin{aligned} z_i^{j+1}(l) &= f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ w_{i+1}^j(l) &= g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)). \end{aligned}$$

with the initial values $z_i^0(l) = z_i(l)$ and $w_0^j(l) = w^j(l)$ respectively.

Proposition 2.15. *The uniform estimates hold:*

$$\begin{aligned} &\max\left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}, \frac{z_i^{j+1}(2)}{z_i^{j+1}(1)}, \frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}, \frac{w_{i+1}^j(2)}{w_{i+1}^j(1)}\right) \\ &\leq M^{2P_{i+j}(\gamma+1)(c)} [\{(z_i(l), w^j(l))\}_{l=1}^2]^{\tilde{c}^{i+1+j}(\gamma+1)}. \end{aligned}$$

In particular if the initial values are the same:

$$z_i(1) = z_i(2), \quad w^j(1) = w^j(2)$$

then they satisfy the uniform estimates:

$$\max\left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}, \frac{z_i^{j+1}(2)}{z_i^{j+1}(1)}, \frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}, \frac{w_{i+1}^j(2)}{w_{i+1}^j(1)}\right) \leq M^{2P_{i+j}(\gamma+1)(c)}.$$

Notice that for the Mealy case ($\alpha = \beta = 0$), ther rates are bounded by:

$$M^{2P_{i+j}(c)} [\{(z_i(l), w^j(l))\}_{l=1}^2]^{\tilde{c}^{i+j+1}}.$$

Proof of proposition 2.17: Let us consider the solutions to the equations:

$$\begin{aligned} x'_l(i, j+1) &= \psi_t^l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i+\alpha, j)) \\ y'_l(i+1, j) &= \phi_t^l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i+\beta, j)) \end{aligned}$$

with the initial values $x'_l(i, 0) = \log_t z_i(l)$ and $y'_l(0, j) = \log_t w^j(l)$ for $l = 1, 2$.

With the same initial values, let us also consider the solutions to another equations:

$$\begin{aligned}x_l(i, j+1) &= \psi^l(y_l(i, j), x_l(i, j), \dots, x_l(i+\alpha, j)) \\ y_l(i+1, j) &= \phi^l(y_l(i, j), x_l(i, j), \dots, x_l(i+\beta, j))\end{aligned}$$

with $x_l(i, 0) = x'_l(i, 0)$ and $y_l(0, j) = y'_l(0, j)$.

By proposition 2.13, the estimates hold for $l = 1, 2$:

$$|x_l(i, j+1) - x'_l(i, j+1)|, |y_l(i+1, j) - y'_l(i+1, j)| \leq P_{i+j(\gamma+1)}(c) \log_t M.$$

On the other hand by lemma 2.14, the estimates hold:

$$\begin{aligned}|x_1(i, j+1) - x_2(i, j+1)|, |y_1(i+1, j) - y_2(i+1, j)| \\ \leq \tilde{c}^{i+1+j(\gamma+1)} |\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|.\end{aligned}$$

Thus combining with these, the estimates hold:

$$\begin{aligned}& |x'_1(i, j) - x'_2(i, j)| \\ & \leq |x'_1(i, j) - x_1(i, j)| + |x_1(i, j) - x_2(i, j)| + |x_2(i, j) - x'_2(i, j)| \\ & \leq 2P_{i+(j-1)(\gamma+1)}(c) \log_t M + \tilde{c}^{i+1+(j-1)(\gamma+1)} |\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|, \\ & |y'_1(i, j) - y'_2(i, j)| \\ & \leq |y'_1(i, j) - y_1(i, j)| + |y_1(i, j) - y_2(i, j)| + |y_2(i, j) - y'_2(i, j)| \\ & \leq 2P_{i-1+j(\gamma+1)}(c) \log_t M + \tilde{c}^{i+j(\gamma+1)} |\{x_i(1), x_i(2)\}_i; \{y^j(1), y^j(2)\}_j|.\end{aligned}$$

Since $\log_t F^l(i, j) = x'_l(i, j)$ and $\log_t G^l(i, j) = y'_l(i, j)$ hold by proposition 2.4, these estimates verify the conclusions. This completes the proof.

2.E Dynamical inequalities: Let φ^1 and φ^2 be relatively $(\max, +)$ -functions with $M = \max(M_{\varphi^1}, M_{\varphi^2})$.

Suppose they are mutually equivalent and so the equality $c = c_{\varphi^1} = c_{\varphi^2}$ holds. Let us denote tropically equivalent functions by g_t^1 and g_t^2 correspondingly.

Notice that there are some cases where the inequalities $g_t^1 \leq g_t^2$ hold. For example if φ^2 has the presentation as $\varphi^2 = \max(\varphi^1, \varphi^1)$, then $g_t^1 \leq g_t^2 = 2g_t^1$ holds.

Let us analyze orbits which admit dynamical inequalities. Here we start from the simple case as in 2.D.2. Let us take $w_0 \in (0, \infty)$ and an initial

sequence $\{z_0, z_1, \dots\} \subset (0, \infty)$. Then consider the solutions $\{w_i^l\}_i$ to the equations with $w_0^l = w_0$:

$$w_{i+1}^l = g_t^l(w_i^l, z_i, \dots, z_{i+\beta}).$$

Lemma 2.16. *With the same w_0 , suppose that another sequence $\{w_i\}_i$ satisfies the dynamical inequality:*

$$g_t^1(w_i, z_i, \dots, z_{i+\beta}) \leq w_{i+1} \leq g_t^2(w_i, z_i, \dots, z_{i+\beta})$$

for all $i \geq 0$. Then the uniform estimates hold:

$$\left(\frac{w_i}{w_i^1}\right)^{\pm 1}, \left(\frac{w_i}{w_i^2}\right)^{\pm 1} \leq M^{2P_{i-1}(c)}.$$

Proof: Let us put $x_i = \log_t z_i$, $y_i = \log_t w_i$, $y_i^l = \log_t w_i^l$ and:

$$\bar{x}_i = (x_i, \dots, x_{i+\beta}).$$

Then the equations hold by proposition 2.4:

$$y_{i+1}^l = \varphi_t^l(y_i^l, \bar{x}_i) \equiv \varphi_t^l(y_i^l, x_i, \dots, x_{i+\beta}).$$

Since $\log_t$ are increasing, the estimates hold:

$$\varphi_t^1(y_i, \bar{x}_i) \leq y_{i+1} \leq \varphi_t^2(y_i, \bar{x}_i).$$

Notice that the equivalent functions take the same values:

$$\varphi^1(y, \bar{x}) = \varphi^2(y, \bar{x}).$$

Now we claim that the estimates:

$$|y_i - y_i^1| \leq 2P_{i-1}(c) \log_t M$$

hold for all $i \geq 0$. For $i = 0$, $y_0 = y_0^1$ holds.

Let us take any $i \geq 0$, and divide into two cases.

Firstly suppose $y_{i+1} \geq y_{i+1}^1$ hold. Then the estimates hold by lemma 2.1:

$$\begin{aligned} 0 \leq y_{i+1} - y_{i+1}^1 &\leq \varphi_t^2(y_i, \bar{x}_i) - \varphi_t^1(y_i^1, \bar{x}_i) = |\varphi_t^2(y_i, \bar{x}_i) - \varphi_t^1(y_i^1, \bar{x}_i)| \\ &\leq |\varphi_t^2(y_i, \bar{x}_i) - \varphi_t^2(y_i^1, \bar{x}_i)| \\ &\quad + |\varphi_t^1(y_i, \bar{x}_i) - \varphi_t^1(y_i^1, \bar{x}_i)| + |\varphi_t^1(y_i^1, \bar{x}_i) - \varphi_t^1(y_i^1, \bar{x}_i)| \\ &\leq 2 \log_t M + c|y_i - y_i^1|. \end{aligned}$$

Conversely suppose $y_{i+1} \leq y_{i+1}^1$ hold. Then the estimates hold:

$$\begin{aligned}
0 \leq y_{i+1}^1 - y_{i+1} &\leq \varphi_t^1(y_i^1, x_i) - \varphi_t^1(y_i, x_i) = |\varphi_t^1(y_i^1, x_i) - \varphi_t^1(y_i, x_i)| \\
&\leq |\varphi_t^1(y_i^1, x_i) - \varphi^1(y_i^1, x_i)| \\
&\quad + |\varphi^1(y_i^1, x_i) - \varphi^1(y_i, x_i)| + |\varphi^1(y_i, x_i) - \varphi_t^1(y_i, x_i)| \\
&\leq 2 \log_t M + c|y_i - y_i^1|.
\end{aligned}$$

Thus the estimates $|y_{i+1} - y_{i+1}^1| \leq 2 \log_t M + c|y_i - y_i^1|$ hold in any case. By iteration,

$$\begin{aligned}
|y_{i+1} - y_{i+1}^1| &\leq 2 \log_t M + c|y_i - y_i^1| \\
&\leq 2 \log_t M + c\{2 \log_t M + c|y_{i-1} - y_{i-1}^1|\} \\
&= 2(1+c) \log_t M + c^2|y_{i-1} - y_{i-1}^1| \\
&\leq \dots \leq 2P_i(c) \log_t M + c^N|y_0 - y_0^1| = 2P_i(c) \log_t M
\end{aligned}$$

hold, since $y_0 = y_0^1 = \log_t w_0$. This verifies the claim.

The left hand side is equal to $\log_t (\frac{w_{i+1}}{w_{i+1}^1})^{\pm 1}$ and the right hand side is equal to $\log_t M^{2P_i(c)}$. Since $\log_t$ are distance increasing, we obtain the estimates:

$$(\frac{w_{i+1}}{w_{i+1}^1})^{\pm 1} \leq M^{2P_i(c)}.$$

The estimates $(\frac{w_{i+1}}{w_{i+1}^1})^{\pm 1} \leq M^{2P_i(c)}$ are obtained by the same way, and we omit repetition. This completes the proof.

2.E.2 Dynamical inequalities for state dynamics: Let (ψ^1, ϕ^1) and (ψ^2, ϕ^2) be pairs of $(\max, +)$ -functions so that $\psi^1 \sim \psi^2$ and $\phi^1 \sim \phi^2$ are pairwise equivalent. Let (f_t^1, g_t^1) and (f_t^2, g_t^2) be the pairs of the corresponding relatively elementary functions.

Let us take the initial data $\{w^j\}_{j \geq 0}$ and $\{z_i\}_{i \geq 0}$ by positive numbers, and consider the solutions to the state systems of the rational dynamics:

$$\begin{aligned}
F^l(i, j+1) &= f_t^l(G^l(i, j), F^l(i, j), \dots, F^l(i+\alpha, j)), \\
G^l(i+1, j) &= g_t^l(G^l(i, j), F^l(i, j), \dots, F^l(i+\beta, j))
\end{aligned}$$

with $F^l(i, 0) = z_i$ and $G^l(0, j) = w^j$ for $l = 1, 2$.

Now we study the dynamical inequalities:

Theorem 2.17. *Let us take another sequences $\{w_i^j\}_{i,j}$ and $\{z_i^j\}_{i,j}$ by positive numbers. Suppose they satisfy the dynamical inequalities:*

$$\begin{aligned} f_t^1(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) &\leq z_i^{j+1} \leq f_t^2(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) \\ g_t^1(w_i^j, z_i^j, \dots, z_{i+\beta}^j) &\leq w_{i+1}^j \leq g_t^2(w_i^j, z_i^j, \dots, z_{i+\beta}^j) \end{aligned}$$

with the same initial values $z_i^0 = z_i$ and $w_0^j = w^j$.

Then the uniform estimates hold for $l = 1, 2$:

$$\left(\frac{F^l(i, j+1)}{z_i^{j+1}}\right)^{\pm 1}, \quad \left(\frac{G^l(i+1, j)}{w_{i+1}^j}\right)^{\pm 1} \leq M^{2P_{i+j(\gamma+1)}(c)}.$$

Proof: Let us put $p_i^j = \log_t z_i^j$ and $q_i^j = \log_t w_i^j$. Then since $\log_t$ are monotone increasing, the estimates:

$$\begin{aligned} \psi_t^1(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) &\leq p_i^{j+1} \leq \psi_t^2(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) \\ \phi_t^1(q_i^j, p_i^j, \dots, p_{i+\beta}^j) &\leq q_{i+1}^j \leq \phi_t^2(q_i^j, p_i^j, \dots, p_{i+\beta}^j) \end{aligned}$$

hold by proposition 2.4, where $p_i^0 = x_i = \log_t z_i$ and $q_0^j = y^j = \log_t w^j$.

Let us put:

$$x'_l(i, j) = \log_t F^l(i, j), \quad y'_l(i, j) = \log_t G^l(i, j).$$

Then $x'_l(i, 0) = p_i^0$ and $y'_l(0, j) = q_0^j$ hold, and they satisfy the state systems of the equations for $l = 1, 2$:

$$\begin{aligned} x'_l(i, j+1) &= \psi_t^l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i+\alpha, j)) \\ y'_l(i+1, j) &= \phi_t^l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i+\beta, j)) \end{aligned}$$

We claim that the uniform estimates hold for $l = 1, 2$:

$$|x'_l(i, j+1) - p_i^{j+1}|, |y'_l(i+1, j) - q_{i+1}^j| \leq 2P_{i+j(\gamma+1)}(c) \log_t M.$$

Firstly suppose $p_i^{j+1} \geq x_1(i, j+1)$ hold. Then the estimates hold:

$$\begin{aligned} 0 &\leq p_i^{j+1} - x'_1(i, j+1) = |p_i^{j+1} - x'_1(i, j+1)| \\ &\leq |\psi_t^2(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) - \psi_t^1(y'_1(i, j), x'_1(i, j), \dots, x'_1(i+\alpha, j))| \\ &\leq 2 \log_t M + |\psi_t^2(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) - \psi_t^1(y'_1(i, j), \dots, x'_1(i+\alpha, j))| \\ &\leq 2 \log_t M + c \max(|q_i^j - y'_1(i, j)|, |p_i^j - x'_1(i, j)|, \dots, |p_{i+\alpha}^j - x'_1(i+\alpha, j)|). \end{aligned}$$

Conversely suppose $p_i^{j+1} \leq x'_1(i, j+1)$ hold. Then the estimates hold:

$$\begin{aligned}
0 &\leq x'_1(i, j+1) - p_i^{j+1} = |x'_1(i, j+1) - p_i^{j+1}| \\
&\leq |\psi_t^1(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) - \psi_t^1(y'_1(i, j), x'_1(i, j), \dots, x'_1(i+\alpha, j))| \\
&\leq 2\log_t M + |\psi^1(q_i^j, p_i^j, \dots, p_{i+\alpha}^j) - \psi^1(y'_1(i, j), \dots, x'_1(i+\alpha, j))| \\
&\leq 2\log_t M + c \max(|q_i^j - y'_1(i, j)|, |p_i^j - x'_1(i, j)|, \dots, |p_{i+\alpha}^j - x'_1(i+\alpha, j)|).
\end{aligned}$$

Thus in any cases, the estimates:

$$|x'_1(i, j+1) - p_i^{j+1}| \leq 2\log_t M + c \max(|q_i^j - y'_1(i, j)|, \dots, |p_{i+\alpha}^j - x'_1(i+\alpha, j)|)$$

hold. By use of the similar argument, one obtains the estimates:

$$|x'_2(i, j+1) - p_i^{j+1}| \leq 2\log_t M + c \max(|q_i^j - y'_2(i, j)|, \dots, |p_{i+\alpha}^j - x'_2(i+\alpha, j)|).$$

By the same way one obtains the estimates for $l = 1, 2$:

$$\begin{aligned}
&|y'_l(i+1, j) - q_{i+1}^j| \\
&\leq 2\log_t M + c \max(|q_i^j - y'_l(i, j)|, |p_i^j - x'_l(i, j)|, \dots, |p_{i+\beta}^j - x'_l(i+\beta, j)|).
\end{aligned}$$

By applying proposition 2.11 for $x_i^j = x'_l(i, j)$ and $y_i^j = y'_l(i, j)$ with $T = \log_t M^2$, one obtains the estimates:

$$|x'_l(i, j+1) - p_i^{j+1}|, |y'_l(i+1, j) - q_{i+1}^j| \leq 2P_{i+j(\gamma+1)}(c) \log_t M.$$

Thus we have verified the claim.

From proposition 2.4, one has the inequalities:

$$|\log_t \frac{z_i^{j+1}}{F^l(i, j+1)}|, |\log_t \frac{w_{i+1}^j}{G^l(i+1, j)}| \leq \log_t M^{2P_{i+j(\gamma+1)}(c)}.$$

By removing $\log_t$ from the both sides, the conclusion follows.

This completes the proof.

Let us consider the case when the initial data take different values. Recall $\tilde{c} = \max(1, c)$.

Let us consider the solutions to the state systems:

$$\begin{aligned}
F^l(i, j+1) &= f_t^l(G^l(i, j), F^l(i, j), \dots, F^l(i+\alpha, j)) \\
G^l(i+1, j) &= g_t^l(G^l(i, j), F^l(i, j), \dots, F^l(i+\beta, j))
\end{aligned}$$

and denote $F^l(i, 0) = z_i(1)$ and $G^l(0, j) = w^j(1)$.

Corollary 2.18. *Suppose another sequences $\{w_i^j\}_{i,j}$ and $\{z_i^j\}_{i,j}$ satisfy the dynamical inequalities:*

$$\begin{aligned} f_t^1(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) &\leq z_i^{j+1} \leq f_t^2(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \\ g_t^1(w_i^j, z_i^j, \dots, z_{i+\beta}^j) &\leq w_{i+1}^j \leq g_t^2(w_i^j, z_i^j, \dots, z_{i+\beta}^j) \end{aligned}$$

with $z_i^0 \equiv z_i(2)$ and $w_0^j \equiv w^j(2)$.

Then the uniform estimates hold for $l = 1, 2$:

$$\begin{aligned} \left(\frac{F^l(i, j+1)}{z_i^{j+1}}\right)^{\pm 1}, \quad \left(\frac{G^l(i+1, j)}{w_{i+1}^j}\right)^{\pm 1} \\ \leq M^{4P_{i+j(\gamma+1)}(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\tilde{c}^{i+1+j(\gamma+1)}}. \end{aligned}$$

Proof: Let us consider another solutions to the systems of the equations:

$$\begin{aligned} J^l(i, j+1) &= f_t^l(K^l(i, j), J^l(i, j), \dots, J^l(i+\alpha, j)), \\ K^l(i+1, j) &= g_t^l(K^l(i, j), J^l(i, j), \dots, J^l(i+\beta, j)) \end{aligned}$$

with the initial values $J^l(i, 0) = z_i(2)$ and $K^l(0, j) = w^j(2)$. Then by proposition 2.17, the estimates hold for $l = 1, 2$:

$$\left(\frac{F^l(i, j+1)}{J^l(i, j+1)}\right)^{\pm 1}, \quad \left(\frac{G^l(i+1, j)}{K^l(i+1, j)}\right)^{\pm 1} \leq M^{2P_{i+j(\gamma+1)}(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\tilde{c}^{i+1+j(\gamma+1)}}.$$

On the other hand by theorem 2.19, the estimates hold:

$$\left(\frac{J^l(i, j+1)}{z_i^{j+1}}\right)^{\pm 1}, \quad \left(\frac{K^l(i+1, j)}{w_{i+1}^j}\right)^{\pm 1} \leq M^{2P_{i+j(\gamma+1)}(c)}.$$

Then by multiplying these, one obtains the desired estimates:

$$\begin{aligned} \left(\frac{F^l(i, j+1)}{z_i^{j+1}}\right)^{\pm 1} &= \left(\frac{J^l(i, j+1)}{z_i^{j+1}}\right)^{\pm 1} \left(\frac{F^l(i, j+1)}{J^l(i, j+1)}\right)^{\pm 1} \\ &\leq M^{2P_{i+j(\gamma+1)}(c)} M^{2P_{i+j(\gamma+1)}(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\tilde{c}^{i+1+j(\gamma+1)}} \\ &= M^{4P_{i+j(\gamma+1)}(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\tilde{c}^{i+1+j(\gamma+1)}}. \end{aligned}$$

The desired estimates for $\left(\frac{G^l(i+1, j)}{w_{i+1}^j}\right)^{\pm 1}$ are obtained by the same way.

This completes the proof.

Let us take two sets of functions $\{f_t^k(l)\}_{k,l=1,2}$ and $\{g_t^k(l)\}_{k,l=1,2}$ such that they are all tropically equivalent respectively:

$$f_t^1(1) \sim f_t^2(1) \sim f_t^1(2) \sim f_t^2(2), \quad g_t^1(1) \sim g_t^2(1) \sim g_t^1(2) \sim g_t^2(2).$$

Let c and M be the biggest numbers of their Lipschitz constants and the numbers of their components respectively.

Suppose the sets of sequences $(\{w_i^j(l)\}_{i,j}, \{z_i^j(l)\}_{i,j})$ satisfy the dynamical inequalities:

$$\begin{aligned} f_t^1(l)(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)) &\leq z_i^{j+1}(l) \leq f_t^2(l)(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ g_t^1(l)(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) &\leq w_{i+1}^j(l) \leq g_t^2(l)(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) \end{aligned}$$

with $z_i^0(l) \equiv z_i(l)$ and $w_0^j(l) \equiv w^j(l)$.

Corollary 2.19. *Supppose the above conditions. Then the uniform estimates hold:*

$$\begin{aligned} \left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}\right)^{\pm 1}, \quad \left(\frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}\right)^{\pm 1} \\ \leq M^{6P_{i+j}(\gamma+1)(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\bar{c}^{i+1+j}(\gamma+1)}. \end{aligned}$$

Proof: Let $(F_k^1(i, j), G_k^1(i, j))$ be as in theorem 2.19. Then the uniform estimates:

$$\begin{aligned} \left(\frac{F_k^1(i, j+1)}{z_i^{j+1}(k)}\right)^{\pm 1}, \quad \left(\frac{G_k^1(i+1, j)}{w_{i+1}^j(k)}\right)^{\pm 1} \\ \leq M^{2P_{i+j}(\gamma+1)(c)}. \end{aligned}$$

On the other hand by proposition 2.17, the uniform estimates hold:

$$\begin{aligned} \left(\frac{F_1^1(i, j+1)}{F_2^1(i, j+1)}\right)^{\pm 1}, \quad \left(\frac{G_1^1(i+1, j)}{G_2^1(i+1, j)}\right)^{\pm 1} \\ \leq M^{2P_{i+j}(\gamma+1)(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\bar{c}^{i+1+j}(\gamma+1)}. \end{aligned}$$

Thus combing with these estimates, we obtain the following:

$$\begin{aligned} \left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}\right)^{\pm 1} &= \left(\frac{z_i^{j+1}(1)}{F_1^1(i, j+1)}\right)^{\pm 1} \left(\frac{F_1^1(i, j+1)}{F_2^1(i, j+1)}\right)^{\pm 1} \left(\frac{F_2^1(i, j+1)}{z_i^{j+1}(2)}\right)^{\pm 1}, \\ &\leq M^{6P_{i+j}(\gamma+1)(c)} [\{(z_i(k), w^j(k))\}_{k=1}^2]^{\bar{c}^{i+1+j}(\gamma+1)}. \end{aligned}$$

We can estimate $\left(\frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}\right)^{\pm 1}$ by the same way. This completes the proof.

3 Asymptotic comparison for PDE

3.A Rough approximations by discrete dynamics: Let us describe our general procedure to approximate solutions to the systems of PDE by the state systems of rational dynamics.

Let $f_t = \frac{a_t}{b_t}$ and $g_t = \frac{c_t}{d_t}$ be relatively elementary functions of $\alpha + 2$ and $\beta + 2$ variables respectively, where a_t, b_t, c_t, d_t are all elementary. Let us consider the state systems of the rational dynamics:

$$\begin{aligned} z_i^{j+1} &= f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \\ w_{i+1}^j &= g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j). \end{aligned}$$

We will be interpreted the dynamical systems as the approximations of the systems of the partial differential equations as below.

Let us choose constants $0 < \epsilon \leq 1$. Let us consider a $C^{\mu+1}$ function $u : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$, and take the Taylor expansions:

$$\begin{aligned} u(x + i\epsilon, s + j\epsilon) &= u + i\epsilon u_x + j\epsilon u_s + \frac{(i\epsilon)^2}{2} u_{2x} + \frac{(j\epsilon)^2}{2} u_{2s} \\ &+ i j \epsilon^2 u_{xs} + \dots + \frac{(i\epsilon)^\mu}{\mu!} u_{\mu x} + \frac{(j\epsilon)^\mu}{\mu!} u_{\mu s} \\ &+ \frac{(i\epsilon)^{(\mu+1)}}{(\mu+1)!} u_{(\mu+1)x}(\xi_{ij}) + \dots + \frac{(j\epsilon)^{(\mu+1)}}{(\mu+1)!} u_{(\mu+1)s}(\xi_{ij}) \end{aligned}$$

where $|(x, s) - \xi_{ij}| \leq (|i| + |j|)\epsilon$ hold.

Let us introduce the change of variables by:

$$i = \frac{x}{\epsilon}, \quad j = \frac{s}{\epsilon}, \quad u(x, s) = z(i, j), \quad v(x, s) = w(i, j)$$

Then we take the difference, and insert the Taylor expansions:

$$\begin{aligned} &z_i^{j+1} - f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) \\ &= u(x, s + \epsilon) - f_t(v(x, s), u(x, s), \dots, u(x + \alpha\epsilon, s)) \\ &= \frac{P_1(\epsilon, t, u, v, u_s, u_x, \dots, u_{\mu x}) + R_1(\epsilon, t, u, v, \dots, u_{(\mu+1)x}(\xi))}{b_t(v(x, s), u(x, s), \dots, u(x + \alpha\epsilon, s))} \\ &\equiv \mathbf{L}_1(\epsilon, t, u, v, u_s, \dots, u_{\mu x}) + \epsilon^{\mu+1} \mathbf{E}_1(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi_{ij})\}_{\bar{a}, i, j}). \end{aligned}$$

where P_1 and R_1 are polynomials, and each monomial in R_1 contains derivatives of u of order $\mu + 1$.

Similarly we have the expansions:

$$\begin{aligned}
& w_{i+1}^j - g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j) \\
&= v(x + \epsilon, s) - g_t(v(x, s), u(x, s), \dots, u(x + \beta\epsilon, s)) \\
&= \frac{P_2(\epsilon, t, u, v, u_x, v_x, \dots, u_{\mu x}) + R_2(\epsilon, t, u, v, \dots, u_{(\mu+1)x}(\xi'))}{d_t(v(x, s), u(x, s), \dots, u(x + \beta\epsilon, s))} \\
&\equiv \mathbf{L}_2(\epsilon, t, u, v, u_x, v_x, \dots, u_{\mu x}) \\
&\quad + \epsilon^{\mu+1} \mathbf{E}_2(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi'_{ij})\}_{\bar{a}, i, j}, \{v_{\bar{a}}(\xi'_{ij})\}_{\bar{a}, i, j})
\end{aligned}$$

where each monomial in R_1 contains derivatives of u or v of order $\mu + 1$.

We say that $\mathbf{L}_i$ and $\mathbf{E}_i$ are the *leading* and *error* terms respectively.

Once one has chosen a pair of relatively elementary functions (f_t, g_t) , then the above process determines a system of PDEs $P_1 = P_2 = 0$, while tropical geometry provides an automaton given by $(\max, +)$ functions (ψ, ϕ) . So the pair (f_t, g_t) plays a role of a bridge to connect between systems of PDEs and automata.

Conversely a pair of $(\max, +)$ functions (ψ, ϕ) and the order μ determine the parametrized systems of PDEs:

$$\begin{aligned}
P_1(\epsilon, t, u, v, u_x, u_s, u_{2x}, \dots, u_{\mu x}) &= 0 \\
P_2(\epsilon, t, v, u, u_x, v_x, u_{2x}, \dots, u_{\mu x}) &= 0
\end{aligned}$$

which are called the *induced systems of partial differential equations* of order μ , with respect to the pairs (ψ, ϕ) .

Remark 3.1: In this paper we treat systems of PDEs with the same order. In particular the Mealy systems are of first order. However a general construction produces systems of PDEs with various orders, where say P_1 has order μ_1 and P_2 order μ_2 . In such cases we will have to choose different scaling parameters as $i = \frac{x}{\epsilon^p}$ and $j = \frac{s}{\epsilon^q}$. Such cases are treated in [K3] in a general form.

3.A.2 Mealy automaton: Let us consider the Mealy dynamics given by:

$$x(i, j + 1) = \psi(y(i, j), x(i, j)), \quad y(i + 1, j) = \phi(y(i, j), x(i, j)).$$

Let f_t and g_t be relatively elementary functions corresponding to ψ and ϕ respectively, and consider the systems of the rational dynamics:

$$z_i^{j+1} = f(w_i^j, z_i^j), \quad w_{i+1}^j = g(w_i^j, z_i^j).$$

Definition 3.1. *The induced first order systems of the equations:*

$$\begin{aligned} \epsilon u_s &= f_t(v, u) - u, \\ \epsilon v_s &= g_t(v, u) - v \end{aligned}$$

are called the hyperbolic Mealy systems.

Notice that the error term are the followings:

$$\mathbf{E}_1 = \frac{1}{2} \frac{\partial^2 u}{\partial s^2}, \quad \mathbf{E}_2 = \frac{1}{2} \frac{\partial^2 v}{\partial x^2}$$

respectively.

3.B Higher distortions: Our main interest here is to study analysis of asymptotic growth of solutions to different PDEs with respect to higher derivatives and initial conditions. In this paper we study globally rough-analytic behaviour of positive solutions to systems of PDEs, in terms of two data introduced below.

Let f_t and g_t be relatively elementary functions of $\alpha+2$ and $\beta+2$ variables respectively.

(A) Let $u, u', v, v' : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ be four functions. The *initial rate* of the pairs with respect to $\epsilon > 0$ is defined by:

$$\begin{aligned} & [(u, v) : (u', v')]_\epsilon \\ &= \max \left[\sup_{(x,s) \in [0, \infty) \times [0, \epsilon]} \max \left\{ \frac{u}{u'}(x, s), \frac{u'}{u}(x, s) \right\}, \right. \\ & \quad \left. \sup_{(x,s) \in [0, \epsilon] \times [0, \infty)} \max \left\{ \frac{v}{v'}(x, s), \frac{v'}{v}(x, s) \right\} \right]. \end{aligned}$$

(B) Let $(u, v) : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)^2$ be a pair of functions of $C^{\mu+1}$ class. We introduce the pointwise norms by:

$$\|(u, v)\|_{\mu, \alpha}^1(x, s) = \begin{cases} \max \left[\sup_{(x,s) \in [x, x+\alpha\epsilon] \times \{s\}} \left| \frac{\partial^{\mu+1} u}{\partial x^{\mu+1}} \right|(x, s), \right. \\ \quad \left. \sup_{(x,s) \in \{x\} \times [s, s+\epsilon]} \left| \frac{\partial^{\mu+1} u}{\partial s^{\mu+1}} \right|(x, s) \right], & \alpha \geq 1, \\ \sup_{(x,s) \in \{x\} \times [s, s+\epsilon]} \left| \frac{\partial^{\mu+1} u}{\partial s^{\mu+1}} \right|(x, s), & \alpha = 0, \end{cases}$$

$$||(u, v)||_{\mu, \beta}^2(x, s) = \begin{cases} \max[\sup_{(x, s) \in [x, x+\beta\epsilon] \times \{s\}} |\frac{\partial^{\mu+1}u}{\partial_x^{\mu+1}}|(x, s), \\ \sup_{(x, s) \in [x, x+\epsilon] \times \{s\}} |\frac{\partial^{\mu+1}v}{\partial_x^{\mu+1}}|(x, s)], & \beta \geq 1, \\ \sup_{(x, s) \in [x, x+\epsilon] \times \{s\}} |\frac{\partial^{\mu+1}v}{\partial_x^{\mu+1}}|(x, s)], & \beta = 0. \end{cases}$$

The *higher distorsion* is given by:

$$K(u, v) \equiv \sup_{(x, s) \in [0, \infty)^2} \max[\frac{||(u, v)||_{\mu, \alpha}^1}{u(x, s + \epsilon)}, \frac{||(u, v)||_{\mu, \beta}^2}{v(x + \epsilon, s)}].$$

The *error constants* C is the minimum number so that the pointwise estimates hold for any functions u, v independently of ϵ and t :

$$|\mathbf{E}_1(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi_{ij})\}_{\bar{a}, i, j})|(x, s) \leq C ||(u, v)||_{\mu, \alpha}^1(x, s),$$

$$|\mathbf{E}_2(\epsilon, t, u, v, \dots, \{u_{\bar{a}}(\xi_{ij})\}_{\bar{a}, i, j}, \{v_{\bar{a}}(\xi'_{ij})\}_{\bar{a}, i, j})|(x, s) \leq C ||(u, v)||_{\mu, \beta}^2(x, s)$$

Example 3.1: For the Mealy case, the error constant is $C = \frac{1}{2}$, and we have the followings:

$$||(u, v)||_{\mu, \alpha}^1 = \sup_{(x, s) \in \{x\} \times [s, s+\epsilon]} |\frac{\partial^2 u}{\partial^2 s}|(x, s),$$

$$||(u, v)||_{\mu, \beta}^2 = \sup_{(x, s) \in [x, x+\epsilon] \times \{s\}} |\frac{\partial^2 v}{\partial^2 x}|(x, s).$$

3.C Asymptotic comparisons: Let us take four relatively elementary functions f^1, f^2 and g^1, g^2 so that $f^1 \sim f^2$ and $g^1 \sim g^2$ are tropically equivalent mutually. For $l = 1, 2$, let:

$$P_1^l(\epsilon, t, u, v, u_s, \dots, u_{\mu x}) = 0,$$

$$P_2^l(\epsilon, t, u, v, v_x, \dots, u_{\mu x}) = 0$$

be the induced systems of PDEs of order μ .

let M be the largest numbers of their components.

Theorem 3.1. For $l = 1, 2$, let C be the bigger one of their error constants.

Let $(u^l, v^l) : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ be the solutions to the above systems respectively, so that the estimates:

$$0 \leq CK(u^l, v^l) \leq (1 - \delta)\epsilon^{-1}$$

are satisfied for some positive $\delta > 0$.

Then they satisfy the asymptotic estimates for all $(x, s) \in [0, \infty) \times [0, \infty)$:

$$\begin{aligned} & \left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s), \quad \left(\frac{v^1}{v^2}\right)^{\pm 1}(x, s) \\ & \leq (N_0 M)^{6P_{\epsilon^{-1}(x+s(\gamma+1))}(\epsilon)} \left[(u^1, v^1) : (u^2, v^2)\right]_{\epsilon}^{\tilde{c}\epsilon^{-1}(x+s(\gamma+1))+1}. \end{aligned}$$

where N_0 is any integer with $N_0 \geq \max(\delta^{-1}, 2 - \delta)$.

Proof: Let us choose $0 \leq a, b \leq 1$, and put the domain lattices by:

$$L_{\epsilon}(a, b) = \{(l_1 + a)\epsilon, (l_2 + b)\epsilon\} \in [0, \infty) \times [0, \infty) : l_1, l_2 \in \mathbb{N}\}.$$

Let us put:

$$\begin{aligned} z_i^j(l) &= u^l((i + a)\epsilon, (j + b)\epsilon), \\ w_i^j(l) &= v^l((i + a)\epsilon, (j + b)\epsilon) \end{aligned}$$

and consider the Taylor expansions:

$$\begin{aligned} & z_i^{j+1}(l) - f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)) \\ &= \mathbf{L}_1(\epsilon, t, u^l, v^l, u_s^l, \dots, u_{\mu x}^l) + \epsilon^{\mu+1} \mathbf{E}_1(\epsilon, t, u^l, v^l, \dots, \{u_{\bar{a}}^l(\xi_{ij})\}_{\bar{a}, i, j}), \end{aligned}$$

$$\begin{aligned} & w_{i+1}^j(l) - g_t(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) \\ &= \mathbf{L}_2(\epsilon, t, u^l, v^l, u_x^l, v_x^l, \dots, u_{\mu x}^l) \\ & \quad + \epsilon^{\mu+1} \mathbf{E}_2(\epsilon, t, u^l, v^l, \dots, \{u_{\bar{a}}^l(\xi'_{ij})\}_{\bar{a}, i, j}, \{v_{\bar{a}}^l(\xi'_{ij})\}_{\bar{a}, i, j}) \end{aligned}$$

By the assumption, both the leading terms satisfy the equalities:

$$\mathbf{L}_1^l(\epsilon, t, u^l, v^l, u_s^l, \dots) = 0, \quad \mathbf{L}_2^l(\epsilon, t, u^l, v^l, u_x^l, \dots) = 0.$$

Moreover the error terms satisfy the estimates:

$$\begin{aligned} |\mathbf{E}_1^l(\epsilon, t, u^l, v^l, \dots)|(x, s) &\leq CK(u^l, v^l) u^l(x, s + \epsilon), \\ |\mathbf{E}_2^l(\epsilon, t, u^l, v^l, \dots)|(x, s) &\leq CK(u^l, v^l) v^l(x + \epsilon, s). \end{aligned}$$

Then combining with these, one obtains the estimates:

$$\begin{aligned} |z_i^{j+1}(l) - f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l))| &\leq \epsilon^{\mu+1} |\mathbf{E}_1(\epsilon, t, u^l, v^l, \dots)| \\ &\leq \epsilon CK(u^l, v^l) z_i^{j+1}(l) \leq (1 - \delta) z_i^{j+1}(l), \end{aligned}$$

$$\begin{aligned} |w_{i+1}^j(l) - g_t(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l))| &\leq \epsilon^{\mu+1} |\mathbf{E}_2(\epsilon, t, u^l, v^l, \dots)| \\ &\leq \epsilon CK(u^l, v^l) w_{i+1}^j(l) \leq (1 - \delta) w_{i+1}^j(l). \end{aligned}$$

In particular there is some integer $N_0 \geq \max(\delta^{-1}, 2 - \delta)$ so that the inequalities hold:

$$\frac{1}{N_0} f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)) \leq z_i^{j+1}(l) \leq N_0 f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l))$$

$$\frac{1}{N_0} g_t(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) \leq w_{i+1}^j(l) \leq N_0 g_t(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)).$$

Now $\frac{1}{N_0} f$ and $N_0 f$ are both tropically equivalent to f , with the bounds $\max(M_{\frac{1}{N_0} f}, M_{N_0 f}) \leq N_0 M_f$. Then it follows from corollary 2.21 that the uniform estimates:

$$\begin{aligned} \left(\frac{z_i^{j+1}(1)}{z_i^{j+1}(2)}\right)^{\pm 1}, \left(\frac{w_{i+1}^j(1)}{w_{i+1}^j(2)}\right)^{\pm 1} &\leq (N_0 M)^{6P_{i+j(\gamma+1)}(c)} [(u^1, v^1) : (u^2, v^2)]_{\epsilon}^{\tilde{c}^{i+1+j(\gamma+1)}} \\ &\leq (N_0 M)^{6P_{\epsilon^{-1}(x+s(\gamma+1))}(c)} [(u^1, v^1) : (u^2, v^2)]_{\epsilon}^{\tilde{c}^{\epsilon^{-1}(x+s(\gamma+1))+1}}} \end{aligned}$$

where $(x, s) = ((i+a)\epsilon, (j+b)\epsilon)$.

Since the right hand side does not depend on a, b , one obtains the uniform estimates:

$$\begin{aligned} \left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s), \left(\frac{v^1}{v^2}\right)^{\pm 1}(x, s) \\ \leq (N_0 M)^{6P_{\epsilon^{-1}(x+s(\gamma+1))}(c)} [(u, v) : (u', v')]_{\epsilon}^{\tilde{c}^{\epsilon^{-1}(x+s(\gamma+1))+1}}} \end{aligned}$$

This completes the proof.

4 State dynamics and their scale transforms

4.A Automaton: Let us recall the state dynamics. Let S and Q be finite sets called the *alphabet set* and the *state set* respectively. Let $\alpha, \beta \geq 0$ and consider a pair of functions:

$$\psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

where we call ψ as the *output map* and ϕ as the *transition map*. In this paper we call such a pair of functions as an *automaton*.

Let:

$$X_S = \{\bar{k} = (k_0, k_1, \dots) : k_i \in S\}$$

be all the set of one-sided strings of infinite length. Then each state $q \in Q$ induces a continuous map:

$$\mathbf{A}_q : X_S \rightarrow X_S$$

given by: $\mathbf{A}_q(k_0, k_1, \dots) = (k'_0, k'_1, \dots)$, where k'_i are inductively defined as below with $q_0 = q$:

$$k'_i = \psi(q_i, k_i, \dots, k_{i+\alpha}), \quad q_{i+1} = \phi(q_i, k_i, \dots, k_{i+\beta}).$$

We call it the *state dynamics* with respect to $\mathbf{A}$.

4.A.2 Automata groups: Let us consider a Mealy automaton:

$$\psi : Q \times S \rightarrow S, \quad \phi : Q \times S \rightarrow Q$$

which is a special case of the state dynamics with $q_0 = q$:

$$k'_i = \psi(q_i, k_i), \quad q_{i+1} = \phi(q_i, k_i).$$

Notice that the Mealy dynamics induce the level-set actions as $\mathbf{A}_q : X_A^{N+1} \rightarrow X_A^{N+1}$, where

$$X_A^{N+1} = \{\bar{k}^* = (k_0, k_1, \dots, k_N) : k_i \in S\}$$

are the set of words of length $N + 1$.

Let $m = \#|A|$ be the cardinality of S , and T_m be the rooted regular m -tree. The set of all vertices of T_m can be identified with $X_A^\infty \equiv \cup_N X_A^N$. Thus $\mathbf{A}_q$ give the actions:

$$\mathbf{A}_q : T_m \rightarrow T_m.$$

Let us say that $\mathbf{A}$ is *invertible*, if $\psi(q, \cdot) : S \cong S$ are one to one and onto for all $q \in Q$. An invertible automaton $\mathbf{A}$ gives automorphisms $\mathbf{A}_q : T_m \cong T_m$.

Definition 4.1. Let $\mathbf{A}$ be invertible. The group generated by the set of states:

$$G(\mathbf{A}) = \text{gen } \{\mathbf{A}_q : T_m \cong T_m : q \in Q\}$$

is called the automata group.

Remark 4.1: (1) Automata groups are subgroups of $\text{Aut}(T_m)$.

(2) General state dynamics give the maps as $\mathbf{A}_q : \partial T_m \rightarrow \partial T_m$.

Later on we will always assume that Mealy automata are invertible, and both S and Q are subsets of the real number:

$$S, Q \subset \mathbb{R}.$$

For $X_Q^L = \{\bar{q}^L = (q^0, \dots, q^L) : q^i \in Q\}$, let us put:

$$\mathbf{A}_{\bar{q}^L} = \mathbf{A}_{q^L} \circ \dots \circ \mathbf{A}_{q^0} : T_m \cong T_m$$

If we denote $\mathbf{A}_{\bar{q}^L}(k_0, k_1, \dots) = (k_0^L, k_1^L, \dots)$, then the state dynamics given by the Mealy automata are exactly given by $\mathbf{A}_{\bar{q}^L}$, so that $\{k_i^j\}_{i,j}$ are the orbits given by the state dynamics with the initial data $(q^0, q^1, \dots)$ and $(k_0, k_1, \dots)$.

4.A.3 Extensions: One of the most basic properties commonly shared among all Mealy automata is that the groups are residually finite, since their restrictions induce the level-set actions. Here we consider a canonical generalization whose groups are not necessarily the case, which arise from the state dynamics.

Let us consider an automaton $\mathbf{A}$ given by the two functions:

$$\psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

and consider the corresponding state dynamics $\mathbf{A}_q : X_S \rightarrow X_S$ for each $q \in Q$. $\mathbf{A}_q : \partial T_m \rightarrow \partial T_m$ does not induce level-set actions in general.

Let us say that $\mathbf{A}$ is *invertible*, if $\mathbf{A}_q : X_S \cong X_S$ are isomorphisms for all $q \in Q$. An invertible automaton $\mathbf{A}$ gives automorphisms $\mathbf{A}_q : \partial T_m \cong \partial T_m$, and the group generated by the set of states is also denoted by $G(\mathbf{A})$:

$$G(\mathbf{A}) = \text{gen } \{\mathbf{A}_q : \partial T_m \cong \partial T_m : q \in Q\}$$

which is a subgroup of the boundary automorphism group $\text{Aut}(\partial T_m)$.

Example 4.1: The following automaton is not Mealy, since dynamics are not determined by every level set of the trees.

Let us choose $\alpha = 1$. Let $S = \{s_0, s_1\}$ and $Q = \{q^0, q^1\}$, and ϵ_0 be the permutation between two elements. Let us consider the functions:

$$\begin{aligned} \psi(q^1, \cdot, s) &= \epsilon_0, & \psi(q^0, \cdot, s) &= \begin{cases} \epsilon_0 & s = s_0 \\ id & s = s_1 \end{cases} \\ \phi(\cdot, s, s') &= \epsilon_0 \end{aligned}$$

where $s, s' \in S$.

Lemma 4.1. *The state dynamics by this automaton is invertible.*

Proof: Let us consider $\psi(q^0, \cdot, \cdot)$, and suppose $(q^0, s, t) \neq (q^0, s', t')$. The equalities $\psi(q^0, s, t) = \psi(q^0, s', t')$ hold only when $(s, s') = (s_1, s_0)$ or (s_0, s_1) and $t \neq t'$.

On the other hand at the next level, $\phi(q^0, s, t) = \phi(q^0, s', t') = q^1$ hold, and so:

$$\psi(q^1, t, \cdot) \neq \psi(q^1, t', \cdot).$$

This implies $\mathbf{A}_{q^0} : \partial T_m \rightarrow \partial T_m$ is injective.

Since $\psi(q^1, \cdot, \cdot)$ does not depend on the second variable, injectivity of $\mathbf{A}_{q^1} : \partial T_m \rightarrow \partial T_m$ follows from the one of $\mathbf{A}_{q^0}$.

Let us consider surjectivity, and take any $(x'_0, x'_1, \dots) \in X_S$. We seek for some elements $(x_0, x_1, \dots) \in X_S$ with $\mathbf{A}_q(x_0, x_1, \dots) = (x'_0, x'_1, \dots)$. Notice that the states change periodically as $(q^0, q^1, q^0, q^1, \dots)$ or $(q^1, q^0, q^1, \dots)$.

Both cases can be considered similarly, and we treat the first case only. Firstly we choose $x_{2i+1} = x'_{2i+1} + 1 \pmod 2$ for all $i \geq 0$. Then we can choose x_{2i} uniquely so that the equalities $\psi(q^0, x_{2i}, x_{2i+1}) = x'_{2i}$ hold for all i . Thus $\mathbf{A}_q$ are surjective, and so they are isomorphisms. This completes the proof.

4.B Extensions of automata: Let $Q, S \subset \mathbb{R}$ be finite sets, and choose an automaton $\psi : Q \times S^{\alpha+1} \rightarrow S$ and $\phi : Q \times S^{\beta+1} \rightarrow Q$.

Let:

$$\tilde{\psi} : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \tilde{\phi} : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}$$

be two maps.

Let us say that the pair of functions $(\tilde{\psi}, \tilde{\phi})$ extends the automaton, if their restrictions coincide with each other:

$$\tilde{\psi}|_{Q \times S^{\alpha+1}} = \psi, \quad \tilde{\phi}|_{Q \times S^{\beta+1}} = \phi.$$

Conversely if there are finite subsets $Q, S \subset \mathbb{R}$ so that their restrictions of the pair of functions $(\tilde{\psi}, \tilde{\phi})$ induce the functions:

$$\psi \equiv \tilde{\psi} : Q \times S^{\alpha+1} \rightarrow S, \quad \phi \equiv \tilde{\phi} : Q \times S^{\beta+1} \rightarrow Q$$

then we say that $(\tilde{\psi}, \tilde{\phi})$ restricts to an automaton (ψ, ϕ) .

Notice that if a pair of bounded functions induces maps as:

$$\tilde{\psi} : \mathbb{Z} \times \mathbb{Z}^{\alpha+1} \rightarrow \mathbb{Z}, \quad \tilde{\phi} : \mathbb{Z} \times \mathbb{Z}^{\beta+1} \rightarrow \mathbb{Z}$$

then they restricts to some automaton for some $Q, S \subset \mathbb{Z}$.

4.B.2 Lamplighter group: Let:

$$\begin{aligned} \psi : \{q^0, q^1\} \times \{s_0, s_1\} &\rightarrow \{s_0, s_1\}, & \psi(q^0, \quad) &= id, \quad \psi(q^1, \quad) = \epsilon, \\ \phi : \{q^0, q^1\} \times \{s_0, s_1\} &\rightarrow \{q^0, q^1\}, & \phi(\quad, s_i) &= q^i \quad (i = 0, 1) \end{aligned}$$

be the pair which gives a Mealy automaton **A**. It is known ([GZ]) that the associated automata group is isomorphic to the Lamplighter group which acts on the rooted binary tree.

Let us extend the automaton to a pair of $(\max, +)$ -functions as below. Let us put $q^0 = s_0 = 0$ and $q^1 = s_1 = 3 \in \mathbb{Z}$.

Now consider the piecewise-linear functions:

$$\begin{aligned} P(k) &= -\max(-3, -\max(0, -3(k-2))), \\ Q(k) &= -\max(-3, -\max(0, 3(k-1))). \end{aligned}$$

Then an extension of the pair (ψ, ϕ) is given by the function:

$$\psi(q, s) = \max(P(s), Q(s-q)), \quad \phi(q, s) = Q(s)$$

as maps from $\mathbb{Z}^2$ to $\mathbb{Z}$.

The extension (ψ, ϕ) holds the following properties:

(1) They are constants on the intervals $[-1, 1]$ and $[2, 4]$. Namely the extension satisfies stability defined in 4.C.

(2) They are bounded.

Now let us put the functions:

$$\alpha_a(z) = (t^{-3} + \frac{1}{1 + z^3 t^a})^{-1}.$$

The tropical pair of the rational functions (f_t, g_t) is given by:

$$g_t(w, z) = \alpha_{-3}(z), \quad f_t(w, z) = \alpha_6(z^{-1}) + \alpha_{-3}(zw^{-1}).$$

The induced system of the partial differential equations is given by:

$$\epsilon u_s = f_t(v, u) - u, \quad \epsilon v_x = g_t(u) - v$$

where f_t and g_t are as above.

4.B.3 Extensions by relatively $(\max, +)$ -functions: Let $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}$ be a map. Let us note that there are two canonical ways of the extensions:

(1) By connecting these integer values by segments, one can extend φ straightforwardly to a piecewise linear map:

$$\varphi : \mathbb{R}^n \rightarrow \mathbb{R}.$$

(2) If $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}$ is equipped with its presentation as a $(\max, +)$ -function, then it is canonically extended as a map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$.

For our purposes in this paper we will choose the method (2) above.

Lemma 4.2. *Let us consider a Mealy automaton $\psi : Q \times S \rightarrow S$, $\phi : Q \times S \rightarrow Q$ with $Q, S \subset \mathbb{R}$. Then there is a bounded extension:*

$$\tilde{\psi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad \tilde{\phi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

so that both functions $\tilde{\psi}$ and $\tilde{\phi}$ are represented by relatively $(\max, +)$ -functions.

Proof: We construct local cone extensions inductively.

Step 1: Let $\{m_1, m_2, m_3\} \subset Q \times S$ be three points which are mutually different, and consider the triangle $\Delta \subset \mathbb{R}^3$ whose vertices are given by:

$$O_i = (m_i, \psi(m_i)) \in \mathbb{R}^3 \quad (i = 1, 2, 3).$$

Let $l_i \subset \Delta$ be segments whose end points are O_i and $O_{i+1} \bmod 3$.

Let us choose planes L_i which contain l_i so that they are represented by graphs of affine linear functions $\varphi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$L_i = \{(x_1, x_2, \varphi_i(x_1, x_2)) : (x_1, x_2) \in \mathbb{R}^2\}.$$

Let $g \in \Delta$ be the barycenter, and $l : [0, \infty)$ be the half lines starting from g which are vertical to $\mathbb{R}^2$ in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. Exactly there are two choices l_b and l_a which go below or above to Δ .

For $i = 1, 2, 3$, let L_i^b and L_i^a be another planes which contain l_i and intersect with l_b and l_a at $l_b(t)$ or $l_a(t)$ respectively. So three places $\{L_1^b, L_2^b, L_3^b\}$ intersect at one point $l_b(t)$. Similar for a . We will choose these planes with sufficiently large $t \gg 1$. Let φ_i^b and φ_i^a be their representations by relative $(\max, +)$ -functions.

Now we have two types of the cones:

$$C_b = \text{graph } \max(\varphi_1^b, \varphi_2^b, \varphi_3^b)$$

and C_a are similar. C_b are concave and C_a are convex. Notice that we may choose arbitrarily sharp slopes of the cones.

Let us choose a large C with $C \geq \max\{|\psi(m)| : m \in Q \times S\}$. Let φ be another affine linear function whose graph contains Δ .

Now we put the bounded functions:

$$\begin{aligned} \psi_\Delta^b &= -\max(-C, -\max(\varphi, \varphi_1^b, \varphi_2^b, \varphi_3^b)), \\ \psi_\Delta^a &= \max(-C, -\max(-\varphi, -\varphi_1^a, -\varphi_2^b, -\varphi_3^b)). \end{aligned}$$

Notice that both graphs contain Δ .

Step 2: Let us order all the points $\{m_{-1}, m_0, m_1, m_2, \dots, m_l\} = Q \times S$, so that there are families of two dimensional polytopes M_i such that:

- (1) M_i are given by unions of M_{i-1} with a single triangle Δ_i which contain the vertices m_i and
- (2) M_i do not contain the sets $\{m_{i+1}, \dots, m_l\}$.

Now we inductively construct relative $(\max, +)$ -functions $\tilde{\psi}_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that the equalities:

$$\tilde{\psi}_i(m_k) = \psi(m_k)$$

hold for all $1 \leq k \leq i$.

We have constructed $\psi_1^a = \psi_{\Delta_1}^a$ by Step 1, with $\Delta_1 = \Delta$.

Suppose we have obtained ψ_i , and let us construct $\tilde{\psi}_{i+1}$. Let us divide into two cases:

Suppose $\tilde{\psi}_i(m_{i+1}) \geq \psi(m_{i+1})$. Let $\psi_{\Delta_{i+1}}^b$ be as in Step 1. By choosing sufficiently sharp slopes, we may assume that the estimates:

$$\psi_{\Delta_{i+1}}^b(m_k) \geq \tilde{\psi}_i(m_k)$$

hold for all $1 \leq k \leq i$. Then we put the bounded function by:

$$\tilde{\psi}_{i+1} = -\max(-\psi_{\Delta_{i+1}}^b, -\tilde{\psi}_i).$$

Next suppose $\tilde{\psi}_i(m_{i+1}) \leq \psi(m_{i+1})$. Let $\psi_{\Delta_{i+1}}^a$ be as in Step 1. Again by choosing sufficiently sharp slopes, we may assume that the estimates:

$$\psi_{\Delta_{i+1}}^a(m_k) \leq \tilde{\psi}_i(m_k)$$

hold for all $1 \leq k \leq i$. Then we put the bounded function by:

$$\tilde{\psi}_{i+1} = \max(\psi_{\Delta_{i+1}}^a, \tilde{\psi}_i).$$

In any cases $\tilde{\psi}_{i+1}$ above satisfy the desired properties. This finishes the induction step. This completes the proof.

4.C Stability and extensions: Let us introduce stability under iterations of maps and study their long-time behaviours.

Let $Q, S \subset \mathbb{R}$ be finite sets, and consider a map $\varphi : Q \times S^a \rightarrow S$ with its extension $\varphi : \mathbb{R} \times \mathbb{R}^a \rightarrow \mathbb{R}$.

Definition 4.2. $\varphi : \mathbb{R} \times \mathbb{R}^a \rightarrow \mathbb{R}$ is stable with respect to (Q, S) , if there are $0 < \delta < 1$ and $0 \leq \mu < 1$ so that the estimates hold:

$$|\varphi(y, \bar{x}) - \varphi(q, \bar{s})| \leq \mu d((y, \bar{x}), (q, \bar{s}))$$

for any $(y, \bar{x}) \in \mathbb{R} \times \mathbb{R}^a$ and $(q, \bar{s}) \in Q \times S^a$ with $d((y, \bar{x}), (q, \bar{s})) < \delta$.

Notice that φ is (δ', μ) -stable for all $0 < \delta' \leq \delta$, whenever it is (δ, μ) -stable.

Let $\mathbf{A}$ be an automaton given by a pair of two functions $\psi : Q \times S^{\alpha+1} \rightarrow S$ and $\phi : Q \times S^{\beta+1} \rightarrow Q$. A stable extension of $\mathbf{A}$ is given by two stable extensions of the functions:

$$\psi : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}.$$

Examples 4.1: (1) $\psi : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}$ is a stable extension of $\psi : Q \times S^{\alpha+1} \rightarrow S$ with $\mu = 0$, if it is locally constant on small neighbourhoods of each $(q, \bar{s}) \in Q \times S^{\alpha+1}$.

(2) $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a stable extension of $\psi : Q \times S \rightarrow S$, if it is of the form with $|\alpha_{q,s}| + |\beta_{q,s}| < 1$ on small neighbourhoods of each $(q, s) \in Q \times S$:

$$\psi(y, x) = \alpha_{q,s}(x - s) + \beta_{q,s}(y - q) + \psi(q, s).$$

Proposition 4.3. *Let $Q, S \subset \mathbb{R}$ be finite sets and (ψ, ϕ) be an automaton:*

$$\psi : Q \times S \rightarrow S, \quad \phi : Q \times S \rightarrow S.$$

Then there is a bounded and stable extension:

$$\tilde{\psi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad \tilde{\phi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

so that both functions $\tilde{\psi}$ and $\tilde{\phi}$ are represented by relative $(\max, +)$ -functions.

Proof: The proof consists of a minor modification of lemma 4.2.

Let us choose disjoint union of squares in $\mathbb{R}^2$ so that each square contains unique point $(q, s) \in Q \times S$ in its interior. Let us denote such square by $D(q, s) \subset \mathbb{R}^2$.

Firstly let us put locally constant functions by:

$$\tilde{\psi}(y, x) = \psi(q, s), \quad \tilde{\phi}(y, x) = \phi(q, s)$$

for $(x, y) \in D(q, s)$. This determines functions $\tilde{\psi}$ and $\tilde{\phi}$ on $\cup_{(q,s) \in Q \times S} D(q, s)$.

For the rest, one can follow the same argument as the proof of lemma 4.2, and extend the domains of these functions inductively.

We omit the repetition. This completes the proof.

4.D Stability and state dynamics: Let $Q, S \subset \mathbb{R}$ be finite sets, and consider an automaton $\mathbf{A}$ defined by $\psi : Q \times S^{\alpha+1} \rightarrow S$ and $\phi : Q \times S^{\beta+1} \rightarrow Q$. For each $q \in Q$, let $\mathbf{A}_q : X_S \rightarrow X_S$ be the continuous dynamics given by:

$$\begin{aligned} \mathbf{A}_q(k_0, k_1, \dots) &= (k'_0, k'_1, \dots), \\ k'_i &= \psi(q_i, k_i, \dots, k_{\alpha+1}), \quad q_{i+1} = \phi(q_i, k_i, \dots, k_{\beta+1}) \end{aligned}$$

with $q_0 = q$. For a sequence $\bar{q}^l = (q^0, q^1, \dots, q^l)$, let us denote the compositions of the corresponding dynamics $\mathbf{A}_{\bar{q}^l} \equiv \mathbf{A}_{q^l} \circ \dots \circ \mathbf{A}_{q^0} : X_S \rightarrow X_S$.

Let us introduce the space of the real sequences:

$$X_{\mathbb{R}} = \{\bar{x} = (x_0, x_1, \dots) : x_i \in \mathbb{R}\}$$

and equip with the uniform distance between two elements $\bar{x} = (x_0, x_1, \dots)$ and $\bar{x}' = (x'_0, x'_1, \dots) \in X_{\mathbb{R}}$ by:

$$d(\bar{x}, \bar{x}') = \sup_{0 \leq i < \infty} |x_i - x'_i| \in [0, \infty]$$

Let:

$$\psi : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}$$

be a stable extension of $\mathbf{A}$ with the constants (δ, μ) . For each $y \in \mathbb{R}$, let us extend the above dynamics as $\mathbf{A}_y : X_{\mathbb{R}} \rightarrow X_{\mathbb{R}}$ given by the same rule:

$$\begin{aligned} \mathbf{A}_y(x_0, x_1, \dots) &= (x'_0, x'_1, \dots), \\ x'_i &= \psi(y_i, x_i, \dots, x_{\alpha+1}), \quad y_{i+1} = \phi(y_i, x_i, \dots, x_{\beta+1}) \end{aligned}$$

with $y_0 = y$. For $\bar{y}^l = (y^0, y^1, \dots, y^l) \in \mathbb{R}^{l+1}$, we put:

$$\mathbf{A}_{\bar{y}^l} \equiv \mathbf{A}_{y^l} \circ \dots \circ \mathbf{A}_{y^0} : X_{\mathbb{R}} \rightarrow X_{\mathbb{R}}.$$

This is exactly the state dynamics in 2.B.

For finite sequences $\bar{q}^l \in X_Q^{l+1}$ and $\bar{y}^l \in X_{\mathbb{R}}^{l+1}$, we also equip the same uniform norms by $d(\bar{q}^l, \bar{y}^l) = \sup_{0 \leq i \leq l} |q^i - y^i| \in [0, \infty)$.

Lemma 4.4. *Let us choose pairs of the sequences, $\bar{y}^l \in \mathbb{R}^{l+1}$ with $\bar{q}^l \in X_Q^{l+1}$, and $\bar{x} \in X_{\mathbb{R}}$ with $\bar{k} \in X_S$, so that they have bounded distances by $0 < \delta < 1$ from each other:*

$$d(\bar{x}, \bar{k}), \quad d(\bar{y}, \bar{q}) < \delta.$$

Then the estimates hold:

$$d(\mathbf{A}_{\bar{q}}(\bar{k}), \mathbf{A}_{\bar{y}}(\bar{x})) < \delta.$$

Proof: We split the proof into two steps.

Step 1: Firstly let us consider the case $l = 0$, and put $y_0 = y^0 (= y_0^0)$. $d(q^0, y^0) < \delta$ hold by the assumption. So let us verify the estimates:

$$d(q_{i+1}, y_{i+1}), \quad d(k'_i, x'_i) < \delta$$

by induction on $i = 0, 1, 2, \dots$

Let us start from the former estimates, and suppose they hold up to i . Then we have the estimates:

$$\begin{aligned} d(q_{i+1}, y_{i+1}) &= d(\phi(q_i, k_i, \dots, k_{\beta+i}), \phi(y_i, x_i, \dots, k_{\beta+i})) \\ &\leq \mu \max_{i \leq j \leq \beta+i} \{d(q_i, y_i), d(k_j, x_j)\} < \mu\delta < \delta. \end{aligned}$$

So it holds also at $i + 1$. Thus $d(q_i, y_i) < \delta$ hold for all i by induction.

Next let us consider the latter estimates. By use of the former ones, we have the desired estimates:

$$\begin{aligned} d(k'_i, x'_i) &= d(\psi(q_i, k_i, \dots, k_{\alpha+i}), \psi(y_i, x_i, \dots, k_{\alpha+i})) \\ &\leq \mu \max_{i \leq j \leq \alpha+i} (d(q_i, y_i), d(k_j, x_j)) < \delta. \end{aligned}$$

Thus we obtain the bounds:

$$d(\mathbf{A}_{q^0}(\bar{k}), \mathbf{A}_{y^0}(\bar{x})) < \delta.$$

Step 2: Let us replace the pairs $(\bar{k}, \bar{x})$ by $(\mathbf{A}_{q^0}(\bar{k}), \mathbf{A}_{y^0}(\bar{x}))$ and (q^0, y^0) by (q^1, y^1) in Step 1. We apply the same process and obtain the estimates:

$$d(\mathbf{A}_{q^1}(\mathbf{A}_{q^0}(\bar{k})), \mathbf{A}_{y^1}(\mathbf{A}_{y^0}(\bar{x}))) < \delta.$$

By iterating this process l times, one obtains the conclusion.

This completes the proof.

4.D.2 Change of automata structure: For $l = 1, 2$, let:

$$\mathbf{A}_l : \psi^l : Q \times S^{\alpha+1} \rightarrow S, \quad \phi^l : Q \times S^{\beta+1} \rightarrow Q$$

be two automata. For each $q \in Q$, let $(\mathbf{A}_l)_q : X_S \rightarrow X_S$ be the continuous dynamics and denote $(\mathbf{A}_l)_q(k_0, k_1, \dots) = (k'_0(l), k'_1(l), \dots)$, where $k'_i(l)$ are inductively determined by:

$$k'_i(l) = \psi^l(q_i(l), k_i(l), \dots, k_{\alpha+1}(l)), \quad q_{i+1}(l) = \phi^l(q_i(l), k_i(l), \dots, k_{\beta+1}(l))$$

with $q_0(l) = q$. It happens quite often that two different automata $\mathbf{A}_1$ and $\mathbf{A}_2$ give the same dynamics. In such situation, the equalities always hold:

$$(k'_0(1), k'_1(1), \dots) = (k'_0(2), k'_1(2), \dots)$$

while the state sequences may differ from each other:

$$(q^0(1), q^1(1), \dots) \neq (q^0(2), q^1(2), \dots).$$

Definition 4.3. *Let us say that $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalent, if they give the same dynamics in the sense as above.*

Let $R \subset Q$ be a subset. If the equalities $(\mathbf{A}_1)_q = (\mathbf{A}_2)_q$ hold for all $q \in R$, then we say that $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalent over R .

Let us give stable extensions of $\mathbf{A}_1$ and $\mathbf{A}_2$ by relative $(\max, +)$ functions (ϕ^1, ψ^1) and (ϕ^2, ψ^2) respectively.

Let us consider the corresponding state dynamics:

$$\begin{aligned} x_l(i, j+1) &= \psi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \alpha, j)), \\ y_l(i+1, j) &= \phi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \beta, j)) \end{aligned}$$

with the initial values $\bar{x}(l) = \{x_i(l)\}_i$ and $\bar{y}(l) = \{y^j(l)\}_j$ respectively. For $l = 1, 2$, let us denote:

$$\bar{x}^j(l) = (x_l(0, j), x_l(1, j), \dots) \in X_{\mathbb{R}}.$$

For an infinite sequence $\bar{q} = (q^0, q^1, \dots) \in X_Q$, we denote its restrictions as $\bar{q}^l = (q^0, q^1, \dots, q^l) \in X_Q^{l+1}$.

Corollary 4.5. *Suppose the following conditions:*

- (1) *There exist subsets $R \subset Q$ so that $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalent over R .*
- (2) *The initial data satisfy the uniform estimates:*

$$d(\bar{x}(l), \bar{k}), \quad d(\bar{y}(l), \bar{q}) < \delta$$

for some $\bar{k} \in X_S$ and $\bar{q} \in X_R$ for $l = 1, 2$.

Then the uniform estimates hold for all $0 \leq j < \infty$:

$$d(\bar{x}^j(1), \bar{x}^j(2)) < 2\delta.$$

Proof: The condition (1) implies that the equalities:

$$(\mathbf{A}_1)_{\bar{q}^l} = (\mathbf{A}_2)_{\bar{q}^l} : X_S \rightarrow X_S \quad (l = 0, 1, 2, \dots)$$

hold for all $\bar{q}^l \in X_R^{l+1}$. Let us denote:

$$\mathbf{A}_{q^j} \circ \dots \circ \mathbf{A}_{q^0}(k_0, k_1, \dots) = (k_0^j, k_1^j, \dots) \equiv \bar{k}^j.$$

By lemma 4.4, the estimates:

$$d(\bar{x}^j(l), \bar{k}^j) < \delta$$

hold for $l = 1, 2$. So the estimates:

$$d(\bar{x}^j(1), \bar{x}^j(2)) \leq d(\bar{x}^j(1), \bar{k}^j) + d(\bar{x}^j(2), \bar{k}^j) < 2\delta$$

hold. This completes the proof.

Remark 4.2: In particular for two different stable extensions of the same automaton, we can still apply this and obtain uniform estimates between their orbits.

4.D.3 Uniform estimates for stable dynamics: Let:

$$\mathbf{A} : \psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

be an automaton, and choose stable extensions with the constants (δ, μ) :

$$\psi : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}$$

represented by $(\max, +)$ -functions. Let (ψ_t, ϕ_t) be the tropical correspondences to (ψ, ϕ) , and M be the bigger one of the numbers of their components.

Let us consider the corresponding systems of the state dynamics:

$$\begin{aligned} x(i, j+1) &= \psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)), \\ y(i+1, j) &= \phi(y(i, j), x(i, j), \dots, x(i+\beta, j)), \end{aligned}$$

$$\begin{aligned} x'(i, j+1) &= \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j)), \\ y'(i+1, j) &= \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta, j)) \end{aligned}$$

with the same initial values $\bar{x} = \{x_i\}_i$ and $\bar{y} = \{y^j\}_j$ respectively.

Lemma 4.6. *Suppose $t_0 \gg 1$ satisfies the estimates:*

$$\mu\delta + 2\log_{t_0} M < \delta.$$

Then for all $t \geq t_0$ and any initial data with the uniform bounds:

$$d(\bar{x}, \bar{k}), \quad d(\bar{y}, \bar{q}) < \frac{\delta}{2}$$

for some $\bar{k} \in X_S$ and $\bar{q} \in X_Q$, their orbits satisfy the uniform estimates:

$$|x(i, j) - x'(i, j)|, \quad |y(i, j) - y'(i, j)| < \frac{\delta}{2}.$$

Proof: Step 1: The first condition can be realized for sufficiently large $t_0 \gg 1$, since $\mu < 1$ holds by stability assumption.

Let us denote the orbits by $(\{k_i^j\}, \{q_i^j\})$ determined by:

$$k_i^{j+1} = \psi(q_i^j, k_i^j, \dots, k_{i+\alpha}^j), \quad q_{i+1}^j = \phi(q_i^j, k_i^j, \dots, k_{i+\beta}^j)$$

with the initial values $\bar{k}$ and $\bar{q}$ as above. By lemma 4.4, the uniform estimate:

$$|x(i, j) - k_i^j|, \quad |y(i, j) - q_i^j| < \frac{\delta}{2}$$

hold for all i, j .

Let us consider the case $y(i, 0)$ and $x(i, 1)$.

$y(0, 0) = y'(0, 0)$ hold by the assumption. So let us verify the estimates:

$$d(y'(i+1, 0), y(i+1, 0)), \quad d(x(i, 1), x'(i, 1)) < \frac{\delta}{2}$$

by induction on $i = 0, 1, 2, \dots$. So suppose they hold up to i . Notice the estimates:

$$d(y'(i, 0), q_i^0) \leq d(y(i, 0), q_i^0) + d(y'(i, 0), y(i, 0)) < \delta.$$

Then we have the estimates:

$$\begin{aligned} d(y(i+1, 0), y'(i+1, 0)) &= d(\phi(y(i, 0), x_i, \dots, x_{i+\alpha}), \phi_t(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\leq d(\phi(y(i, 0), x_i, \dots, x_{i+\alpha}), \phi(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\quad + d(\phi(y'(i, 0), x_i, \dots, x_{i+\alpha}), \phi_t(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\leq \mu d(y(i, 0), y'(i, 0)) + \log_t M < \mu \frac{\delta}{2} + \log_t M < \frac{\delta}{2}. \end{aligned}$$

So we have verified the estimates $d(y'(i, 0), y(i, 0)) < \frac{\delta}{2}$ for all i by induction.

Then we have the estimates:

$$\begin{aligned} d(x(i, 1), x'(i, 1)) &= d(\psi(y(i, 0), x_i, \dots, x_{i+\alpha}), \psi_t(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\leq d(\psi(y(i, 0), x_i, \dots, x_{i+\alpha}), \psi(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\quad + d(\psi(y'(i, 0), x_i, \dots, x_{i+\alpha}), \psi_t(y'(i, 0), x_i, \dots, x_{i+\alpha})) \\ &\leq \mu d(y(i, 0), y'(i, 0)) + \log_t M < \mu \frac{\delta}{2} + \log_t M < \frac{\delta}{2}. \end{aligned}$$

So we have verified the estimates $d(x'(i, 1), x(i, 1)) < \frac{\delta}{2}$.

Step 2: Let us put the sequences $\bar{x}^j \equiv (x(0, j), x(1, j), \dots)$ and similar for others. Let us verify the estimates:

$$d((\bar{y}')^j, \bar{y}^j), \quad d((\bar{x}')^{j+1}, \bar{x}^{j+1}) < \frac{\delta}{2}$$

by induction on $j = 0, 1, 2, \dots$

We are done for $j = 0$ at Step 1. Suppose the above estimates hold up to $j - 1$.

Let us start from the former estimates. $y(0, j) = y'(0, j)$ hold by the assumption. Suppose the estimates $d(y'(i, j), y(i, j)) < \frac{\delta}{2}$ hold up to i . Then we have the estimates:

$$\begin{aligned} & d(y(i+1, j), y'(i+1, j)) = \\ & d(\phi(y(i, i), x(i, j), \dots, x(i, i+\beta)), \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta))) \\ & \leq d(\phi(y(i, j), x(i, j), \dots, x(i, i+\beta)), \phi(y'(i, j), x'(i, j), \dots, x'(i+\beta))) \\ & + d(\phi(y'(i, j), x'(i, j), \dots, x'(i, i+\beta)), \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha))) \\ & \leq \mu \max_{i \leq l \leq \beta+i} \{d(y(i, j), y'(i, j)), d(x(l, j), x'(l, j))\} + \log_t M \\ & < \mu \frac{\delta}{2} + \log_t M < \frac{\delta}{2}. \end{aligned}$$

So we have verified the estimates $d((\bar{y}')^j, \bar{y}^j) < \frac{\delta}{2}$ for all i by induction.

Then we have the estimates:

$$\begin{aligned} & d(x(i, j+1), x'(i, j+1)) = \\ & d(\psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)), \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))) \\ & \leq d(\psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)), \psi(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))) \\ & + d(\psi(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j)), \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j))) \\ & \leq \mu \max_{i \leq l \leq \alpha+i} \{d(y(i, j), y'(i, j)), d(x(l, j), x'(l, j))\} + \log_t M \\ & < \mu \frac{\delta}{2} + \log_t M < \frac{\delta}{2}. \end{aligned}$$

So we have verified the estimates $d((\bar{x}')^{j+1}, \bar{x}^{j+1}) < \frac{\delta}{2}$.

So we have finished the induction step on j . This completes the proof.

4.D.4 Rational dynamics and change of automata: Let $S \subset \mathbb{R}$ be a

finite set, and take $\bar{a} = (a_0, a_1, \dots) \in X_S$. The *exponential sequence* is given by the sequence of positive numbers parametrized by $t > 1$:

$$t^{\bar{a}} = (t^{a_0}, t^{a_1}, \dots).$$

Its $C \geq 1$ neighbourhood is given by the set:

$$N_C(t^{\bar{a}}) = \{\bar{z} \in X_{\mathbb{R}} : C^{-1}t^{a_i} < z_i < Ct^{a_i}\} \supset t^{\bar{a}}.$$

For $l = 1, 2$, let:

$$\mathbf{A}_l : \psi^l : Q \times S^{\alpha+1} \rightarrow S, \quad \phi^l : Q \times S^{\beta+1} \rightarrow Q$$

be two automata, and choose their stable extensions:

$$\psi^l : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi^l : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R} \quad (l = 1, 2)$$

with the constants (δ, μ) by relative $(\max, +)$ -functions.

Let (f_t^l, g_t^l) and (ψ_t^l, ϕ_t^l) be the tropical correspondences to (ψ^l, ϕ^l) respectively, and M be the bigger one of their numbers of the components.

Let us consider the state systems of the rational dynamics:

$$\begin{aligned} z_i^{j+1}(l) &= f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ w_{i+1}^j(l) &= g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) \end{aligned}$$

with the initial values $\bar{z}(l) = \{z_i(l)\}_i$ and $\bar{w}(l) = \{w^j(l)\}_j$.

Theorem 4.7. *Suppose $\mathbf{A}_1$ and $\mathbf{A}_2$ are equivalent over some $R \subset Q$, and choose stable extensions with the constants (δ, μ) .*

Then for any large $C \gg 1$, there exists $t_0 > 1$ so that for all $t \geq t_0$ and any initial values which are contained in:

$$\bar{z}(l) \in N_C(t^{\bar{k}}), \quad \bar{w}(l) \in N_C(t^{\bar{q}}) \quad (l = 1, 2)$$

for some $\bar{k} \in X_S$ and $\bar{q} \in X_R$, then the uniform estimates hold:

$$\max\left\{\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)}\right\} < C^4.$$

Proof: We split the proof into two steps.

Step 1: Notice that the pairs (ψ^l, ϕ^l) are (δ', μ) -stable for all $0 < \delta' \leq \delta$. Let us choose large C so that the estimates hold:

$$M < C^{1-\mu}.$$

Then choose $t_0 \gg 1$ so that the estimates hold:

$$\log_{t_0} C \leq \frac{\delta}{2}.$$

Now let us choose and fix any $t \geq t_0$, and put $\delta' = \log_t C$. Then the estimates:

$$\mu\delta' + \log_{t_0} M \leq \mu \log_t C + \log_t M < \log_t C = \delta'$$

hold by the above inequality. So the condition in lemma 4.6 is satisfied.

Let us regard that the pairs (ψ^l, ϕ^l) are $(2\delta', \mu)$ -stable.

Step 2: Let us put:

$$x_i(l) = \log_t z_i(l), \quad y^j(l) = \log_t w^j(l) \quad (l = 1, 2)$$

and consider the corresponding systems of the state dynamics:

$$\begin{aligned} x_l(i, j+1) &= \psi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \alpha, j)), \\ y_l(i+1, j) &= \phi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \beta, j)), \end{aligned}$$

$$\begin{aligned} x'_l(i, j+1) &= \psi'_l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i + \alpha, j)), \\ y'_l(i+1, j) &= \phi'_l(y'_l(i, j), x'_l(i, j), \dots, x'_l(i + \beta, j)) \end{aligned}$$

with the initial values $\bar{x}(l) = \{x_i(l)\}_i$ and $\bar{y}(l) = \{y^j(l)\}_j$ respectively.

Then the estimates:

$$d(\bar{x}(l), \bar{k}) = \sup_i |x_i(l) - k_i| = \sup_i \log_t \left(\frac{z_i(l)}{t^{k_i}} \right)^{\pm 1} < \log_t C = \delta'$$

hold. Similarly the estimates $d(\bar{y}(l), \bar{q}) < \delta'$ hold.

By corollary 4.5, the estimates:

$$|x_1(i, j) - x_2(i, j)| < 2\delta'$$

hold for all i, j .

On the other hand by lemma 4.6,

$$|x_l(i, j) - x'_l(i, j)| < \delta'$$

hold for $l = 1, 2$ and all i, j .

Then combining with these, we have the estimates:

$$\begin{aligned} |x'_1(i, j) - x'_2(i, j)| &\leq |x'_1(i, j) - x_1(i, j)| + \\ &\quad |x_1(i, j) - x_2(i, j)| + |x_2(i, j) - x'_2(i, j)| \\ &< 4\delta' = \log_t C^4. \end{aligned}$$

Since the equalities:

$$|x'_1(i, j) - x'_2(i, j)| = \log_t \left(\frac{z_i^j(1)}{z_i^j(2)} \right)^{\pm 1}$$

hold, this verifies the desired estimates. This completes the proof.

4.D.5 Dynamical inequalities under change of automata: For $l = 1, 2$, let:

$$\mathbf{A}_l : \psi^l : Q \times S^{\alpha+1} \rightarrow S, \quad \phi^l : Q \times S^{\beta+1} \rightarrow Q$$

be automata which are equivalent over $R \subset Q$.

For each l , let us take two pairs of their stable extensions with the constants (δ, μ) by $(\max, +)$ -functions:

$$\psi^{l,m} : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi^{l,m} : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R} \quad (m = 1, 2)$$

Suppose that for each $l = 1, 2$:

$$(\psi^{l,1}, \phi^{l,1}) \sim (\psi^{l,2}, \phi^{l,2})$$

are pairwise tropically equivalent. Let $(f_t^{l,m}, g_t^{l,m})$ and $(\psi_t^{l,m}, \phi_t^{l,m})$ be the tropical correspondences to $(\psi^{l,m}, \phi^{l,m})$ respectively.

Corollary 4.8. *Assume the above conditions. Then for any large $C \geq 1$, there exists $t_0 > 1$ and $D \geq 1$ so that the following holds for all $t \geq t_0$:*

Suppose two sequences $\{w_i^j(l)\}_{i,j}$ and $\{z_i^j(l)\}_{i,j}$ satisfy the dynamical inequalities:

$$\begin{aligned} f_t^{l,1}(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)) &\leq z_i^{j+1}(l) \leq f_t^{l,2}(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ g_t^{l,1}(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) &\leq w_{i+1}^j(l) \leq g_t^{l,2}(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) \end{aligned}$$

Moreover suppose the initial values are contained in the same C neighbourhoods:

$$\bar{z}(l) = (z_0(l), z_1(l), \dots) \in N_C(t^{\bar{k}}), \quad \bar{w}(l) = (w^0(l), w^1(l), \dots) \in N_C(t^{\bar{q}})$$

for some $\bar{k} = (k_0, k_1, \dots) \in X_S$ and $\bar{q} = (q^0, q^1, \dots) \in X_R$.

Then the uniform estimates hold:

$$\max\left\{\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)}\right\} < D.$$

Proof: The proof is long and we split it into several steps, but the idea is quite parallel to theorem 2.19 and corollary 2.21.

Let us choose and fix large $t \gg 1$. By replacing δ by a smaller one as in Step 1 in the proof of theorem 4.7, we may assume the followings:

(1) $\delta = \log_t C$ and the estimates:

$$M^2 < C^{1-\mu}$$

holds. In particular the estimates $2 \log_t M + \mu \delta < \delta$ holds.

(2) $(\psi^{l,m}, \phi^{l,m})$ are all $(3\delta, \mu)$ -stable.

Let us put $p_i^j(l) = \log_t z_i^j(l)$ and $o_i^j(l) = \log_t w_i^j(l)$. Then we have the estimates:

$$\begin{aligned} \psi_t^{l,1}(o_i^j(l), p_i^j(l), \dots, p_{i+\alpha}^j(l)) &\leq p_i^{j+1}(l) \leq \psi_t^{l,2}(o_i^j(l), p_i^j(l), \dots, p_{i+\alpha}^j(l)), \\ \phi_t^{l,1}(o_i^j(l), p_i^j(l), \dots, p_{i+\beta}^j(l)) &\leq o_{i+1}^j(l) \leq \phi_t^{l,2}(o_i^j(l), p_i^j(l), \dots, p_{i+\beta}^j(l)) \end{aligned}$$

where $p_i^0(l) = x_i(l) \equiv \log_t z_i(l)$ and $o_0^j(l) = y^j(l) \equiv \log_t w^j(l)$.

Notice that both the equalities:

$$\psi^{l,1} = \psi^{l,2}, \quad \phi^{l,1} = \phi^{l,2}$$

hold as functions by the assumption. We will use the notation ψ^l and ϕ^l when no cofusion occurs.

Let us consider another solutions to the state systems:

$$\begin{aligned} k_i^{j+1}(l) &= \psi^l(q_i^j(l), k_i^j(l), \dots, k_{i+\alpha}^j(l)) \\ q_{i+1}^j(l) &= \phi^l(q_i^j(l), k_i^j(l), \dots, k_{i+\beta}^j(l)) \end{aligned}$$

with $k_i^0(l) = k_i$ and $q_0^j(l) = q^j$.

We verify the uniform estimates:

$$|p_i^j(l) - k_i^j(l)|, \quad |o_i^j(l) - q_i^j(l)| < \delta$$

by several steps.

Step 1: Let us introduce another state systems:

$$\begin{aligned} x'_{l,m}(i, j+1) &= \psi_t^{l,m}(y'_{l,m}(i, j), x'_{l,m}(i, j), \dots, x'_{l,m}(i + \alpha, j)), \\ y'_{l,m}(i+1, j) &= \phi_t^{l,m}(y'_{l,m}(i, j), x'_{l,m}(i, j), \dots, x'_{l,m}(i + \beta, j)), \end{aligned}$$

$$\begin{aligned} x_l(i, j+1) &= \psi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \alpha, j)), \\ y_l(i+1, j) &= \phi^l(y_l(i, j), x_l(i, j), \dots, x_l(i + \beta, j)), \end{aligned}$$

with the same initial values $x'_{l,m}(i, 0) = x_l(i, 0) = x_i(l)$ and $y'_{l,m}(0, j) = y_l(0, j) = y^j(l)$.

By lemma 4.4, the estimates hold:

$$|x_l(i, j) - k_i^j(l)|, \quad |y_l(i, j) - q_i^j(l)| < \delta.$$

On the other hand by lemma 4.6, another estimates hold:

$$|x_l(i, j) - x'_{l,m}(i, j)|, \quad |y_l(i, j) - y'_{l,m}(i, j)| < \delta.$$

So combining with these estimates, one obtains the estimates:

$$|x'_{l,m}(i, j) - k_i^j(l)|, \quad |y'_{l,m}(i, j) - q_i^j(l)| < 2\delta.$$

Step 2: Next let us verify the uniform estimates for $l = 1, 2$:

$$|x'_{l,m}(i, 1) - p_i^1(l)|, \quad |y'_{l,m}(i, 0) - o_i^0(l)| < \delta.$$

Notice the equalities $y'_{l,m}(0, 0) = o_0^0(l)$ and $x'_{l,m}(i, 0) = p_i^0(l)$.

Let us verify $|y'_{l,m}(i, 0) - o_i^0(l)| < \delta$ by induction on $i = 0, 1, 2, \dots$. Assume that the estimates hold up to i .

Firstly suppose $o_{i+1}^0(l) \geq y'_{l,1}(i+1, 0)$ hold. Then the estimates hold:

$$\begin{aligned}
0 &\leq o_{i+1}^0(l) - y'_{l,1}(i+1, 0) = |o_{i+1}^0(l) - y'_{l,1}(i+1, 0)| \\
&\leq |\phi_t^{l,2}(o_i^0(l), p_i^0(l), \dots, p_{i+\beta}^0(l)) - \phi_t^{l,1}(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0))| \\
&\leq 2\log_t M + \\
&|\phi^l(o_i^0(l), p_i^0(l), \dots, p_{i+\beta}^0(l)) - \phi^l(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0))| \\
&\leq 2\log_t M + \mu|o_i^0(l) - y'_{l,1}(i, 0)| \leq 2\log_t M + \mu\delta < \delta.
\end{aligned}$$

Conversely suppose $o_{i+1}^0(l) \leq y'_{l,1}(i+1, 0)$ hold. Then the estimates hold:

$$\begin{aligned}
0 &\leq y'_{l,1}(i+1, 0) - o_{i+1}^0(l) = |y'_{l,1}(i+1, 0) - o_{i+1}^0(l)| \\
&\leq |\phi_t^{l,1}(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0)) - \phi_t^{l,1}(o_i^0(l), p_i^0(l), \dots, p_{i+\beta}^0(l))| \\
&\leq 2\log_t M + \\
&|\phi^l(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0)) - \phi^l(o_i^0(l), p_i^0(l), \dots, p_{i+\beta}^0(l))| \\
&\leq 2\log_t M + \mu|o_i^0(l) - y'_{l,1}(i, 0)| \leq 2\log_t M + \mu\delta < \delta.
\end{aligned}$$

Thus in any cases, the estimates $|y'_{l,1}(i+1, 0) - o_{i+1}^0(l)| < \delta$ hold. By the induction step, we have verified the claim. By the same way the estimates $|y'_{l,2}(i, 0) - o_i^0(l)| < \delta$ also hold for all i .

Then by use of the above estimates, we follow a parallel argument as below. Suppose $p_i^1(l) \geq x'_{l,1}(i, 1)$ hold for some i . Then we have the estimates:

$$\begin{aligned}
0 &\leq p_i^1(l) - x'_{l,1}(i, 1) = |p_i^1(l) - x'_{l,1}(i, 1)| \\
&\leq |\psi_t^{l,2}(o_i^0(l), p_i^0(l), \dots, p_{i+\alpha}^0(l)) - \psi_t^{l,1}(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0))| \\
&\leq 2\log_t M + \\
&|\psi^l(o_i^0(l), p_i^0(l), \dots, p_{i+\alpha}^0(l)) - \psi^l(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0))| \\
&\leq 2\log_t M + \mu|o_i^0(l) - y'_{l,1}(i, 0)| \leq 2\log_t M + \mu\delta < \delta.
\end{aligned}$$

Conversely suppose $p_i^j(l) \leq x'_{l,1}(i, 1)$ hold. Then we have the estimates:

$$\begin{aligned}
0 &\leq x'_{l,1}(i, 1) - p_i^1(l) = |x'_{l,1}(i, 1) - p_i^1(l)| \\
&\leq |\psi_t^{l,1}(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0)) - \psi_t^{l,1}(o_i^0(l), p_i^0(l), \dots, p_{i+\alpha}^0(l))| \\
&\leq 2\log_t M + \\
&|\psi^l(y'_{l,1}(i, 0), x'_{l,1}(i, 0), \dots, x'_{l,1}(i+\beta, 0)) - \psi^l(o_i^0(l), p_i^0(l), \dots, p_{i+\alpha}^0(l))| \\
&\leq 2\log_t M + \mu|o_i^0(l) - y'_{l,1}(i, 0)| \leq 2\log_t M + \mu\delta < \delta.
\end{aligned}$$

Thus in any cases we have the estimates $|x'_{l,1}(i, 1) - p_i^1(l)| < \delta$ for all $i = 0, 1, 2, \dots$. The estimates $|x'_{l,2}(i, 1) - p_i^1(l)| < \delta$ are obtained by the same way.

Step 3: Let us verify the estimates for all $i, j \geq 0$:

$$|x'_{l,m}(i, j) - p_i^j(l)|, |y'_{l,m}(i, j) - \sigma_i^j(l)| < \delta$$

Let us put the sequences:

$$\bar{x}'_{l,m}(j) = (x'_{l,m}(0, j), x'_{l,m}(1, j), x'_{l,m}(2, j), \dots)$$

and similar for $\bar{y}'_{l,m}(j)$, $\bar{p}^j(l)$ and $\bar{\sigma}^j(l)$.

Let us verify the estimates:

$$d(\bar{x}'_{l,m}(j+1), \bar{p}^{j+1}(l)), d(\bar{y}'_{l,m}(j), \bar{\sigma}^j(l)) < \delta$$

by induction on $j = 0, 1, 2, \dots$. We have verified the estimates for $j = 0$ at step 2. So assume they hold up to $j - 1$.

The initial conditions $\sigma_0^j(l) = y'_{l,1}(0, j)$ hold. Let us verify the estimates $|\sigma_i^j(l) - y'_{l,1}(i, j)| < \delta$ by induction on i . Suppose they hold up to i .

Firstly suppose $\sigma_{i+1}^j(l) \geq y'_{l,1}(i+1, j)$ hold. Then the estimates hold:

$$\begin{aligned} 0 &\leq \sigma_{i+1}^j(l) - y'_{l,1}(i+1, j) = |\sigma_{i+1}^j(l) - y'_{l,1}(i+1, j)| \\ &\leq |\phi_t^{l,2}(\sigma_i^j(l), p_i^j(l), \dots, p_{i+\beta}^j(l)) - \phi_t^{l,1}(y'_{l,1}(i, j), x'_{l,1}(i, j), \dots, x'_{l,1}(i+\beta, j))| \\ &\leq 2 \log_t M + \\ &|\phi^l(\sigma_i^j(l), p_i^j(l), \dots, p_{i+\beta}^j(l)) - \phi^l(y'_{l,1}(i, j), x'_{l,1}(i, j), \dots, x'_{l,1}(i+\beta, j))| \\ &\leq 2 \log_t M + \\ &\mu \max(|\sigma_i^j(l) - y'_{l,1}(i, j)|, |p_i^j(l) - x'_{l,1}(i, j)|, \dots, |p_{i+\beta}^j(l) - x'_{l,1}(i+\beta, j)|) \\ &\leq 2 \log_t M + \mu \delta < \delta. \end{aligned}$$

The converse case can be estimated by the same way as step 2, and we omit repetition. So by the induction step we have the estimates $|y'_{l,1}(i, j) - \sigma_i^j(l)| < \delta$ for all i . By the same way we can verify the estimates $|y'_{l,2}(i, j) - \sigma_i^j(l)| < \delta$.

Then we have the estimates $|x'_{l,m}(i, j+1) - p_i^{j+1}(l)| < \delta$ for all $i = 0, 1, 2, \dots$ again by the same way as step 2.

This completes the induction step on j , and so we have obtained the desired estimates:

$$|x'_{l,m}(i, j) - p_i^j(l)|, |y'_{l,m}(i, j) - \sigma_i^j(l)| < \delta$$

for all $i, j = 0, 1, 2, \dots$

Step 4: Combining step 1 $\sim$ 3, we obtain the estimates:

$$|p_i^j(l) - k_i^j(l)| < 3\delta.$$

In particular all the values of these sequences lie within μ -Lipschitz constants of $\psi^{l,m}$ and $\phi^{l,m}$.

Let us consider the state systems of the rational dynamics:

$$\begin{aligned} F^l(i, j+1) &= f_t^{l,1}(G^l(i, j), F^l(i, j), \dots, F^l(i + \alpha, j)) \\ G^l(i, j+1) &= g_t^{l,1}(G^l(i, j), F^l(i, j), \dots, F^l(i + \beta, j)) \end{aligned}$$

with the initial values $F^l(i, 0) = z_i(l)$ and $G^l(0, j) = w^j(l)$ respectively. Then we apply theorem 2.17, and obtain the uniform estimates:

$$\max\left(\frac{F^l(i, j+1)}{z_i^{j+1}(l)}, \frac{z_i^{j+1}(l)}{F^l(i, j+1)}\right) \leq M^{2P_{i+j(\gamma+1)}(\mu)} \leq C' \quad (l = 1, 2)$$

for some C' , since $P_i(\mu)$ are uniformly bounded for $0 < \mu < 1$. On the other hand by theorem 4.7, the estimates hold:

$$\max\left(\frac{F^1(i, j+1)}{F^2(i, j+1)}, \frac{F^2(i, j+1)}{F^1(i, j+1)}\right) \leq C^4.$$

Thus combining with these, one obtains the desired estimates:

$$\left(\frac{z_i^j(1)}{z_i^j(2)}\right)^{\pm 1} = \left(\frac{z_i^j(1)}{F^1(i, j)}\right)^{\pm 1} \left(\frac{F^2(i, j)}{z_i^j(2)}\right)^{\pm 1} \leq C' C^4 \equiv D.$$

This completes the proof.

4.E Quasi recursivity: Let us consider recursivity for rational dynamics. We start from a simpler case which is time-independent and one dimensional.

Let φ be a $(\max, +)$ function of n variable, and consider the one dimensional dynamics:

$$x_N = \varphi(x_{N-n}, \dots, x_{N-1}).$$

It is *recursive* over $X \subset \mathbb{R}$, if there is some $M \geq 0$ so that any orbits $\{x_i\}_i$ with any initial values $\bar{x}_0 = (x_0, \dots, x_{n-1}) \in X^n \subset \mathbb{R}^n$, are periodic of period M . Namely equalities $x_{i+M} = x_i$ hold for all $i \geq 0$.

Let f_t be the tropical correspondence to φ , and consider the corresponding dynamics:

$$z_N = f_t(z_{N-n}, \dots, z_{N-1})$$

with the initial data $x_0 = \log_t z_0, \dots, x_{n-1} = \log_t z_{n-1}$.

It is known that φ is always recursive over $\mathbb{R}$ whenever f_t is the case over $(0, \infty)$ for all $t > 1$. However the converse is not true.

Let us say that f_t is a *quasi recursive* of period M over $X_t \subset \mathbb{R}$, if there are constants $C \geq 1$ and $M \geq 0$ independently of t and initial values $(z_0, \dots, z_{n-1}) \in X_t^n \subset \mathbb{R}^n$ so that the uniform estimates hold:

$$\max\left(\frac{z_{N+M}}{z_N}, \frac{z_N}{z_{N+M}}\right) \leq C, \quad N = 0, 1, \dots$$

Proposition 4.9 (K2). *Suppose f_t corresponds to a relative $(\max, +)$ -function φ . Then f_t is quasi recursive of minimum period M , over $(0, \infty)$ if and only if φ is recursive of the same minimum period over $\mathbb{R}$.*

For example:

$$\begin{aligned} \varphi^1(x_0, x_1) &= \max(0, x_1) - x_0, \\ \varphi^2(x_0, x_1) &= \max(x_1, -x_1) - x_0 \end{aligned}$$

are recursive of periods 5 and 9 over $\mathbb{R}$ respectively. The corresponding relatively elementary functions are given by:

$$\begin{aligned} f_t^1(w, z) &= w^{-1}(1 + z), \\ f_t^2(w, z) &= w^{-1}(z^{-1} + z) \end{aligned}$$

respectively. It turns out that the former is also recursive over $(0, \infty)$, but the second is not the case, and so it only satisfies quasi recursivity.

4.E.2 Finite order group elements: Let:

$$\mathbf{A} : \psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

be an automaton, and $\psi : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}$, $\phi : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}$ be stable extensions with the constants (δ, μ) .

Let (f_t, g_t) and (ψ_t, ϕ_t) be the corresponding functions to (ψ, ϕ) respectively, and M be the biggest one of the numbers of their components.

For $\bar{q}^m = (q^0, \dots, q^m) \in X_Q^{m+1}$, let us denote l times iterations of $\bar{q}^m$ by $l\bar{q}^m \equiv (q^0, \dots, q^m, q^0, \dots, q^m, \dots, q^0, \dots, q^m) \in X_Q^{(m+1)l}$. We also denote the infinite times iterations of $\bar{q}^m$ by:

$$\bar{q}_{per}^m \equiv (q^0, \dots, q^m, q^0, \dots, q^m, \dots, q^0, \dots, q^m, \dots) \in X_Q.$$

Let us consider the state dynamics:

$$z_i^{j+1} = f_t(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \quad w_{i+1}^j = g_t(w_i^j, z_i^j, \dots, z_{i+\beta}^j).$$

Proposition 4.10. *Suppose $\mathbf{A}_{\bar{q}^m} : X_S \rightarrow X_S$ is of finite order with period p :*

$$\mathbf{A}_{p\bar{q}^m} = (\mathbf{A}_{\bar{q}^m}) \circ \dots \circ (\mathbf{A}_{\bar{q}^m}) \equiv (\mathbf{A}_{\bar{q}^m})^p = id.$$

Then for any $C \geq 1$, there exists $t_0 > 1$ so that for all $t \geq t_0$ and any initial values:

$$\{z_i\}_i \subset N_C(t^S), \quad \{w^j\}_j \subset N_C(t^{\bar{q}_{per}^m})$$

the uniform bounds hold for all $i, j, l = 0, 1, 2, \dots$:

$$\left(\frac{z_i^j}{z_i^{j+p(m+1)l}}\right)^{\pm 1} \leq C^4$$

Proof: Let us choose large $t_0 > 1$ so that the estimates hold:

$$\log_{t_0} C, \quad \mu \frac{\delta}{2} + \log_{t_0} M < \frac{\delta}{2}.$$

Recall that the pair (ψ, ϕ) is (δ', μ) -stable for any $0 < \delta' \leq \delta$. Let us fix $t \geq t_0$. Then by replacing δ by $\delta' = 2 \log_t C$, one may assume the equality $\delta = 2 \log_t C$.

By the assumption, there is $\bar{k} = (k_0, k_1, \dots) \in X_S$ so that the initial value is contained as $\{z_i\}_i \in N_C(t^{\bar{k}})$. Let us rewrite $\bar{q}_{per}^m = (q^0, q^1, \dots)$

Let us consider three state systems of dynamics:

$$\begin{aligned} x(i, j+1) &= \psi(y(i, j), x(i, j), \dots, x(i+\alpha, j)), \\ y(i+1, j) &= \phi(y(i, j), x(i, j), \dots, x(i+\beta, j)), \\ x'(i, j+1) &= \psi_t(y'(i, j), x'(i, j), \dots, x'(i+\alpha, j)), \\ y'(i+1, j) &= \phi_t(y'(i, j), x'(i, j), \dots, x'(i+\beta, j)), \\ k_i^{j+1} &= \psi(q_i^j, k_i^j, \dots, k_{i+\alpha}^j), \\ q_{i+1}^j &= \phi(q_i^j, k_i^j, \dots, k_{i+\beta}^j), \end{aligned}$$

with the initial values $x(i, 0) = x'(i, 0) = \log_t z_i$, $y(0, j) = y'(0, j) = \log_t w^j$, and $k_i^0 = k_i$, $q_0^j = q^j$. By the condition,

(1) The estimates $|x(i, 0) - k_i|, |y(0, j) - q^j| < \frac{\delta}{2}$ hold.

(2) periodicity $k_i^j = k_i^{j+p(m+1)l}$ hold for all i, j, l .

By lemma 4.4, the estimates $|x(i, j) - k_i^j| < \frac{\delta}{2}$ hold. By lemma 4.6, the estimates $|x(i, j) - x'(i, j)| < \frac{\delta}{2}$ hold. Combining with these, we obtain the estimates:

$$|x'(i, j) - k_i^j| < \delta.$$

Then we have the estimates:

$$\begin{aligned} & |x'(i, j) - x'(i, j + p(m+1)l)| \\ & \leq |x'(i, j) - k_i^j| + |x'(i, j + p(m+1)l) - k_i^{j+p(m+1)l}| < 2\delta = \log_t C^4. \end{aligned}$$

Since the left hand side is equal to $\log_t(\frac{z_i^j}{z_i^{j+p(m+1)l}})^{\pm 1}$, the conclusion holds.

This completes the proof.

4.E.3 Infinite quasi-recursive rational dynamics: The *Burnside problem* asks existence of finitely generated and infinite torsion groups. The solutions in the class of automata groups are given by Aleshin-Grigorchuk as follows:

Lemma 4.11. *There exist Mealy automata such that the groups generated by some states of them are infinite and torsion.*

Let (f, g) be a pair of (unparametrized) rational functions, and let us restate the state dynamics so that it is compatible with the expression of the dynamics by automata. For $w \in \mathbb{R}$, let:

$$\begin{aligned} \bar{\mathbf{A}}_w : \mathbb{R}^{\mathbb{N}} &\rightarrow \mathbb{R}^{\mathbb{N}}, \\ (z_0, z_1, \dots) &\rightarrow (z'_0, z'_1, \dots) \end{aligned}$$

be given by the rule:

$$z'_i = f(w_i, z_i, \dots, z_{i+\alpha}), \quad w_{i+1} = g(w_i, z_i, \dots, z_{i+\beta})$$

where $w_0 = w$. For finite sequences $\bar{w} = (w^0, \dots, w^l) \in \mathbb{R}^{l+1}$, we denote:

$$\bar{\mathbf{A}}_{\bar{w}} \equiv \bar{\mathbf{A}}_{w^l} \circ \dots \circ \bar{\mathbf{A}}_{w^0} : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}.$$

Let:

$$X, Y \subset \mathbb{R}$$

be two subsets. The state system is said to be *recursive* over (X, Y) , if for any finite sequences:

$$\bar{w} = (w^0, \dots, w^l) \in Y^{l+1} \subset \mathbb{R}^{l+1}$$

there exist some $p = p(\bar{w}) \in \mathbb{N}$ so that the rational dynamics induce:

$$\bar{\mathbf{A}}_{\bar{w}} : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$$

which are periodic of period p :

$$(\bar{\mathbf{A}}_{\bar{w}})^p = \text{id} .$$

The *rational Burnside problem* asks existence of pairs of rational functions whose state dynamics are recursive such that $\bar{\mathbf{A}}_{\bar{w}}$ are mutually different with respect to infinitely many words $\bar{w} \in Y^*$.

So far it seems not known whether such a pair of rational functions exist.

Let us introduce a variant of the rational Burnside problem, where we need to use pairs of parametrized rational functions.

Firstly we introduce some notations. Let us consider two dynamics:

$$\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2 : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$$

and denote their orbits by:

$$(\bar{\mathbf{A}}_k)^j(\bar{z}) = (z_0^j(k), z_1^j(k), \dots)$$

for $k = 1, 2$. Let us say that:

(1) $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are *C-close* over $X \subset \mathbb{R}$, if for any $\bar{z} \in X^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$, the corresponding orbits satisfy the uniform estimates:

$$\max\left(\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)} \right) \leq C$$

hold for all $i, j = 0, 1, 2, \dots$

(2) $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ are C' -separated over X , if there is some $\bar{z} \in X^{\mathbb{N}}$ so that the bounds hold for some i :

$$\max\left(\frac{z_i^1(1)}{z_i^1(2)}, \frac{z_i^1(2)}{z_i^1(1)}\right) \geq C'$$

Let (f_t, g_t) be a pair of relatively elementary functions, and consider the state system of the rational dynamics.

Definition 4.4. *The state system of the parametrized rational dynamics by (f_t, g_t) is infinite quasi-recursive if there exist a constant $C > 0$ and subsets:*

$$X_t, Y_t \subset \mathbb{R}$$

so that for any $C' \geq 1$, the followings hold for all $t \geq t_0 = t_0(C')$:

(1) For any words $\bar{w} \in Y_t^{l+1}$, there exist some $p = p(\bar{w}) \in \mathbb{N}$ such that $(\bar{\mathbf{A}}_{\bar{w}})^p$ and the identity are C -close over X_t .

(2) There are infinitely many words $\bar{w}^j \in Y_t^*$ so that $\bar{\mathbf{A}}_{\bar{w}^j}$ and $\bar{\mathbf{A}}_{\bar{w}^{j'}}$ are mutually C' -separated over X_t for any $j \neq j'$.

Let $S = \{s_0, s_1\} \subset \mathbb{Z}$ and $R = \{q^0, q^1\} \subset \mathbb{Z}$ be any embeddings, and denote:

$$N_C(t^S) = \{z \in (0, \infty) : C^{-1}t^s \leq z \leq Ct^s : s \in S\}.$$

Now we have the existence of parametrized infinitely quasi-recursive dynamics:

Theorem 4.12. *There exists a pair of relatively elementary functions (f_t, g_t) so that for any $C > 0$, the state dynamics is infinitely quasi-recursive over:*

$$(X_t, Y_t) = (N_C(t^S), N_C(t^R)).$$

Proof: Let us choose any embeddings $S = \{s_0, s_1\} \subset \mathbb{Z}$ and $Q = \{q^0, \dots, q^7\} \subset \mathbb{Z}$. Let us introduce *quasi periodic exponential lattices*:

$$\bar{N}_C(t^Q) = \{(w^0, w^1, \dots) : C^{-1}t^{q^j} \leq w^j \leq Ct^{q^j}, q^j \in Q\},$$

$$\bar{N}_C^{per}(t^Q) = \{(w^0, w^1, \dots) : C^{-1}t^{q^j} \leq w^j \leq Ct^{q^j}, \bar{q} \in X_Q \text{ are periodic}\}.$$

Notice that if two points $z, z' \in \bar{N}_C(t^s)$, then their ratios satisfy the estimates $(\frac{z}{z'})^{\pm 1} \leq C^2$ for $s \in S$.

If $z \in \bar{N}_C(t^{s_0})$ and $z' \in \bar{N}_C(t^{s_1})$, then the estimates hold:

$$\left(\frac{z}{z'}\right)^{\pm 1} \geq C^{-2}t.$$

In particular if $t \gg 1$ is sufficiently large, then $C^{-2}t \geq C'$ holds.

Let us consider the automaton $\mathbf{A}$ which produce the infinite torsion groups in lemma 4.11. By proposition 4.3, there exists a stable extension of $\mathbf{A}$. Let (f_t, g_t) be the pair of the corresponding relatively elementary functions.

Because the group generated by $\{q^0, q^1\}$ is infinite torsion, the conclusion follows from proposition 4.10. This completes the proof.

4.E.4 Estimates for dynamical inequalities: Let $\mathbf{A} : \psi : Q \times S^{\alpha+1} \rightarrow S$, $\phi : Q \times S^{\beta+1} \rightarrow Q$ be an automaton, and choose embeddings $Q, S \subset \mathbb{Z}$.

Let us take a stable extension with the constants (δ, μ) :

$$\begin{aligned} \psi &: \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \\ \phi &: \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}. \end{aligned}$$

Let us represent the pair of functions by two relatively $(\max, +)$ -functions (ψ^l, ϕ^l) for $l = 1, 2$, and let (f_t^l, g_t^l) be the tropical correspondences respectively. Thus $(f_t^1, g_t^1) \sim (f_t^2, g_t^2)$ are pairwise tropically equivalent.

Corollary 4.13. *Suppose $\mathbf{A}_{\bar{q}^m} : X_S \rightarrow X_S$ is of finite order with period p for some $\bar{q}^m \in X_Q^{m+1}$.*

Then for any $C \geq 1$, there exists $t_0 > 1$ and $D \geq 1$ so that for all $t \geq t_0$ and any pair of sequences $\{(z_i^j, w_i^j)\}_{i,j}$ which satisfy the dynamical inequalities:

$$\begin{aligned} f_t^1(w_i^j, z_i^j, \dots, z_{i+\alpha}^j) &\leq z_i^{j+1} \leq f_t^2(w_i^j, z_i^j, \dots, z_{i+\alpha}^j), \\ g_t^1(w_i^j, z_i^j, \dots, z_{i+\beta}^j) &\leq w_{i+1}^j \leq g_t^2(w_i^j, z_i^j, \dots, z_{i+\beta}^j) \end{aligned}$$

with the initial values $\{z_i\}_i \subset N_C(t^S)$, $\{w_j\}_j \subset N_C(t^{\bar{q}_{per}^m})$, then they satisfy the uniform bounds for all $i, j, l = 0, 1, 2, \dots$:

$$\left(\frac{z_i^j}{z_{i+j+p(m+1)l}^j}\right)^{\pm 1} \leq D.$$

Proof: Let us consider the state dynamics:

$$\begin{aligned} F_i^{j+1} &= f_t^1(G_i^j, F_i^j, \dots, F_{i+\alpha}^j), \\ G_{i+1}^j &= g_t^1(G_i^j, F_i^j, \dots, F_{i+\beta}^j) \end{aligned}$$

with the initial values $F_i^0 = z_i$ and $G_0^j = w^j$.

By corollary 4.8, the uniform estimates hold:

$$\left(\frac{z_i^j}{F_i^j}\right)^{\pm 1} \leq D.$$

By proposition 4.10, the uniform bounds:

$$\left(\frac{F_i^j}{F_i^{j+p(m+1)l}}\right)^{\pm 1} \leq C^4$$

hold for all $i, j, l \geq 0$.

Combining with these, we obtain the desired estimates:

$$\left(\frac{z_i^j}{z_i^{j+p(m+1)}}\right)^{\pm 1} = \left(\frac{z_i^j}{F_i^j}\right)^{\pm 1} \left(\frac{F_i^j}{F_i^{j+p(m+1)}}\right)^{\pm 1} \left(\frac{F_i^{j+p(m+1)}}{z_i^{j+p(m+1)}}\right)^{\pm 1} \leq C^4 D^2.$$

This completes the proof.

5 Analysis of hyperbolic systems of PDEs

In section 5, we develop basic analysis of the hyperbolic Mealy systems of PDE in our sense, and then apply the previous results to the large scale analysis of them.

5.A PDE systems and equivalent automata: For $l = 1, 2$, let

$$\mathbf{A}_l : \psi^l : Q \times S^{\alpha+1} \rightarrow S, \quad \phi^l : Q \times S^{\beta+1} \rightarrow Q$$

be equivalent automata over $R \subset Q$, and choose embeddings $Q, S \subset \mathbb{R}$.

Let us take their stable extensions with the constants (δ, μ) :

$$\psi^l : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}, \quad \phi^l : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R} \quad (l = 1, 2).$$

and (f_t^l, g_t^l) and (ψ_t^l, ϕ_t^l) be the corresponding functions respectively.

Let us consider the state systems of the rational dynamics:

$$\begin{aligned} z_i^{j+1}(l) &= f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ w_{i+1}^j(l) &= g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)). \end{aligned}$$

Let us follow the process in 3.B, and induce the systems of partial differential equations of order μ :

$$\begin{aligned} P_1^l(\epsilon, t, u^l, v^l, u_x^l, u_s^l, \dots, u_{\mu s}^l) &= 0 \\ P_2^l(\epsilon, t, u^l, v^l, u_x^l, v_x^l, \dots, u_{\mu x}^l) &= 0 \end{aligned}$$

with the scaling parameters:

$$i = \frac{x}{\epsilon}, \quad j = \frac{s}{\epsilon}, \quad u^l(x, s) = z_i^j(l), \quad v^l(x, s) = w_i^j(l).$$

Let us fix any positive number $0 < \tau < 0.5$, and put the domains:

$$\begin{aligned} D_1(\epsilon, \tau) &= \{ i\epsilon + a \in [0, \infty) : i \in \mathbb{N}, |a| \leq \epsilon\tau \}, \\ D(\epsilon, \tau) &= D_1(\epsilon, \tau) \times D_1(\epsilon, \tau) \subset [0, \infty) \times [0, \infty). \end{aligned}$$

$D(\epsilon)$ are the disjoint unions of the squares.

We also put the initial domains:

$$I_x(\epsilon, \tau) = [0, \infty) \times [0, \tau\epsilon), \quad I_s(\epsilon, \tau) = [0, \tau\epsilon) \times [0, \infty).$$

Proposition 5.1. *Let C_0 be the bigger one of their error constants. Let $u^l, v^l : (0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ be solutions to the above systems respectively. Then for any $C \geq 1$, there exists $t_0 > 1$ and $D \geq 1$ so that the followings hold for all $t \geq t_0$:*

Suppose two conditions: (1) The estimates:

$$0 \leq CK(u^l, v^l) \leq (1 - \mu)\epsilon^{-1}$$

are satisfied for some positive $\mu > 0$.

(2) The inclusions hold:

$$u^l|_{I_x(\epsilon, \tau)} \subset N_C(t^S), \quad v^l|_{I_s(\epsilon, \tau)} \subset N_C(t^Q).$$

Then they satisfy the uniform bounds:

$$\left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s) \leq D$$

for all $(x, y) \in D(\epsilon, \tau)$.

Proof: We follow a similar argument as the proof of theorem 3.2. Let $N_0 \geq \max(\mu^{-1}, 2 - \mu)$ be an integer.

Let us fix $0 \leq a, b \leq \tau$, and put the domain lattices by:

$$L_{\epsilon, \tau}(a, b) = \{((l_1 + a)\epsilon, (l_2 + b)\epsilon) \in [0, \infty) \times [0, \infty) : l_1, l_2 \in \mathbb{N}\}.$$

Let us put:

$$z_i^j(l) = u^l((i + a)\epsilon, (j + b)\epsilon), \quad w_i^j(l) = v^l((i + a)\epsilon, (j + b)\epsilon)$$

Then by the same way as the proof of theorem 3.2, we obtain the inequalities:

$$\begin{aligned} \frac{1}{N_0} f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)) &\leq z_i^{j+1}(l) \leq N_0 f_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\alpha}^j(l)), \\ \frac{1}{N_0} g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)) &\leq w_{i+1}^j(l) \leq N_0 g_t^l(w_i^j(l), z_i^j(l), \dots, z_{i+\beta}^j(l)). \end{aligned}$$

Notice that $\frac{1}{N_0} f$ and $N_0 f$ are both tropically equivalent to f .

By the assumption, the initial values satisfy:

$$z_i^0(l) = u^l((i + a)\epsilon, b\epsilon) \in N_C(t^S), \quad w_0^j(l) = v^l(a\epsilon, (j + b)\epsilon) \in N_C(t^R).$$

It follows from corollary 4.8 that the uniform estimates:

$$\max\left(\frac{z_i^j(1)}{z_i^j(2)}, \frac{z_i^j(2)}{z_i^j(1)}\right) \leq D.$$

Since the right hand side does not depend on a, b , one obtains the uniform bounds:

$$\left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s) \leq D$$

for all $(x, y) \in D(\epsilon, \tau)$. This completes the proof.

5.B. Hyperbolic Mealy systems: In this section we study the basic analysis of 1st order hyperbolic systems of PDE with 2 variables.

Let (f_t, g_t) be a pair of relatively elementary functions, corresponding to the pair of relative $(\max, +)$ -functions (ψ, ϕ) .

Let us consider the corresponding hyperbolic Mealy systems:

$$\begin{aligned}\epsilon u_s &= f_t(v, u) - u, \\ \epsilon v_x &= g_t(v, u) - v.\end{aligned}$$

Recall the higher distortion for the Mealy systems:

$$K(u, v) \equiv \sup_{(x,s) \in [0,\infty)^2} \max \left[\frac{\|(u, v)\|_{\mu,0}^1}{u(x, s + \epsilon)}, \frac{\|(u, v)\|_{\mu,0}^2}{v(x + \epsilon, s)} \right].$$

Definition 5.1. *The pair (f_t, g_t) is admissible, if there are $0 < \delta < L < L'$, $0 < \mu$ and constants A so that there are solutions with the initial values:*

$$u, v : [0, \infty) \times [0, \infty) \rightarrow [t^L, t^{L'}],$$

$$\|u_x\| C^0([0, \infty) \times \{0\}), \quad \|v_s\| C^0(\{0\} \times [0, \infty)) \leq A$$

which satisfy the following estimates hold for all $0 \leq \alpha \leq 1$:

$$|(f_t(v, u) - u)((f_t)_u(v, u) - 1)| + |v_s(f_t)_v(u, v)|(x, s) < (2 - \mu)u(x, s + \alpha),$$

$$|(g_t(v, u) - v)((g_t)_v(v, u) - 1)| + |u_x(g_t)_u(u, v)|(x, s) < (2 - \mu)v(x + \alpha, s).$$

Corollary 5.2. *For $l = 1, 2$, let (f_t^l, g_t^l) be admissible pairs, and $(u^l, v^l) : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ be the solutions to the above systems respectively.*

Then they satisfy the asymptotic estimates:

$$\begin{aligned} & \left(\frac{u^1}{u^2}\right)^{\pm 1}(x, s), \quad \left(\frac{v^1}{v^2}\right)^{\pm 1}(x, s) \\ & \leq (M_0)^{6P_{\epsilon^{-1}(x+s(\gamma+1))}(c)} \left[(u^1, v^1) : (u^2, v^2)\right]_{\epsilon}^{\tilde{c}\epsilon^{-1}(x+s(\gamma+1))+1} \end{aligned}$$

for some M_0 and all $(x, s) \in [0, \infty) \times [0, \infty)$.

Proof: For the Mealy systems, the error constants are always $\frac{1}{2}$. Moreover we have the equalities:

$$\begin{aligned} \frac{\partial^2 u}{\partial_s^2} &= (f_t(v, u) - u)((f_t)_u(v, u) - 1) + (f_t)_v(u, v)v_s, \\ \frac{\partial^2 v}{\partial_x^2} &= (g_t(v, u) - v)((g_t)_v(v, u) - 1) + (g_t)_u(u, v)u_x. \end{aligned}$$

So in order to apply theorem 3.1, the required consitions on the higher distorsion are:

$$\frac{|(f_t(v, u) - u)((f_t)_u(v, u) - 1)| + |v_s(f_t)_v(u, v)|(x, s)}{2u(x, s + \alpha)} < (1 - \delta)\epsilon^{-1},$$

$$\frac{|(g_t(v, u) - v)((g_t)_v(v, u) - 1)| + |u_x(g_t)_u(u, v)|(x, s)}{2v(x + \alpha, s)} < (1 - \delta)\epsilon^{-1}$$

for some $0 < \delta < 1$ and $0 < \epsilon \leq 1$, which follow from admissibility.

This completes the proof.

In 5.D we have concrete examples of admissible pairs with the Lipschitz constants 1. In particular we obtain the exponential estimates for their solutions.

5.B.3 Refinement: Let:

$$\mathbf{A} : \psi : Q \times S^{\alpha+1} \rightarrow S, \quad \phi : Q \times S^{\beta+1} \rightarrow Q$$

be an automaton over $R \subset Q$. For $\bar{q} = (q^0, q^1, \dots) \in X_Q$ and $\bar{k} = (k_0, k_1, \dots) \in X_S$, let us denote the orbits by $\{k_i^j\}$ and $\{q_i^j\}$.

Let $\tilde{\psi} : \mathbb{R} \times \mathbb{R}^{\alpha+1} \rightarrow \mathbb{R}$ and $\tilde{\phi} : \mathbb{R} \times \mathbb{R}^{\beta+1} \rightarrow \mathbb{R}$ be two maps. Suppose the restrictions induce maps as below for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$:

$$(\tilde{\psi}, \tilde{\phi}) : [a, b]^2 \rightarrow [a, b]^2.$$

For $\bar{y} = (y^0, y^1, \dots), \bar{x} = (x_0, x_1, \dots) \in X_{[a, b]}$, let us denote the orbits by $\{x_i^j\}$ and $\{y_i^j\}$. Let us choose embeddings $S, Q \subset [a, b]$.

Definition 5.2. $(\tilde{\psi}, \tilde{\phi})$ is an ϵ -refinement of the pair (ψ, ϕ) , if there is a positive number N so that for any $\bar{q} \in X_Q$ and $\bar{k} \in X_S$, there are paths $y : \{0, 1, \dots\} \rightarrow \mathbb{R}$ and $x : \{0, 1, \dots\} \rightarrow \mathbb{R}$ with:

$$y(jN) = q^j, \quad |y(j+1) - y(j)| \leq \epsilon,$$

$$x(iN) = k_i, \quad |x(i+1) - x(i)| \leq \epsilon$$

for all $i, j \in \{0, 1, \dots\}$, such that the equalities hold:

$$x_{iN}^{jN} = k_i^j, \quad y_{iN}^{jN} = q_i^j.$$

(ψ, ϕ) is refinable, if there is an ϵ -refinement for any small $\epsilon > 0$.

An ϵ refinement is almost diagonal, if moreover they satisfy the estimates for all $(x, y) \in [a, b]$:

$$|(x, y) - (\tilde{\psi}(x, y), \tilde{\phi}(x, y))| \leq \epsilon.$$

By use of refinement, we verify existence of admissible solutions in 5.D, and their exponential asymptotic estimates, which we will state below.

5.B.4 Asymptotic comparisons with group actions: Let $\mathbf{A}$ be a Mealy automaton with 2 alphabets, equipped with a representative (ψ, ϕ) by relatively $(\max, +)$ -functions.

For any $(s_0, s_1, \dots) \in X_2$, and $(q^0, q^1, \dots) \in X_{m+1}$, let:

$$\mathbf{A}_{(q^0, \dots, q^{l-1})}(s_0, s_1, \dots) = ((s_0^l, s_1^l, \dots) \in X_2$$

be the orbits of the automata group actions.

For its refinement $(\bar{\psi}, \bar{\phi})$ and their tropical correspondences $(\bar{f}_t, \bar{g}_t)$, let us consider the hyperbolic Mealy systems:

$$u_s = \bar{f}_t(v, u) - u, \quad v_x = \bar{g}_t(v, u) - v.$$

Theorem 5.3. *For any $C > 0$ and any $t \geq t(C) > 1$, there are refinements $(\bar{\psi}, \bar{\phi})$ of (ψ, ϕ) with the pairs of tropical correspondences $(\bar{f}_t, \bar{g}_t)$ so that:*

- (1) $(\bar{f}_t, \bar{g}_t)$ admits admissible solutions,
- (2) for any another pairs (f_t, g_t) toropically equivalent to $(\bar{f}_t, \bar{g}_t)$, any admissible solutions to the equations:

$$u_s = f_t(v, u) - u, \quad v_x = g_t(v, u) - v$$

whose initial values satisfy the inclusions for all $k = 0, 1, 2, \dots$:

$$d(u(Nk, 0), S), \quad d(v(0, Nk), Q) \leq C$$

then they satisfy the asymptotic estimates for some $M, c \geq 1$:

$$\left(\frac{u(Ni, Nj)}{s_i^j} \right)^{\pm 1} \leq M^{P_{N(i+j)+1}(c)}$$

Proof: This follows from proposition 5.12 below.

Remark: (1) We can choose $c = 1$ and hence obtain the exponential estimates, if we can represent the transition functions by 1-Lipschitz functions.

(2) It would be quite possible to obtain better estimates, by tracing the orbits in detail, but one will be required more analysis to achieve it.

It would be of interest to apply group-theoretic results to the analysis of PDE. Compared with the dynamical Burnside problem, we would like to propose the following.

Let $X \subset [0, \infty)$ be a net and $Y \subset [0, \infty)$ be a periodic subset. Let us take parametrized functional subsets $A_X, B_Y \subset C[0, \infty)$.

Conjecture 5.1: There exists a Mealy automaton so that the corresponding hyperbolic system is infinitely quasi-recursive over (A_X, B_Y) in the following sense; there exists $C \geq 1$ so that for any $C' > 0$, the followings hold:

(1) There are solutions $u, v : [0, \infty)^2 \rightarrow (0, \infty)$ for any initial values $u(x, 0) \in A_X$ and $v(0, s) \in B_Y$.

(2) For each $n \in B_Y$, there is a minimal p such that any solutions (u, v) with $u(\cdot, 0) \in A_X$ and $v(0, \cdot) = n$ satisfy quasi periodicity:

$$\left(\frac{u(x, s)}{u(x, s + kp)}\right)^{\pm 1} \leq D$$

for any $(x, s), (x, s + kp) \in X \times Y$.

(3) For each $n \neq n' \in B_Y$, there is some $m \in A_X$ so that for any solutions (u, v) and (u', v') with $u(x, 0) = u'(x, 0) = m$ with $v(0, y) = n$ and $v'(0, y) = n'$, the estimates hold:

$$\sup_x \left(\frac{u(x, 1)}{u'(x, 1)}\right)^{\pm 1} \geq C'.$$

On ϵ -controll of higher distortions: In order to apply the construction in section 3, we need to controll the second derivatives of solutions.

Let $(u, v) : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)^2$ be a solution to the hyperbolic Mealy system:

$$\begin{aligned} \epsilon u_s &= f_t(v, u) - u, \\ \epsilon v_x &= g_t(v, u) - v \end{aligned}$$

with finite higher distortion $K(u, v) < \infty$.

In order to apply the asymptotic comparisons for solutions to the PDE systems, their solutions are required to satisfy some bounds:

$$CK(u, v) \leq (1 - \delta)\epsilon^{-1}$$

for some $0 < \delta < 1$, where C is the error constant.

For small $0 < \tau$, let us consider the reparametrized pair of the functions:

$$(\tilde{u}, \tilde{v})(x, s) \equiv (u, v)(\tau x, \tau s)$$

which satisfy the equations:

$$\begin{aligned}\epsilon\tau^{-1} \tilde{u}_s &= f_t(\tilde{v}, \tilde{u}) - \tilde{u}, \\ \epsilon\tau^{-1} \tilde{v}_x &= g_t(\tilde{v}, \tilde{u}) - \tilde{v}.\end{aligned}$$

Lemma 5.4. *The pairs $(\tilde{u}, \tilde{v})$ satisfy the condition above on the higher distortion if we choose sufficiently small $\tau > 0$.*

Proof: Notice that the higher distortions are related as:

$$\tau^2 K_2(u, v) = K_2(\tilde{u}, \tilde{v}).$$

The required condition with respect to the reparametrized pair is given by:

$$CK_2(\tilde{u}, \tilde{v}) \leq (1 - \delta)\tau\epsilon^{-1}$$

which can be rewritten as:

$$\tau CK_2(u, v) \leq (1 - \delta)\epsilon.$$

So if we choose sufficiently small $\tau > 0$, then the pairs $(\tilde{u}, \tilde{v})$ satisfy the required condition. This completes the proof.

On the other hand the estimates $0 < \epsilon\tau^{-1} \leq 1$ are required, and so at best we can take $\tau = \epsilon$. So for our later purpose this reparametrization is not so effective. We will take:

$$\epsilon = 1$$

for the rest of this paper.

5.C Basic PDE-analysis, existence and uniqueness: Let us start from some analytic properties of the hyperbolic Mealy systems.

Lemma 5.5. *(1) Suppose ψ and ϕ both take bounded values from both below and above. Then f_t and g_t also satisfy the same properties for each $t > 1$.*

Let us fix $t > 1$, and $(u, v) : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^2$ be a solution to the hyperbolic Mealy system, which take positive initial values on $x, s \in [0, \infty)$:

$$u(x, 0), v(0, s) \in (0, \infty).$$

(2) Suppose that there are constants $r < R$ so that f_t and g_t both satisfy the uniform bounds:

$$r \leq f_t(a, b), \quad g_t(b, a) \leq R.$$

Then they take positive and bounded values on all the domain:

$$u, v : [0, \infty) \times [0, \infty) \rightarrow (0, \infty).$$

(3) If f_t and g_t both satisfy the bounds:

$$r \leq f_t(a, b), \quad g_t(b, a) \leq R$$

for any $r \leq b \leq R$ and all $a > 0$, and if we choose the initial values with their ranges as:

$$r \leq u(x, 0), v(0, s) \leq R$$

then their values are also contained in the range:

$$r \leq u(x, s), v(x, s) \leq R$$

for all $(x, s) \in [0, \infty) \times [0, \infty)$.

Proof: (1) follows from lemma 2.1 and proposition 2.4.

Let us fix $x \in [0, \infty)$ and $v(x, s)$. Then we regard $u(x, \cdot) : [0, \infty) \rightarrow \mathbb{R}$ satisfy the ODE on s variable:

$$\epsilon u_s(x, \cdot) = f_t(v(x, s), u(x, \cdot)) - u(x, \cdot).$$

For (2) and (3), because the range of f_t is away from 0, if $u(x, \cdot)$ take small values, then $u_s(x, \cdot)$ become positive and so their values must increase. Conversely if $u(x, \cdot)$ take large values, then $u_s(x, \cdot)$ become negative and their values must decrease.

For $v(\cdot, s)$ case, we can use the same argument. From these observations, the conclusions follow immediately. This completes the proof.

Let us say that the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$ (for a fixed $t > 1$), if there is some $0 < q < r$ so that they satisfy the bounds for all $a > 0$:

$$\begin{aligned} f_t(a, b) - b, \quad g_t(b, a) - b & \begin{cases} \geq q & b \leq r + q \\ \leq -q & b \geq R - q \end{cases} \\ |f_t(a, b) - b|, \quad |g_t(b, a) - b| & \leq R - r \quad r \leq b \leq R. \end{aligned}$$

If (f_t, g_t) satisfies the uniform bounds as $r \leq f_t, g_t \leq R$, then the pair restricts to a self-dynamics over $[r - 2q, R + 2q]$.

By lemma 5.5, if we take the initial values as $r + q \leq u(x, 0), v(0, s) \leq R - q$, then their values are also contained in the range:

$$r + q \leq u(x, s), v(x, s) \leq R - q$$

for all $(x, s) \in [0, \infty) \times [0, \infty)$.

5.C.2 Existence and uniqueness: Let us study the existence of solutions to the hyperbolic Mealy systems. We use the automatic version of the Picard iteration method of successive approximation.

Let (f_t, g_t) be a pair of relatively elementary functions of 2 variables, which correspond to the relative $(\max, +)$ -functions (ψ, ϕ) . Let us assume that the pair restricts to a self-dynamics over $[r, R]$ with $q < r$.

Let us introduce the following:

$$D = D_{r,R}(f_t, g_t) = \sup_{(v,u) \in [r,R]^2} \{|f_t(v, u) - u|, |g_t(v, u) - v|\}.$$

Let us put $\bar{f}_t(v, u) = f_t(v, u) - u$ and $\bar{g}_t(v, u) = g_t(v, u) - v$ so that the hyperbolic Mealy equations can be written as:

$$u_s = \bar{f}_t(v, u), \quad v_x = \bar{g}_t(v, u).$$

Now let us fix $t > 1$ and give the initial values:

$$u \mid [0, \infty) \times \{0\}, \quad v \mid \{0\} \times [0, \infty)$$

which satisfy the followings:

(1) Their ranges are uniformly bounded both from above and below by:

$$r + q \leq u(x, 0), v(0, s) \leq R - q.$$

(2) Both have uniformly bounded C^2 norms.

With u and v as the initial values, let us solve the equations of the hyperbolic Mealy systems by the following constructions (1),(2),(3), (4):

(1): Let us choose small $\tau > 0$ so that:

(a) the Lipschitz constants of both $\bar{f}_t$ and $\bar{g}_t$ are smaller than $(2\tau)^{-1}$,

(b) the estimates holds:

$$\tau \leq D^{-1}q.$$

(2): For $m = 0, 1, 2, \dots$, let us put the small domains:

$$D_m = [m\tau, (m+1)\tau] \times [0, \tau] \subset [0, \infty) \times [0, \infty)$$

and construct solutions inductively on $m = 0, 1, \dots$.

Suppose given solutions over D_{m-1} . With the original initial values, we have determined the values of solutions as:

$$u|_{D_{m-1} \cup [0, \infty) \times \{0\}}, \quad v|_{D_{m-1} \cup \{0\} \times [0, \infty)}.$$

(3): For $x_0 = m\tau$, we have the initial values as:

$$u|[x_0, x_0 + \tau] \times \{0\}, \quad v|\{x_0\} \times [0, \tau]$$

and let us extend the solutions over D_m .

For $(x, s) \in D_m$, let us put:

$$(u_0, v_0) = (u(x, 0), v(x_0, s))$$

and define the sequences inductively by:

$$(u_n, v_n) = (u_0, v_0) + \left(\int_0^s \bar{f}_t(v_{n-1}, u_{n-1}) dt, \int_{x_0}^x \bar{g}_t(v_{n-1}, u_{n-1}) dy \right).$$

Lemma 5.6. *Suppose that the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$. Then the sequences $\{(u_n, v_n)\}_n$ converge uniformly on D_m , and:*

$$(u, v) = \lim_{n \rightarrow \infty} (u_n, v_n)$$

give the solutions which coincide with the given initial values:

$$u|[x_0, x_0 + \tau] \times \{0\}, \quad v|\{x_0\} \times [0, \tau].$$

Moreover they satisfy the estimates:

$$|(u, v) - (u_n, v_n)| \leq \frac{1}{2^n} \max(|u_1 - u_0|, |v_1 - v_0|).$$

Proof: Step 1: We claim that the ranges of u_n and v_n are in the region $[r, R]$. We verify it only for u_n . v_n can be considered similarly.

By the assumption, the uniform bounds $r + q \leq u_0 \leq R - q$ hold. Firstly we have the estimates:

$$|u_1 - u_0| \leq \int_0^\tau |f_t(v_0, u_0) - u_0| \leq \tau D \leq D^{-1} q D = q.$$

So u_1 admits the bounds $r \leq u_1 \leq R$.

Let us consider:

$$u_n = u_0 + \int_0^s [f_t(v_{n-1}, u_{n-1}) - u_{n-1}] dt$$

and assume the uniform bounds $r \leq u_{n-1} \leq R$. Then we have the estimates:

$$|u_n - u_0| \leq \int_0^\tau |f_t(v_{n-1}, u_{n-1}) - u_{n-1}| \leq \tau D \leq D^{-1} q D = q.$$

So we have verified the uniform bounds $r \leq u_n \leq R$ for all n by the induction step. Since the estimates are independent of choice of $x \in [x_0, x_0 + \tau]$, this verifies the claim.

Step 2: By step 1, both u_n and v_n take their values in $[r, R]$. So the Lipschitz constants $L_{\bar{f}}$ of $\bar{f}_t$ and $L_{\bar{g}}$ are both uniformly bounded at (u_n, v_n) .

Let us put $V_n = (u_n, v_n)$. Then:

$$\begin{aligned} & |V_{n+1} - V_n| \\ &= |(\int_0^s (\bar{f}_t(v_n, u_n) - \bar{f}_t(v_{n-1}, u_{n-1})) dt, \int_{x_0}^x (\bar{g}_t(v_n, u_n) - \bar{g}_t(v_{n-1}, u_{n-1})) dy)| \\ &\leq \max(s L_{\bar{f}} |V_n - V_{n-1}|, (x - x_0) L_{\bar{g}} |V_n - V_{n-1}|) \\ &\leq \tau \max(L_{\bar{f}}, L_{\bar{g}}) |V_n - V_{n-1}| < \frac{1}{2} |V_n - V_{n-1}|. \end{aligned}$$

Thus they are contracting:

$$\max\{|u_n - u_{n-1}|, |v_n - v_{n-1}|\} \leq \frac{1}{2^n} \max\{|u_1 - u_0|, |v_1 - v_0|\}$$

and $(u, v) = \lim_{n \rightarrow \infty} (u_n, v_n)$ exist uniformly. One also obtains the estimates:

$$\max\{|u_n - u|, |v_n - v|\} \leq \frac{1}{2^n} \max\{|u_1 - u_0|, |v_1 - v_0|\}.$$

Then they satisfy the integral equations:

$$(u, v) = (u_0, v_0) + \left(\int_0^s \bar{f}_t(v, u) dt, \int_{x_0}^x \bar{g}_t(v, u) dy \right)$$

which are equivalent to the hyperbolic Mealy equations.

Let us check that these satisfy the boundary conditions. On $[x_0, x_0 + \tau] \times \{0\} \cup \{x_0\} \times [0, \tau]$,

$$\begin{aligned} (u, v)(x, 0) &= (u(x, 0), v(x_0, 0) + \int_{x_0}^x \bar{g}_t(v, u) dy) = (u(x, 0), v(x, 0)) \\ (u, v)(x_0, s) &= (u(x_0, 0) + \int_0^s \bar{f}_t(v, u) dy, v(x_0, s)) = (u(x_0, s), v(x_0, s)). \end{aligned}$$

So the pair (u, v) certainly satisfies the boundary condition.

This completes the proof.

(4): Let us continue the construction of solutions on $[0, \infty) \times [0, \infty)$. By lemma 5.4, let us extend solutions on D_m inductively, and one obtains solutions on $[0, \infty) \times [0, \tau]$.

Since the initial values $u|_{[0, \infty) \times \{0\}}$ and $v|_{\{0\} \times [0, \infty)}$ take their ranges between $[r - q, R + q]$, the range of both u, v on $[0, \infty) \times [0, \tau]$ also the same, since the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$.

Next let us regard that:

$$[0, \infty) \times \{\tau\} \cup \{0\} \times [\tau, 2\tau]$$

is the boundary equipped with the boundary condition. Let us iterate the same construction over $[0, \infty) \times [\tau, 2\tau]$. By the above observation, the range of the initial conditions are also contained in $[r + q, R - q]$. So one can repeat the above process by the same way.

By repeating this process, one finally obtains the solutions with the given boundary condition:

$$u, v : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}.$$

This completes the construction of solutions to the hyperbolic Mealy systems.

Let us put the initial domain:

$$I_0 = [0, \infty) \times \{0\} \cup \{0\} \times [0, \infty).$$

Theorem 5.7. *Suppose that the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$, and give the positive initial values:*

$$u, v : I_0 \rightarrow [r + q, R - q].$$

Then:

(1) *there exists a positive solution*

$$u, v : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$$

with the uniform bounds:

$$r + q \leq u(x, s), \quad v(x, s) \leq R - q$$

hold.

(2) *The solution is unique.*

Proof: The first statements follow from the above construction with lemma 5.2.

Let us verify uniqueness. Suppose two solutions (u, v) and (u', v') exist with the same initial values. Let us put:

$$a = |u - u'|, \quad b = |v - v'| : [0, \infty)^2 \rightarrow [0, \infty)$$

and verify $a = b \equiv 0$. Notice that $a|_{[0, \infty) \times \{0\}}$ and $b|_{\{0\} \times [0, \infty)}$ both vanish. Firstly, at $x = 0$, the ODE:

$$u_s(0, s) = f_t(v(0, s), u(0, s)) - u(0, s)$$

has the unique solution on $s \in [0, \infty)$, where $v(0, s)$ is the given initial value. So $u(x, 0) = u'(x, 0)$ hold for all $x \in [0, \infty)$.

Similarly $v(0, s) = v'(0, s)$ holds. In particular:

$$u(x, s) = u'(x, s), \quad v(x, s) = v'(x, s)$$

hold for all $(x, s) \in [0, \infty) \times \{0\} \cup \{0\} \times [0, \infty)$.

The solutions satisfy the integral equations:

$$u(x, s) = u(x, 0) + \int_0^s \bar{f}_t(v, u) ds, \quad v(x, s) = v(0, s) + \int_0^x \bar{g}_t(v, u) dx.$$

Since both $\bar{f}_t$ and $\bar{g}_t$ are Lipschitz, there exists L so that the estimates hold:

$$a \leq L \int_0^s (a+b), \quad b \leq L \int_0^x (a+b).$$

Let us denote the broken lines:

$$\gamma_{x,s} = \{(\alpha, s) \cup (x, \beta) : 0 \leq \alpha \leq x, 0 \leq \beta \leq s\}.$$

Then combination of the above inequalities gives the following inequalities:

$$(a+b)(x, s) \leq L \int_{\gamma_{x,s}} (a+b).$$

Let us choose (x, s) so that the estimate $L(x+s) < \delta < 1$ holds.

Now let $D \subset [0, \infty)^2$ be the rectangle whose boundary is given by:

$$\partial D = \gamma_{x,s} \cup [0, x] \times \{0\} \cup \{0\} \times [0, s].$$

Let us choose some point $(x_0, s_0) \in D$ so that the equality $(a+b)(x_0, s_0) = \sup_{(p,q) \in D} (a+b)(p, q)$ holds. Then for $(x, s) \in D$, the estimates hold:

$$(a+b)(x, s) \leq (a+b)(x_0, s_0) \leq L \int_{\gamma_{x_0, s_0}} (a+b) \leq \delta (a+b)(x_0, s_0).$$

Since $\delta < 1$, this implies $(a+b)(x_0, s_0) = 0$ and hence $(a+b)(x, s) \equiv 0$ on D .

By changing the domains by parallel transport as in the above construction, we can follow the same argument as above. By iterating the same process, we conclude that $a = b \equiv 0$ hold on $[0, \infty)^2$.

This completes the proof.

5.C.3 Energy estimates: Let us study C^1 estimates of solutions. It is well known as the energy estimates for hyperbolic equations. Here we also give concrete estimates of the constants which appear in the asymptotic growth of C^1 derivatives of solutions.

For functions $u : [0, \infty)^2 \rightarrow [0, \infty)$, let us denote the norms of the first derivative by:

$$\|u\|_{\bar{C}^1} = \sup_{(x,s) \in [0, \infty)^2} \max\left\{\left|\frac{\partial u}{\partial x}\right|(x, s), \left|\frac{\partial u}{\partial s}\right|(x, s)\right\}.$$

The following is elementary:

Lemma 5.8. *Let us give any discrete initial values:*

$$u : \mathbb{N} \times \{0\} \rightarrow \mathbb{R}, \quad v : \{0\} \times \mathbb{N} \rightarrow \mathbb{R}$$

such that the estimates holds for all $n = 0, 1, 2, \dots$:

$$|u(n+1) - u(n)|, |v(n+1) - v(n)| \leq \mu.$$

Then there is a constant C independent of choice of the initial values and μ so that there are extensions of the initial values as:

$$u : [0, \infty) \times \{0\} \rightarrow \mathbb{R}, \quad v : \{0\} \times [0, \infty) \rightarrow \mathbb{R}$$

equipped with uniform $\bar{C}^1$ bounds:

$$\|u\|_{\bar{C}^1([0, \infty) \times \{0\})}, \|v\|_{\bar{C}^1(\{0\} \times [0, \infty))} \leq C\mu.$$

Proof: Let us extend the initial values by connecting the discrete values by segments. Then its C^1 approximations give the desired property.

This completes the proof.

Recall the number $D = D_{r,R}(f_t, g_t)$ in 5.C.2 and introduce another one:

$$B = \max(\|(f_t)_u - 1\|_{C^0}, \|(f_t)_v\|_{C^0}, \|(g_t)_v - 1\|_{C^0}, \|(g_t)_u\|_{C^0}).$$

The next gives the exponential energy estimates:

Proposition 5.9. *Suppose that the pair (f_t, g_t) restricts to a self-dynamics over $[r, R]$, and give the initial values:*

$$u(\cdot, 0), v(0, \cdot) : [0, \infty) \rightarrow [r+q, R-q]$$

with uniformly bounded C^1 norms:

$$\|u_x\|_{C^0([0, \infty) \times \{0\})}, \|v_s\|_{C^0(\{0\} \times [0, \infty))} \leq A < \infty.$$

Then there is a constant C so that solutions $u, v : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ have the asymptotic C^1 bounds:

$$\begin{aligned} \left\| \frac{\partial u}{\partial x} \right\|_{C^0([0, \infty) \times \{m\})}, \left\| \frac{\partial v}{\partial s} \right\|_{C^1(\{m\} \times [0, \infty))} &\leq 2^{\tau^{-1}m} (A + 2D), \\ \left\| \frac{\partial u}{\partial s} \right\|_{C^0}, \left\| \frac{\partial v}{\partial x} \right\|_{C^0} &\leq D \end{aligned}$$

where:

$$\tau \text{Lip}_{\bar{f}_t, \bar{g}_t} \leq \frac{1}{2}, \quad \tau \leq D^{-1}q, \quad \delta \equiv \tau B \leq \frac{1}{4}.$$

Proof: Let us use the notations in 5.C.2. Firstly we split the domains into periodic stripes. Then we verify some uniform estimates on the first derivatives of the approximated solutions (u_n, v_n) over each stripe, which are independent of n . Secondly we verify that they converge to the solutions uniformly in C^1 over each stripe.

Step 1: By theorem 5.5, the range of solutions (u, v) are uniformly bounded between r to R . Then by the defining equations, both uniform bounds hold:

$$\|u_s\|C^0, \|v_x\|C^0 \leq D.$$

Let us estimate u_x as follows. By theorem 5.7, the solutions are unique with respect to the given initial values.

Let us recall the inductive construction of solutions over $[0, \infty) \times [0, \tau]$ in 5.C.2. We split the domain into periodic squares $D_m = [m\tau, (m+1)\tau] \times [0, \tau]$, and construct solutions successively on each D_m as $\lim_{n \rightarrow \infty} (u_n, v_n)$, where:

$$\begin{aligned} u_n &= u_0 + \int_0^s (f_t(v_{n-1}, u_{n-1}) - u_{n-1}) dt, \\ v_n &= v_0 + \int_{x_0}^x (g_t(v_{n-1}, u_{n-1}) - v_{n-1}) dy \end{aligned}$$

for $0 \leq s \leq \tau$ and $x_0 = m\tau \leq x \leq (m+1)\tau$.

Let us denote $u'_n = \frac{\partial u_n}{\partial x}$. Then:

$$\begin{aligned} u'_n &= u'_0 + \int_0^s ([(f_t)_u(v_{n-1}, u_{n-1}) - 1]u'_{n-1} + (f_t)_v(v_{n-1}, u_{n-1})v'_{n-1}) dt, \\ v'_n &= g_t(v_{n-1}, u_{n-1}) - v_{n-1}. \end{aligned}$$

By lemma 5.6, the estimates:

$$|v - v_n|, |u - u_n| \leq \frac{1}{2^n} \max(|u_1 - u_0|, |v_1 - v_0|) \leq \frac{q}{2^n}$$

hold. Since the equality $v_x = g_t(v, u) - v$ holds, we obtain the estimates:

$$\begin{aligned} |v'_n| &\leq |v'| + |v'_n - v'| \leq D + |(g_t(v, u) - v) - (g_t(v_{n-1}, u_{n-1}) - v_{n-1})| \\ &\leq D + \frac{q}{2^{n-1}} Lip_{\bar{g}_t} \leq D + \frac{q}{2^n \tau}. \end{aligned}$$

Let us put:

$$\tau B = \delta \leq \frac{1}{4}.$$

Then we have the estimates:

$$\begin{aligned}
|u'_n| &\leq |u'_0| + \tau B[|u'_{n-1}| + D + \frac{q}{2^n \tau}] \\
&= |u'_0| + (\delta D + B \frac{q}{2^n}) + \delta |u'_{n-1}| \leq \dots \\
&\leq (1 + \delta + \delta^2 + \dots)(|u'_0| + D) + (\frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots) Bq \\
&\leq 2(|u'_0| + D + \frac{Bq}{2^n}) \leq 2(A + D + \frac{Bq}{2^n}).
\end{aligned}$$

Similarly we have the estimates $|\frac{\partial v_n}{\partial s}| \leq 2(A + D + \frac{Bq}{2^n})$ on $[0, \tau] \times [0, \infty)$. By lemma 5.6, we have the estimates:

$$|\frac{\partial u_n}{\partial s}|, \quad |\frac{\partial v_n}{\partial x}| \leq D.$$

Notice that so far we have required the conditions on $\tau > 0$ as:

$$\tau \text{ Lip}_{\bar{f}_t, \bar{g}_t} \leq \frac{1}{2}, \quad \tau \leq D^{-1}q, \quad \delta = \tau B \leq \frac{1}{4}.$$

Step 2: Let us verify C^1 uniform convergence of $\{u_n, v_n\}_n$. For simplicity of the argument, we assume that the Lipschitz constants of $(\bar{f}_t)_u, (\bar{g}_t)_v$ and $(f_t)_v, (g_t)_u$ are all finite and bounded by B' .

Let us consider the estimates:

$$|v'_n| \leq D, \quad |u'_n| \leq 2A + 2D + \frac{2Bq}{2^n} \equiv \alpha_n.$$

on the domains $[0, \infty) \times [0, \tau]$. Then we have the estimates:

$$\begin{aligned}
&|(\bar{f}_t)_u(v_{n-1}, u_{n-1})u'_{n-1} - (\bar{f}_t)_u(v_{n-2}, u_{n-2})u'_{n-2}| \\
&\leq |(\bar{f}_t)_u(v_{n-1}, u_{n-1})u'_{n-1} - (\bar{f}_t)_u(v_{n-1}, u_{n-1})u'_{n-2}| \\
&\quad + |(\bar{f}_t)_u(v_{n-1}, u_{n-1})u'_{n-2} - (\bar{f}_t)_u(v_{n-2}, u_{n-2})u'_{n-2}| \\
&\leq B|u'_{n-1} - u'_{n-2}| + B'(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|)|u'_{n-2}| \\
&\leq B|u'_{n-1} - u'_{n-2}| + B'\alpha_{n-2}(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|),
\end{aligned}$$

$$\begin{aligned}
& |(f_t)_v(v_{n-1}, u_{n-1})v'_{n-1} - (f_t)_v(v_{n-2}, u_{n-2})v'_{n-2}| \\
& \leq B|v'_{n-1} - v'_{n-2}| + B'(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|)|v'_{n-2}| \\
& \leq B(g_t(v_{n-2}, u_{n-2}) - g_t(v_{n-3}, u_{n-3})| + |v_{n-2} - v_{n-3}|) \\
& \quad + B'D(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|) \\
& \leq B\tau^{-1}(|u_{n-2} - u_{n-3}| + |v_{n-2} - v_{n-3}|) \\
& \quad + B'D(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|).
\end{aligned}$$

By lemma 5.6, the estimates hold:

$$|u_n - u_{n-1}|, |v_n - v_{n-1}| \leq \frac{q}{2^n}.$$

Thus we have the estimates:

$$\begin{aligned}
|u'_n - u'_{n-1}| & \leq \tau[B|u'_{n-1} - u'_{n-2}| + \\
& B'(\alpha_{n-2} + D)(|u_{n-1} - u_{n-2}| + |v_{n-1} - v_{n-2}|)] \\
& + B(|u_{n-2} - u_{n-3}| + |v_{n-2} - v_{n-3}|) \\
& \leq \delta|u'_{n-1} - u'_{n-2}| + \frac{C}{2^n}
\end{aligned}$$

for some constant C . Then we have the estimates:

$$\begin{aligned}
|u'_n - u'_{n-1}| & \leq \delta|u'_{n-1} - u'_{n-2}| + \frac{C}{2^n} \leq \frac{1}{4}|u'_{n-1} - u'_{n-2}| + \frac{C}{2^{n+1}} \\
& \leq \frac{1}{2^4}|u'_{n-2} - u'_{n-3}| + \frac{C}{2^{n+2}} \cdots \leq \frac{1}{2^{n-1}}|u'_1 - u'_0| + \frac{C}{2^{2n-1}}.
\end{aligned}$$

Similarly we have the estimates:

$$|\frac{\partial v_n}{\partial s} - \frac{\partial v_{n-1}}{\partial s}| \leq \frac{1}{2^{n-1}}|v'_1 - v'_0| + \frac{C}{2^{2n-1}}$$

on $[0, \tau] \times [0, \infty)$.

So the convergence is uniform.

Step 3: By step 1 and 2, we have the uniform estimates:

$$|\frac{\partial u}{\partial x}| \leq 2A + 2D$$

on $[0, \infty) \times [0, \tau]$, and:

$$|\frac{\partial v}{\partial s}| \leq 2A + 2D$$

on $[0, \tau] \times [0, \infty)$ by letting $n \rightarrow \infty$.

Let us repeat the same process of extensions of the solutions, on $[0, \infty) \times [\tau, 2\tau]$ for u and on $[\tau, 2\tau] \times [0, \infty)$ for v .

Notice that the initial norms have to replace from A by $2A + 2D$. Then successively we have the estimates for $N = 0, 1, 2, \dots$:

$$\begin{cases} |\frac{\partial u}{\partial x}| \leq 2^N A + (2^N - 1)2D & \text{on } [0, \infty) \times [(N-1)\tau, N\tau] \\ |\frac{\partial v}{\partial s}| \leq 2^N A + (2^N - 1)2D & \text{on } [(N-1)\tau, N\tau] \times [0, \infty) \end{cases}$$

So in total, we have the following estimates:

$$\begin{aligned} |\frac{\partial u}{\partial x}|(x, s) &\leq 2^{\tau^{-1}s}(A + 2D) \\ |\frac{\partial v}{\partial s}|(x, s) &\leq 2^{\tau^{-1}x}(A + 2D), \\ |\frac{\partial u}{\partial s}|, |\frac{\partial v}{\partial x}| &\leq D. \end{aligned}$$

This completes the proof.

Example 5.1: Let us consider the hyperbolic systems of the form:

$$u_s = \frac{au}{1+u} - u, \quad v_x = g_t(u, v) - v.$$

Suppose the initial conditions satisfy $u(1, 0) = a - 1$. Then along the half line $\{(1, s) : s \geq 0\}$, the ODE $u_s = \frac{au}{1+u} - u$ has the unique solution $u(1, s) \equiv a - 1$. Then by differentiating the first equality by x variable, one obtains the equation $u_{xs} = (a^{-1} - 1)u_x$, whose solutions are given by:

$$u_x(1, s) = \exp((a^{-1} - 1)s)u_x(1, 0).$$

They are uniformly bounded if $a > 1$ hold.

If we choose $a < 1$, u_x grow exponentially, even though u take negative values.

For our purpose of the estimates on the second derivatives, this may not be so useful.

Let us induce the uniform energy estimates by assuming negative coefficients on derivatives as below.

Firstly we have the general estimates:

Lemma 5.10. *Let $w(s)$ satisfy the estimates:*

$$-aw(s) + b \leq w'(s) \leq -cw(s) + d$$

for some positive $a, b, c, d > 0$. Then the estimates hold:

$$\frac{b}{a} + (w(0) - \frac{b}{a}) \exp(-as) \leq w(s) \leq \frac{d}{c} + (w(0) - \frac{d}{c}) \exp(-as).$$

Proof: Let us rewrite the inequalities as:

$$-a(w(s) - \frac{b}{a}) \leq (w(s) - \frac{b}{a})'.$$

Then we obtain the left hand side estimates of the conclusions immediately.

The right hand side can be treated similarly. This completes the proof.

Let us consider the hyperbolic system of PDE:

$$u_s = f_t(v, u) - u, \quad v_x = g_t(v, u) - v.$$

Proposition 5.11. *Assume negativities:*

$$-a \leq (f_t)_u - 1, \quad (g_t)_v - 1 \leq -c$$

for some $0 < a, b$. Then the uniform estimates hold:

$$\begin{aligned} \frac{b}{a} + (u_x(x, 0) - \frac{b}{a}) \exp(-as) &\leq u_x(x, s) \leq \frac{b}{c} + (u_x(x, 0) - \frac{b}{c}) \exp(-as), \\ \frac{d}{a} + (v_s(0, s) - \frac{d}{a}) \exp(-ax) &\leq v_s(x, s) \leq \frac{d}{c} + (v_s(0, s) - \frac{d}{c}) \exp(-as) \end{aligned}$$

where:

$$b = \sup |(f_t)_v(g_t - v)|, \quad d = \sup |(g_t)_u(f_t - u)|.$$

In particular $|u_x|$ and $|v_s|$ are both uniformly bounded.

Proof: By differentiations, let us consider the equations:

$$\begin{aligned} u_{xs} &= ((f_t)_u - 1)u_x + (f_t)_v(g_t - v), \\ v_{xs} &= ((g_t)_v - 1)v_s + (g_t)_u(f_t - u). \end{aligned}$$

Then the conclusions follow by applying lemma 5.9.

This completes the proof.

5.D Refinement and the higher distortions: Let us consider a Mealy automaton:

$$\mathbf{A} : \quad \psi : Q \times S \rightarrow S, \quad \phi : Q \times S \rightarrow Q$$

with two alphabets $S = \{s_0, s_1\} = \{L, L+1\}$. Our aim in 5.D is to verify the following:

Proposition 5.12. *Let $\mathbf{A}$ be a Mealy automaton with 2 alphabets.*

There is an refinement of $\mathbf{A}$ with the pair of functions $(\bar{\phi}, \bar{\psi})$ so that the corresponding relatively elementary functions $(\tilde{f}_t, \tilde{g}_t)$ satisfy the estimates:

$$\begin{aligned} [|(\tilde{f}_t(v, u) - u)((\tilde{f}_t)_u(v, u) - 1)| + |(\tilde{f}_t)_v(u, v)||v_s|](x, s + \alpha) &< 2u(x, s + 1), \\ [|(\tilde{g}_t(v, u) - v)((\tilde{g}_t)_v(v, u) - 1)| + |(\tilde{g}_t)_u(u, v)||u_x|](x + \alpha, s) &< 2v(x + 1, s). \end{aligned}$$

for any solutions (u, v) and all $0 \leq \alpha \leq 1$.

Proof requires constructions of $(\max, +)$ -functions by several steps and occupies 5.D. We also need some general estimates of rational functions with positive coefficients.

5.D.2 Prototype : Let us describe a prototype of rational functions, which appear by refinement.

Let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be another relatively $(\max, +)$ -function, given by:

$$\begin{aligned} \xi(x) &= \max(\min(x + \delta, L), x - \delta) \\ &= \max(-\max(-(x + \delta), -L), x - \delta). \end{aligned}$$

$\xi(x)$ is increasing for $x < L$ and decreasing for $x > L$. So for any $q > \delta > 0$, ξ gives the functions as:

$$\xi : [L - q, L + q] \rightarrow [L - q, L + q].$$

Let f_t be the corresponding relatively elementary functions to ξ :

$$f_t(z) = t^{-\delta}z + \frac{1}{N_0} \frac{t^L}{t^{L-\delta} + z}z$$

where $N_0 = N_0(t^\delta) \geq 1$ are chosen so that the estimates hold:

$$\mu \equiv 1 - t^{-\delta} - \frac{t^\delta}{N_0} > 0.$$

Let us consider the equalities:

$$(f_t(z) - z)' = t^{-\delta} - 1 + \frac{1}{N_0} \frac{t^{2L-\delta}}{(t^{L-\delta} + z)^2},$$

$$f_t(z) - z = (t^{-\delta} - 1 + \frac{1}{N_0} \frac{t^L}{t^{L-\delta} + z}) z.$$

Then the estimates hold:

$$\frac{t^{2L-\delta}}{(t^{L-\delta} + z)^2}, \quad \frac{t^L}{t^{L-\delta} + z} \leq t^\delta.$$

Thus we have the estimates:

$$-1 < t^{-\delta} - 1 < (f_t(z) - z)' \leq t^{-\delta} - 1 + \frac{t^\delta}{N_0} = -\mu < 0,$$

$$-z < (t^{-\delta} - 1)z < f_t(z) - z \leq (t^{-\delta} - 1 + \frac{t^\delta}{N_0}) z = -\mu z.$$

So in total we have the estimates:

$$|(f_t(z) - z)'| |f_t(z) - z| \leq (1 - t^{-\delta}) z.$$

This will be one of the required estimates for our asymptotic estimates of solutions to PDE systems.

5.D.3 Construction of exit functions: Let us construct admissible pairs concretely. Such functions arise from *almost diagonal functions* as below. Let:

$$\xi(x) = \max(\min(x + \delta, L), x - \delta)$$

be the relative $(\max, +)$ -functions in 5.D. Given the initial value $x_0 = 0$, let us iterate it as $x_{n+1} = \xi(x_n)$. It is easy to see:

$$x_n \equiv L \quad \text{if } x_0 = L, L + 1 \quad \text{and } n \geq \delta^{-1}.$$

Let us have another relative $(\max, +)$ -functions with ‘two-step stairs’ by:

$$\xi_2(x) = \max(\min(\xi(x), L + 1), x - 3\delta).$$

This satisfies the properties:

$$\xi_2(x) \begin{cases} > x & x < L \\ = L & L - \delta \leq x \leq L + \delta \\ < x & x > L \end{cases}$$

The iterations satisfy the same properties, $x_n \equiv L$ for all $n \geq \delta^{-1}$ for $x_0 = L, L + 1$. On the other hand let us translate ξ_2 . Then its properties change as:

$$\xi_2(x + 2\delta) \begin{cases} > x & x < L + 1 \\ = L + 1 & L + 1 - \delta \leq x \leq L + 1 + \delta \\ < x & x > L + 1 \end{cases}$$

In particular the orbits behave differently. The iterations satisfy:

$$x_n \equiv L + 1 \quad \text{if } x_0 = L, L + 1 \text{ and } n \geq \delta^{-1}.$$

Inductively let us have relative $(\max, +)$ -functions with n -step stairs by:

$$\xi_n(x) = \max(\min(\xi_{n-1}(x), L + n - 1), x - (2n - 1)\delta).$$

To represent exit functions, one can use ξ_l when we have l alphabets. In fact if we want an action which exchanges $L + a$ and $L + a + 1$ with $a < l$, then the translations:

$$\begin{aligned} \xi_l(-2a\delta) : L + a, L + a + 1 &\rightarrow L + a, \\ \xi_l(-2(a + 1)\delta) : L + a, L + a + 1 &\rightarrow L + a + 1 \end{aligned}$$

can play such roles. Notice that all ξ_n are 1-Lipschitz functions.

Let us fix n , and choose a large number $N_0 = N_0(n)$ which will be determined later.

Let f_t be the rational functions corresponding to ξ above:

$$f_t(z) = t^{-\delta}z + \frac{1}{N_0} \frac{t^L z}{z + t^{L-\delta}}.$$

For $0 \leq m \leq n$ let us put rational functions inductively by:

$$f_t^m(z) = t^{-(2m-1)\delta}z + \frac{1}{N_0} \frac{t^{L+m-1} f_t^{m-1}(z)}{t^{L+m-1} + f_t^{m-1}(z)}$$

which correspond to ξ_m defined above.

Lemma 5.13. *let us choose $N_0 \geq 1$ so that the estimates:*

$$t^{-\delta} + \frac{t^\delta}{N_0} \equiv 1 - \mu < 1$$

holds. Then the following estimates hold:

$$\begin{aligned} -1 + t^{-(2n-1)\delta} &< (f_t^n(z) - z)' \leq t^{-\delta} - 1 + \frac{t^\delta}{N_0} = -\mu < 0, \\ (-1 + t^{-(2n-1)\delta})z &< f_t^n(z) - z \leq (t^{-\delta} - 1 + \frac{t^\delta}{N_0})z = -\mu z. \end{aligned}$$

In particular the estimates:

$$|(f_t(z) - z)'||f_t(z) - z| \leq (-1 + t^{-(2n-1)\delta})^2 z.$$

Proof: We have already verified the conclusions for $m = 1$.

Suppose f_t^{m-1} satisfy the conclusions for $m \leq n$.

Let us consider:

$$\begin{aligned} |(f_t^m)'(z)| &= |t^{-(2m-1)\delta} + (\frac{t^{L+m-1}}{t^{L+m-1} + f_t^{m-1}(z)})^2 \frac{(f_t^{m-1})'(z)}{N_0}| \\ &\leq t^{-(2m-1)\delta} + \frac{|(f_t^{m-1})'(z)|}{N_0} \leq t^{-(2m-1)\delta} + \frac{1}{N_0} \leq t^{-\delta} + \frac{t^\delta}{N_0} < 1 - \mu. \end{aligned}$$

So we have the estimates $-1 < (f_t^n(z) - z)' \leq -\mu < 0$.

Next we have the estimates:

$$\begin{aligned} f_t^m(z) &= t^{-(2m-1)\delta}z + \frac{1}{N_0} \frac{t^{L+m-1}}{t^{L+m-1} + f_t^{m-1}(z)} f_t^{m-1}(z) \\ &\leq t^{-(2m-1)\delta}z + \frac{1}{N_0} f_t^{m-1}(z) \leq (t^{-(2m-1)\delta} + \frac{1-\mu}{N_0})z \\ &\leq (t^{-\delta} + \frac{t^\delta}{N_0})z < (1 - \mu)z. \end{aligned}$$

Thus we have verified the conclusions for m . This completes the proof.

Corollary 5.14. *By choosing $N_0 = N_0(t^\delta)$ as above, the solutions to the equation:*

$$u_s = f_t^n(u) - u$$

satisfy the following estimates:

$$|(f_t^n(u) - u)((f_t^n)_u(u) - 1)|(s + \alpha) < \frac{\tau^2}{1 - \tau^2} u(s + 1)$$

for $\tau = 1 - t^{-(2n-1)\delta}$.

Proof: Let us put $\tau = 1 - t^{-(2n-1)\delta}$. By lemma 5.12, the estimates $|f_t^n(z) - z| < \tau^2 z$ hold.

Let us choose $0 \leq \alpha_0 \leq 1$ so that $\sup_{0 \leq s \leq 1} u(s) = u(\alpha_0)$ holds. Then we have the inequalities:

$$u(s + \alpha_0) \leq u(s + 1) + \int_{s+\alpha_0}^{s+1} |u_s| < u(s + 1) + \tau^2 \int_s^{s+1} u(a) da.$$

By the mean value theorem, we have the estimates:

$$\int_s^{s+1} u(a) da = u(s + \beta) \leq u(s + \alpha_0).$$

Thus combining with these estimates, we obtain the following:

$$(1 - \tau^2)u(s + \alpha_0) \leq u(s + 1).$$

Now finally we have the desired estimates:

$$\begin{aligned} |(f_t^n(u) - u)((f_t^n)_u(u) - 1)|(s + \alpha) &< \tau^2 u(s + \alpha) \\ &\leq \tau^2 u(s + \alpha_0) \leq \frac{\tau^2}{1 - \tau^2} u(s + 1). \end{aligned}$$

This completes the proof.

5.D.4 Transition functions with 2 alphabets: Now let us consider a Mealy automaton:

$$\mathbf{A} : \quad \psi : Q \times S \rightarrow S, \quad \phi : Q \times S \rightarrow Q.$$

We will construct the transition functions by relatively $(\max, +)$ -functions with two alphabets $S = \{s_0, s_1\}$. The reason for this restriction is just for simplicity of the notations. The general case can be constructed similarly.

Let us embed as $s_0 = L$ and $s_1 = L + 1$ for some large $L \gg 1$. Let $Q = \{q^0, \dots, q^l\}$ be the set of the states. We change the set, prepare twice of the number of them as $\{q^0, \bar{q}^0, \dots, q^l, \bar{q}^l\}$, and embed them as:

$$\bar{Q} \equiv \{q^0, \bar{q}^0, \dots, q^l, \bar{q}^l\} \subset \delta \mathbb{Z}$$

so that:

$$q^j = \bar{q}^j + 4\delta, \quad q^{j+1} = q^j + 8\delta$$

hold for $j = 0, \dots, l$.

Lemma 5.15. *There exists a refinement by relatively $(\max, +)$ -functions $\bar{\psi}, \bar{\phi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.*

Proof: **Construction of $\bar{\psi}$:** Let us put the function:

$$\begin{aligned}\mu(y) &= \min[2\delta, \max(\tau(y - q^0), \dots, \tau(y - q^l))], \\ \tau(y) &= \min[\max(0, y - \delta), \max(0, -y + 7\delta)].\end{aligned}$$

For $q \in Q$, these functions satisfy the following properties:

$$\mu(y) = \begin{cases} 0 & |y - q^j| \leq \delta, \\ 2\delta & |\bar{q}^j - y| \leq \delta, \\ 0 & y \geq q^l + 7\delta \text{ or } y \leq q^0 + \delta \end{cases}$$

Now we put the desired functions by:

$$\bar{\psi}(y, x) = \xi_2(x + \mu(y)).$$

Construction of $\bar{\phi}$: Denote $\bar{*}^j = \bar{q}^j$ for $* = q^j$ or $\bar{q}^j$, and let ϵ_0 be the permutation between L and $L + 1$. Let:

$$\bar{\phi} : \mathbb{R} \times \mathbb{R} \rightarrow [q^0, q^l + 7\delta]$$

be the relatively $(\max, +)$ -function which satisfy the following properties:

- (1) $\bar{\phi}(*, \delta l) = *$ if $\delta l \neq \mathbb{Z}$ for $l \in \mathbb{Z}$ and $* \in \bar{Q}$.
- (2) $|\bar{\phi}(y, x) - y| \leq k\delta$ for all $x \in \mathbb{R}$ and $y \in [q^0, q^l + 7\delta]$ for some k .
- (3) Suppose $\phi(q', s) = q$ and let ϵ_0 be the permutation between $s_0 = L$ and $s_1 = L + 1$. Then for $*' = q'$ or $\bar{q}'$:

$$\bar{\phi}(*', L) = \begin{cases} q & \text{if } \psi(q, \quad) = \epsilon_0, \\ \bar{q} & \text{if } \psi(q, \quad) = id \end{cases}$$

$$\bar{\phi}(*', L + 1) = \begin{cases} \bar{q} & \text{if } \psi(q, \quad) = \epsilon_0, \\ q & \text{if } \psi(q, \quad) = id \end{cases}$$

If we choose δ^{-1} equal to some integer l_0 , then these constructions give the refinement. This completes the proof.

Example: Let us consider the one state case $Q = \{q\}$. Then we have representations of $\bar{\phi}$ by:

$$\bar{\phi}(y, x) = \begin{cases} \min[\bar{q}, \max(y, y + x - (L + 1 - \delta))] & \text{if } \psi(q, \quad) = \epsilon_0, \\ \min[\bar{q}, \max(y, y - x + L + \delta)] & \text{if } \psi(q, \quad) = \text{id} \end{cases}$$

5.D.5 Estimates of relatively elementary functions: Let us give C^0 and C^1 estimates for relatively elementary functions in terms of the corresponding $(\max, +)$ -functions. Let $f_t(z_0, \dots, z_{n-1})$ correspond to $\varphi(x_0, \dots, x_{n-1})$, and fix $\bar{y}^0 \equiv (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$. Then we consider:

$$\varphi \equiv \varphi(\quad, y_1, \dots, y_{n-1}) : \mathbb{R} \rightarrow \mathbb{R}.$$

Assume that φ has the form:

$$\varphi(x) = L_0 + \alpha(x - x_0)$$

as a function on some neighbourhood $I_0 = (x_0 - \tau, x_0 + \tau) \in \mathbb{R}$.

For $\bar{w}^0 = t^{\bar{y}^0} \in \mathbb{R}_{>0}^{n-1}$ and $z_0 = t^{x_0}$, let us put :

$$f_t \equiv f_t(\quad, \bar{w}^0) : (0, \infty) \rightarrow (0, \infty).$$

Lemma 5.16. *Let M be the number of the components of f_t .*

(1) *For any $t^{x_0-\tau} \leq z \leq t^{x_0+\tau}$, the estimates hold:*

$$\left(\frac{f_t(z)}{t^{L_0} z_0^{-\alpha} z^\alpha} \right)^{\pm 1} \leq M.$$

(2) *Suppose φ takes bounded values from both sides:*

$$L \leq \varphi \leq L + a.$$

Then there exists some constant C so that the estimates:

$$\left| \frac{\partial f_t}{\partial z} \right|(z) \leq C t^a \frac{t^L}{z}$$

hold for all $t > 1$, where $C = C(\deg h_t, \deg k_t, M)$ with $f_t = t^L \frac{h_t}{k_t}$.

(3) *Suppose $a > 0$. Then there exists an integer N so that the estimates:*

$$t^{L-a} \leq N f_t(z) \leq t^{L+3a}$$

hold for all $t^{L-a} \leq z \leq t^{L+3a}$ and all sufficiently large $t \gg 1$.

Proof: (1) Let us put $x = \log_t z$ and $\varphi_t(x) = \log_t f_t(z)$. By lemma 2.1, the estimates hold for all $x \in I_0$:

$$\begin{aligned} |\varphi_t(x) - \varphi(x)| &= |\log_t f_t(z) - \log_t(t^{L_0 + \alpha(x - x_0)})| \\ &= \log_t\left(\frac{f_t(z)}{t^{L_0} z_0^{-\alpha} z^\alpha}\right)^{\pm 1} \leq \log_t M \end{aligned}$$

By removing $\log_t$ from both sides, one obtains the desired estimates.

(2) Notice that f_t is a parametrized rational functions. For each $t > 1$, f_t takes bounded values from both sides, since:

$$M^{-1}t^{|\varphi(x)|} \leq t^{|\varphi(x)|}t^{-|\varphi_t(x) - \varphi(x)|} \leq |f_t(z)| \leq t^{|\varphi(x)|}t^{|\varphi_t(x) - \varphi(x)|} \leq Mt^{|\varphi(x)|}.$$

In particular the degree of f_t must be equal to 0 with respect to z . So the derivative of f_t with respect to z has negative degree.

Let us denote $f_t = t^L \frac{h_t}{k_t}$ by polynomials. Then by the above estimates, we obtain the uniform bounds:

$$M^{-1} \leq \frac{h_t}{k_t} \leq Mt^a$$

where the coefficients of both h_t and k_t are rational in t . Notice that all the coefficients take positive values, which is a characteristic in tropical geometry.

Suppose h_t has degree $n_0 \geq 1$ in z . Then its derivative satisfies the estimates:

$$0 \leq z \frac{\partial h_t}{\partial z}(z) \leq c(n_0)h_t$$

for some constant $c(n_0)$ which is independent of $t > 1$.

Now suppose $\deg h_t = \deg k_t = n_0$, and denote $h'_t = \frac{\partial h_t}{\partial z}$. Then the estimates hold:

$$\left| \frac{\partial f_t}{\partial z}(z) \right| = t^L \left| \left\{ \frac{h'_t}{k_t} - \frac{h_t k'_t}{k_t^2} \right\} \right| \leq 2t^L Mt^a \frac{c(n_0)}{z} = 2c(n_0)Mt^a \frac{t^L}{z}.$$

(3) Firstly we show that there exists an integer N and some $L \leq L' \leq L+a$ so that the estimates:

$$t^{L'} \leq N f_t(t^L) \leq t^{L+a}$$

hold for all sufficiently large $t \gg 1$.

There is some L' so that the estimates $(\frac{f_t(t^L)}{t^{L'}})^{\pm 1} \leq M$ hold by (1) for some $L \leq L' \leq L + a$ and all $t > 1$. Since $\frac{f_t(t^L)}{t^{L'}}$ is a rational function in $t > 1$, there is a limit:

$$M^{-1} \leq \lim_{t \rightarrow \infty} t^{-L'} f_t(t^L) \equiv \kappa \leq M.$$

One may assume the estimates $NM \leq t^a$ with $N = [\kappa^{\pm 1}] + 1$ for all large $t \gg 1$, where $[\kappa^{\pm 1}] = \max([\kappa], [\kappa^{-1}]) \geq 0$.

Then one obtains the estimates $1 \leq Nt^{-L'} f_t(t^L) \leq NM \leq t^a$, and so:

$$t^L \leq t^{L'} \leq Nf_t(t^L) \leq t^{L'+a} \leq t^{L+2a}$$

since $L \leq L' \leq L + a$ hold. It follows from (2) that the estimates:

$$NM^{-1}t^L \leq Nf_t(t^x) \leq Nf_t(t^L) + CNt^{L+a} \log t^{4a}$$

hold for all $L - a \leq x \leq L + 3a$.

One may assume the estimates for all sufficiently large $t \gg 1$:

$$N^{-1}M \leq t^a, \quad CNt^{L+a} \log t^{4a} \leq t^{L+2a}.$$

Then combining with these estimates:

$$t^{L-a} \leq Nf_t(t^x) \leq 2t^{L+2a} \leq t^{L+3a}$$

hold for all $L - a \leq x \leq L + 3a$. This completes the proof.

5.D.6 Proof of proposition 5.12:

Step 1: Let $\tau(y)$ be in 5.D.4 and l_t correspond to τ :

$$l_t(w) = [(1 + t^{-\delta}w)^{-1} + (1 + t^{7\delta}w^{-1})^{-1}]^{-1}.$$

The inequalities hold:

$$(1 + t^{6\delta})^{-1} \leq l_t(w)^{-1} \leq 2, \quad |l_t(w)|' \leq 4(1 + t^{-\delta}).$$

Let:

$$h_t(w) = K^{-1}[t^{-2\delta} + (l_t(t^{-q^0}w) + \cdots + l_t(t^{-q^l}w))^{-1}]^{-1}$$

be corresponding to μ , where $K \in \mathbb{N}$ are chosen so that the estimates $h_t(w) \leq 1$ hold. Notice that the estimates hold:

$$[(l+1)(1+t^{6\delta})]^{-1} + t^{-2\delta} \leq (Kh_t(w))^{-1} \leq \frac{2}{l+1} + t^{-2\delta}.$$

By inceasing the number of l if necessarily, one may assume the lower bound $1 - \chi \leq h_t(w)$ for sufficiently small $0 < \chi < 1$.

We have the estimates:

$$\begin{aligned} |h_t(w)'| &= \frac{t^{-q^0+2\delta}}{K} \left| \frac{\sum_{j=0}^l t^{-(q^j-q^0)} l_t'(t^{-q^j}w)}{(t^{2\delta} + l_t(t^{-q^0}w) + \dots + l_t(t^{-q^l}w))^2} \right| \\ &\leq \frac{t^{2\delta}}{t^{q^0}} \frac{4}{K(l+1)} \sup_{w \in \mathbb{R}} |l_t'(w)| \leq CK^{-1}t^{-q^0} \end{aligned}$$

where C is independent of $t > 1$.

Now let $\bar{\psi}$ be in 5.D.3, and recall ξ with corresponding f_t . Then corresponding to $\bar{\psi}$ is the following:

$$f_t(w, z) = f_t(zh_t(w)).$$

By lemma 5.13, both the estimates:

$$\begin{aligned} (-1 + t^{-3\delta})z &< f_t(w, z) - h_t(w)z < -\mu z, \\ -1 + t^{-3\delta} &< (f_t)_z(w, z) - h_t(w) < -\mu. \end{aligned}$$

In particular we have the estimates:

$$\begin{aligned} -z &< (-2 + t^{-3\delta} + h_t(w))z < f_t(w, z) - z < -\mu z, \\ -1 &< -2 + t^{-3\delta} + h_t(w) < (f_t)_z(w, z) - 1 < -\mu z, \\ |(f_t)_w(w, z)| &= |h_t'(w)|z|f_t'(zh_t(w))| \leq CK^{-1}t^{-q^0}z. \end{aligned}$$

Step 2: Let $g_t(w, z)$ correspond to $\bar{\phi}$, and estimate:

$$|(f_t)_w(g_t - w)|, \quad |(g_t)_z(f_t - z)|.$$

Let us modify $g_t(w, z)$ so that there are C and some a independent of $t > 1$, and the estimates hold:

$$\begin{aligned} |(g_t)_z|, \quad |(g_t)_w| &\leq Ct^{a\delta}, \\ -w &< g_t(w, z) - w < -\mu w, \\ -1 &< (g_t)_w - 1 < -\mu \end{aligned}$$

for all $t^{-a\delta} < w \leq t^{a\delta}$.

Let us denote $\tilde{\phi}(y, x) = \bar{\phi}(y, x) - y$. Then there is some a with the estimates $|\tilde{\phi}(y, x)| \leq a\delta$. So we have the equality $\tilde{\phi}(y, x) = \max(\tilde{\phi}(y, x), -a\delta)$.

Let $\tilde{g}_t(w, z)$ correspond to $\tilde{\phi}$. Then $\tilde{g}'_t(w, z) = \frac{\tilde{g}_t}{N_0} + t^{-a\delta}$ are tropically equivalent to $\tilde{g}_t$ for $N_0 \in \mathbb{N}$. By lemma 5.16, $\tilde{g}_t$ is uniformly bounded and so one may assume the estimates:

$$1 - \chi < \tilde{g}'_t(w, z) < 1 + \chi$$

for sufficiently small $0 < \chi$. Moreover:

$$\left| \frac{\partial \tilde{g}_t}{\partial w} \right| \leq Ct^{2a\delta} t^{-q^0}$$

hold. Now we have the equality:

$$\frac{\partial \tilde{g}'_t}{\partial w} - 1 = \tilde{g}'_t - 1 + \frac{w}{N_0} \frac{\partial \tilde{g}_t}{\partial w}$$

where the estimates $\left| \frac{w}{N_0} \frac{\partial \tilde{g}_t}{\partial w} \right| \leq Ct^{2a\delta} \frac{w}{N_0 t^{q^0}} \leq \frac{Ct^{3a\delta}}{N_0} \ll 1$ hold, and so we obtain the desired estimates.

Since

$$\bar{g}'_t(w, z) = \frac{w}{2} \tilde{g}'_t(w, z)$$

are tropically equivalent to $\bar{g}_t(w, z)$, we obtain the desired rational functions.

So we obtain the followings:

$$|(f_t)_w(g_t - w)| \leq CK^{-1}t^{a\delta}z, \quad |(g_t)_z(f_t - z)| \leq Ct^{a\delta}z.$$

Step 3: By proposition 5.11 and step 2, we have the following estimates:

$$\begin{aligned} |(f_t)_v(v, u)||v_s| &= |h'_t(w)|z|f'_t(zh_t(w))||v_s| \\ &\leq CK^{-1}t^{-q^0}zt^{d\delta}z = CK^{-1}t^{-q^0+a\delta}z^2 \\ |(g_t)_u(v, u)||u_x| &\leq Ct^{2a\delta}z. \end{aligned}$$

If we take $q^0 \geq L + 1$ and large $t \gg 1$, then we obtain the estimates:

$$|(\tilde{f}_t)_v(v, u)||v_s|(x, s) \leq \mu' u(x, s), \quad |(\tilde{g}_t)_u(v, u)||u_x|(x, s) \leq \mu' v(x, s)$$

for sufficiently small $0 < \mu' \ll 1$.

Now combining with corollary 5.14, we obtain the estimates:

$$\begin{aligned} |(\tilde{f}_t)_v(v, u)| |v_s|(x, s + \alpha) &\leq \mu'' u(x, s + 1), \\ |(\tilde{g}_t)_u(v, u)| |u_x|(x + \alpha, s) &\leq \mu'' v(x + 1, s) \end{aligned}$$

for some $0 < \mu'' < 2$. This completes the proof of proposition 5.15.

Basically the constructions of the refinement are quite general, and it would be reasonable to expect the following:

Conjecture 5.2: For any pairs of relatively $(\max, +)$ -functions (ϕ, ψ) , there are refinements $(\bar{\phi}, \bar{\psi})$ by 1-Lipschitz functions so that the corresponding relatively elementary functions $(\bar{f}_t, \bar{g}_t)$ are admissible.

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Tsuyoshi Kato
 Department of Mathematics
 Faculty of Science, Kyoto University
 Kyoto 606-8502 Japan