

THE SUBORDINATION THEOREM FOR λ -SPIRALLIKE FUNCTIONS OF ORDER α

OH SANG KWON AND SHIGEYOSHI OWA

ABSTRACT. We proved a subordination relation for a subclass of the class of λ -spirallike functions of order α .

1. Introduction

Let A denote the class of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent function $f(z)$ in unit disk U .

A function $f(z)$ in S is said to be convex if

$$(1.1) \quad \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in U).$$

And we denote by K the class of all convex functions.

Definition 1.1. A function $f(z)$ in S is said to be λ -spirallike of order α , ($0 \leq \alpha < 1$), if

$$(1.2) \quad \operatorname{Re} \left\{ e^{i\lambda} z \frac{f'(z)}{f(z)} \right\} > \alpha \cos \lambda \quad (z \in U),$$

for some real λ ($|\lambda| < \frac{\pi}{2}$). The class of the functions is denoted by $S_p^\alpha(\lambda)$.

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Definition 1.2. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in U , then their Hadamard product, $f * g$ is function defined by the power series

$$(1.3) \quad (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f * g$ is also analytic in U .

Definition 1.3. Let f be analytic in U , g analytic and univalent in U and $f(0) = g(0)$. Then by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U , we shall mean that $f(U) \subset g(U)$.

Definition 1.4. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$ is regular, univalent and convex in U , we have

$$(1.4) \quad \sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad \text{in } U.$$

Lemma 1.5. The sequence $\{b_n\}_{n=1}^{\infty}$ is subordinating factor sequence if and only if

$$(1.5) \quad \operatorname{Re} \left[1 + 2 \sum_{n=1}^{\infty} b_n z^n \right] > 0 \quad (z \in U).$$

The above lemma is due to (Wilf [2]).

In this paper, we prove a subordination relation for a subclass of the class of λ -spirallike functions of order α .

2. Main results

Before proving our next results, we need the following the Lemmas.

Lemma 2.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic with $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \beta$ for $0 \leq \beta < 1$ and $z \in U$. Then $f(z) \in S_p^\alpha(\lambda)$ for $|\lambda| \leq \cos^{-1} \left(\frac{1-\beta}{1-\alpha} \right)$.

Proof. We may write $\frac{zf'(z)}{f(z)} - 1 = (1 - \beta)w(z)$, where $|w(z)| < 1$ for $z \in U$. Thus

$$\begin{aligned} \operatorname{Re} \left[e^{i\lambda} \frac{zf'(z)}{f(z)} \right] &= \operatorname{Re}[e^{i\lambda}(1 + (1 - \beta)w(z))] \\ &= \cos \lambda + (1 - \beta)\operatorname{Re}\{e^{i\lambda}w(z)\} \\ &\geq \cos \lambda - (1 - \beta)|e^{i\lambda}w(z)| \\ &> \cos \lambda - (1 - \beta) \geq \alpha \cos \lambda \end{aligned}$$

for $|\lambda| \leq \cos^{-1} \frac{1 - \beta}{1 - \alpha}$, and the proof is complete.

Lemma 2.2. If $\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1 - \alpha) \cos \lambda$, then $f \in S_p^\alpha(\lambda)$.

Proof. Set $\beta = 1 - (1 - \alpha) \cos \lambda$ in Lemma 2.1.

Theorem 2.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If

$$(2.1) \quad \sum_{n=2}^{\infty} \left\{ 1 + \frac{n-1}{1-\alpha} \sec \lambda \right\} |a_n| < 1,$$

then $f(z) \in S_p^\alpha(\lambda)$.

Proof. By Lemma 2.2, it suffices to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1 - \alpha) \cos \lambda$. We have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} (n-1)a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n} \right| < \frac{\sum_{n=2}^{\infty} (n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n||z^n|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty} (n-1)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}. \end{aligned}$$

Thus last expression is bounded above by $(1 - \alpha) \cos \lambda$, if

$$(2.2) \quad \sum_{n=2}^{\infty} (n-1)|a_n| \leq (1 - \alpha) \cos \lambda \left(1 - \sum_{n=2}^{\infty} |a_n| \right).$$

which is equivalent to

$$(2.3) \quad \sum_{n=2}^{\infty} \left\{ 1 + \frac{n-1}{1-\alpha} \sec \lambda \right\} |a_n| \leq 1.$$

Remark 1. Taking $\lambda = 0$ in (2.1), we obtain a sufficient condition for $f(z)$ to be starlike of order α (H. Silberman [1]).

Let us denote by $G(\lambda, \alpha)$, the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ whose coefficients satisfy the condition (2.1).

Theorem 2.4. Let $f \in G(\lambda, \alpha)$. Then

$$(2.4) \quad \frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} (f * g)(z) \prec g(z) \quad \text{for } z \in U$$

for every function $g(z)$ in the class K .

In particular

$$(2.5) \quad \operatorname{Re} f(z) > -\frac{2(1-\alpha) + \sec \lambda}{(1-\alpha) + \sec \lambda} \quad \text{for } z \in U.$$

The constant $\frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)}$ cannot be replaced by any larger one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $G(\lambda, \alpha)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be in K . Then

$$(2.6) \quad \frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} (f * g)(z) = \frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right).$$

Thus, by definition 1.4, the assertion of our theorem will hold if the sequence

$$(2.7) \quad \left(\frac{\{(1-\alpha) + \sec \lambda\} a_n}{2(2(1-\alpha) + \sec \lambda)} \right)_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.5, this will be the case if and only if

$$(2.8) \quad \operatorname{Re} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} a_n z^n \right] > 0 \quad \text{for } z \in U.$$

$$\begin{aligned}
& \operatorname{Re} \left[1 + \sum_{n=1}^{\infty} \frac{(1-\alpha) + \sec \lambda}{2(1-\alpha) + \sec \lambda} a_n z^n \right] \\
&= \operatorname{Re} \left[1 + \frac{(1-\alpha) + \sec \lambda}{2(1-\alpha) + \sec \lambda} z + \frac{1-\alpha}{2(1-\alpha) + \sec \lambda} \sum_{n=2}^{\infty} \left(1 + \frac{\sec \lambda}{1-\alpha} \right) a_n z^n \right] \\
&\geq 1 - \frac{(1-\alpha) + \sec \lambda}{2(1-\alpha) + \sec \lambda} r - \frac{1-\alpha}{2(1-\alpha) + \sec \lambda} \sum_{n=2}^{\infty} \left(1 + \frac{(n-1) \sec \lambda}{1-\alpha} \right) |a_n| r^n \quad (|z| = r) \\
&\geq 1 - \frac{(1-\alpha) + \sec \lambda}{2(1-\alpha) + \sec \lambda} r - \frac{1-\alpha}{2(1-\alpha) + \sec \lambda} r \quad (\text{by (2.1)}) \\
&> 0.
\end{aligned}$$

Thus (2.8) holds true in U . Thus prove the first assertion. That $\operatorname{Re} f(z) > -\frac{2(1-\alpha) + \sec \lambda}{(1-\alpha) + \sec \lambda}$ for $f(z) \in G(\lambda, \alpha)$ follows by taking $g(z) = \frac{z}{1-z}$ in (2.4).

To prove the sharpness of the constant $\{(1-\alpha) + \sec \lambda\}/2(2(1-\alpha) + \sec \lambda)$, we consider the function

$$(2.10) \quad f_0(z) = z - \frac{(1-\alpha)}{(1-\alpha) + \sec \lambda} z^2 \quad \text{for } (|\lambda| < \frac{\pi}{2}),$$

which is a member of the class $G(\lambda, \alpha)$. Thus from the relation (2.4), we obtain

$$(2.11) \quad \frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} f_0(z) < \frac{z}{1-z}.$$

If can be verified that

$$(2.12) \quad \min_{|z| \leq 1} \operatorname{Re} \left[\frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)} f_0(z) \right] = -\frac{1}{2}.$$

This shows that the constant $\frac{(1-\alpha) + \sec \lambda}{2(2(1-\alpha) + \sec \lambda)}$ is best possible.

Taking $\lambda = 0$, we obtain the following corollary.

Corollary 2.5. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is regular in U and satisfies the condition

$$(2.13) \quad \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \leq 1$$

then for every function g in K , we have

$$(2.14) \quad \frac{2-\alpha}{2(3-2\alpha)}(f * g)(z) \prec g(z).$$

In particular, $\operatorname{Re} f(z) > -\frac{3-2\alpha}{2-\alpha}$, $z \in U$. The constant $\frac{2-\alpha}{2(3-2\alpha)}$ is best possible.

Remark 2. Putting $\alpha = 0$ in Theorem 2.4, we get the result in S. Singh [3].

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Department of Mathematics
 Kyungsung University
 Pusan 608-736, Korea
 e-mail : oskwon@star.kyungsung.ac.kr

Department of Mathematics
 Kinki University
 Higashi-Osaka, Osaka 577-8502, Japan
 e-mail : owa@math.kindai.ac.kr