

Yukawa couplings of Certain Calabi-Yau 3-folds

by

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1 Mirror Symmetry Conjecture for Calabi-Yau 3-folds

Mirror symmetry appeared in the supersymmetric string theory where a Calabi-Yau 3-fold M plays a role as a background for string propagation. It said that a “mirror pair” of Calabi-Yau 3-folds (X, Y) apparently produce isomorphic physical theories. (See [G-P], [Mo2]).

Mathematically, mirror symmetry related certain geometric invariants of a Calabi-Yau 3-fold to a completely different set of geometric invariants of the mirror partner. Mathematician had not found such a symmetry before physisits’ predecions and calculations. (For more historical backgrounds of mirror symmetry, please consult [Mo2] and references therein). The two models are called A-model and B-model respectively. A-model

on a Calabi-Yau 3-fold X has a correlation function related to the so-called Gromov-Witten invariants of a Calabi-Yau 3-folds, which is essentially related to the number of holomorphic rational curves on X . On the other hand, B-model correlation functions on a Calai-Yau 3-fold Y is given by period integrals of holomorphic forms. A 3-dimensional complex projective manifold X is called a Calabi-Yau 3-fold if $K_X \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. For a compact Kähler manifold Y , we set $H^{pq}(Y) = H^q(Y, \Omega_Y^p)$ and $h^{pq}(Y) = \dim_{\mathbb{C}} H^{pq}(Y)$.

Let X be a Calabi-Yau 3-fold and consider the following classical cubic form in A-model:

$$I^{1,1} : H^{1,1}(X) \times H^{1,1}(X) \times H^{1,1}(X) \longrightarrow \mathbb{C}$$

defined by usual cup product

$$I^{1,1}(L_1, L_2, L_3) = \int_X L_1 \wedge L_2 \wedge L_3.$$

Let Y be another Calabi-Yau 3-fold and fix a nowhere vanishing holomorphic 3-form ω and define a classical cubic form in B-model

$$I_{\omega}^{2,1} : H^{2,1}(Y) \times H^{2,1}(Y) \times H^{2,1}(Y) \longrightarrow \mathbb{C}$$

by

$$I_{\omega}^{2,1}(\theta_1, \theta_2, \theta_3) = \int_Y \omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \omega.$$

Here ∇ is the Gauss-Manin connection associated to the Hodge structure on $H^3(Y, \mathbb{C})$.

In A-model side, the quantum deformation of the classical cubic form can be given by

$$\begin{aligned} I_Q^{1,1} : H^{1,1}(X) \times H^{1,1}(X) \times H^{1,1}(X) &\longrightarrow \mathbb{C}, \\ I_Q^{1,1}(L_1, L_2, L_3) &= \int_Y L_1 \wedge L_2 \wedge L_3 \\ &\quad + \sum_{0 \neq \eta \in H_2(Y, \mathbb{Z})} \Phi_{\eta}(L_1, L_2, L_3) \frac{q^{\eta}}{1-q^{\eta}} \end{aligned}$$

This cubic form is called A-model correlation functions in [Mo2]. Here $\Phi_{\eta}(L_1, L_2, L_3)$ denote Gromov-Witten invariants. The definitions of Gromov-Witten invariants and the symbol q^{η} are given in lecture 3 and 4 in [Mo2]. An axiomatic treatment of Gromov-Witten invariants are in [K-M] and the construction of Gromov-Witten invariants using symplectic geometry and pseudo holomorphic curve is given by Ruan [R]. The notion of stable maps due to Kontsevich ([K-M], [Kon]) are used for another definition of Gromov-Witten invariants. For detail, readers may refer to [Kon], [Fu-Pa] and [B-M]. Givental recently has proved that for Calabi-Yau complete intersections in toric varieties, the predicted enumerative formulas which one calculates by using Batyrev-Borisov candidate mirror partner are in fact correct evaluations of the Gromov-Witten invariants. ([Giv1], [Giv2], [Giv3]). His proof involves stationary phase integrals, equivariant Gromov-Witten invariant and quantum Toda lattices.

In B-model side, the quantum deformation of $I_{\omega}^{2,1}$ is defined to be

$$I_{Q, \omega}^{2,1} = I_{\omega}^{2,1},$$

that is, we keep $I_{\omega}^{2,1}$ not deformed. From the view point of Mirror symmetry, the asymptotic behavior of the B-model correlation function $I_{\omega}^{2,1}$ is very important when the complex structure of Y tends to the large complex structure limit. Readers may consult lecture 6 and lecture 7 in [Mo2].

2 A review on Mordell-Weil groups

In this section, we recall the Mordell-Weil group of abelian scheme over rational function field of a complex projective curve. Let C be a smooth projective curve

defined over $\mathbf{C}$, and $K = \mathbf{C}(C)$ the field of rational functions on C . Let A/K denote an abelian variety defined over K . Then the theory of Néron model says that there exist group scheme

$$\pi : \mathcal{A}^0 \longrightarrow C$$

whose generic fiber $\mathcal{A}_\eta^0$ is isomorphic to A over K . The Mordell-Weil group of A is defined to be

the group of K -rational points of A

$$MW(A/K) = A(K).$$

The Néron universal property ensures that

$$MW(A/K) = \{\sigma : C \longrightarrow \mathcal{A}^0, \quad \text{regular section of } \pi\}.$$

In general, the group $MW(A/K)$ is not finitely generated, for A may have nontrivial $K/\mathbf{C}$ -trace B . However, a theorem of Lang implies that the Mordell-Weil group of the quotient abelian variety A/B_K is finitely generated.

In [Man1], Manin construct a height pairing on the Mordell-Weil group $MW(A/K)$. In order to construct the height, he needed to suppose that $\mathcal{A}^0\pi C$ has a smooth relative compactification

$$\begin{array}{ccc} \mathcal{A}^0 & \hookrightarrow & \mathcal{A} \\ \pi \searrow & & \swarrow h \\ & C & \end{array}$$

such that $\mathcal{A}$ is smooth projective.

The group $D_{\mathbf{C}}(\mathcal{A})$ of all divisors defined over $\mathbf{C}$ on $\mathcal{A}$ splits into the direct sum of the two subgroups

$$D_{\mathbf{C}}(\mathcal{A}) = D^C \oplus D^A$$

where D^C is generated by the irreducible components of the fibers of h and D^A generated by irreducible divisors which maps onto C .

Moreover we suppose that:

1. all translation automorphisms of $\mathcal{A}^0/C$ extend to biregular automorphisms of $\mathcal{A}/C$, and
2. the map $\gamma : D_{\mathbf{C}}(\mathcal{A}) \rightarrow D_K(A)$ which maps each $\mathbf{C}$ -divisors of $\mathcal{A}$ into the divisors induced by it on the generic fiber A is null on D^A and defines an A -isomorphism between D^A and $D_K(A)$. Further for a rational function $f \in K(A)$ we have

$$\gamma((f)_{\mathcal{A}}) = (f)_A,$$

where $(f)_{\mathcal{A}}$ (resp. $(f)_A$) is a principal divisor on $\mathcal{A}$ (resp. on A).

Let $X \in D_K(A)$, then the Tate height on $MW(A/K)$ relative to a divisor X is defined in [Man1]

$$\hat{h}_X : MW(A/K) \longrightarrow \mathbf{R},$$

and moreover if X is symmetric, that is, $X^- = X$, where X^- is the image of the inversion map on A , $\hat{h}_X$ is really quadratic. The following theorem is due to Manin ([Man1], Theorem 4.).

Theorem 2.1 *Let $D = D^C$ /principal divisors on C . On the subgroup $MW^1(A/K)$ which acts trivially on D , the Tate height may be computed by the formula*

$$\hat{h}_X(\sigma) = (\gamma^{-1}(X), \sigma(C)) - (\gamma^{-1}(X), 0(C)).$$

Here the pairings $(\ , \)$ in the right hand side denote the intersection pairing of divisors and curves and $\sigma \in MW(A/K)$ is considered as a section $\sigma : C \rightarrow \mathcal{A}$. (We also denote by 0 the zero element of $MW(A/K)$.)

Corollary 2.1 *If $h : \mathcal{A} \longrightarrow C$ has only irreducible fibers, and $X \in D_K(A)$ is symmetric and $(\gamma^{-1}(X), 0(X)) = 0$, the Tate height is quadratic integral-valued and given by*

$$\hat{h}_X(\sigma) = (\gamma^{-1}(X), \sigma(C)) = (L, C).$$

where L is a line bundle on $\mathcal{A}$ algebraically equivalent to $\gamma^{-1}(X)$.

Assume that $MW(A/K)$ is finitely generated and torsion-free. Under the assumption of corollary, the associated symmetric bilinear form

$$\langle, \rangle : MW(A/C) \times MW(A/K) \longrightarrow \mathbf{Z}$$

gives a natural lattice structure on $MW(A/K)$. In case when A/K is an elliptic curve, one has a good minimal model $h : \mathcal{A} \longrightarrow C$ by using minimal model theory of projective surfaces.¹

Shioda [Sh1], [Sh3] actually showed that this lattice structure can be calculated by using the intersection theory of the surface $\mathcal{A}$, and he called the lattice *Mordell-Weil lattice*.

Later, he extended his results to the case where A is the Jacobian variety of a higher genus curve T over K ([Sh2]).

For example, if $h : \mathcal{A} \longrightarrow \mathbf{P}^1$ is a minimal rational elliptic surface with only irreducible fibers, one has isometry

$$MW(A/K) \simeq E_8$$

where E_8 is the famous even unimodular lattice of rank 8.

¹ Actually, in this case, one does not need the assumption on the irreducibility of fibers and other condition in the corollary

3 Calabi-Yau 3-folds with abelian fibration

In this section, we shall give two examples of Calabi-Yau 3-folds with fibrations of abelian surfaces.

First of all, we make the following

Definition 3.1 *A fibration of curves $f : S \longrightarrow C$ from a smooth projective surface S to a curve C is called a Lefschetz fibration if all cloed fibers has at most one node as its singularity.*

Example I (Example of C. Schoen [Sch].)

Let $f_i : S_i \longrightarrow \mathbf{P}^1$ ($i = 1, 2$) is minimal rational elliptic surfaces with sections and assume that both of f_i are Lefschetz fibrations. Moreover assume that the sets of critical values of f_i has no common elements. Then consider the fiber product $W = S_1 \times_{\mathbf{P}^1} S_2$.

$$\begin{array}{ccc} & W & \\ p_1 \swarrow & & \searrow p_2 \\ S_1 & & S_2 \\ f_1 \searrow & \mathbf{P}^1 & \swarrow f_2 \end{array} .$$

Then it is easy to see that W is a nonsingular Calabi-Yau threefold and induced fibration $h : W \longrightarrow B\mathbf{P}^1$ is a fibration of the product of two elliptic curves. Moreover we fix zero sections 0_1 and 0_2 of S_1 and S_2 respectively and let F_i denote the class of general fiber of f_i , then set

$$L_i = 0_i(\mathbf{P}^1) + F_i.$$

It is easy to see that the Tate height of each section $\sigma \in MW(S_i/\mathbf{P}^1)$ is given by

$$\langle \sigma, \sigma \rangle = (L_i, \sigma(\mathbf{P}^1)),$$

and the Mordell-Weil lattices are isometric to E_8 . Then one can easily prove

Proposition 3.1 *The Mordell-Weil lattice structure of $MW(W/\mathbf{P}^1)$ with respect to line bundle $L = p_1^*(L_1) + p_2^*(L_2)$ is isometric to*

$$(MW(W/\mathbf{P}^1), \langle, \rangle) \simeq E_8 \oplus E_8$$

The Hodge diamond of W is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 19 & & 0 & \\ 1 & & 19 & & 19 & & 1 \\ & 0 & & 19 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Example II([Sa-Sak].)

We recall a construction of genus g fibrations in [Sa-Sak]. For $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$, we let $p_i : \Sigma_0 \rightarrow \mathbf{P}^1$ ($i = 1, 2$) be the natural projections onto i -th factor and $F_i = p_i^*(\text{apoint})$ the divisor class of a fiber of p_i . A curve B in Σ_0 is said to be of bidegree (a, b) if B is linear equivalent to $aF_1 + bF_2$.

Let B be a smooth curve in Σ_0 of bidegree $(2, 2g + 2)$ and let $\pi : X \rightarrow \Sigma_0$ be the double covering whose branch locus is B . Set $f = \pi p_1$ and $\varphi = \pi p_2$. Then we have the following two fibrations:

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow \varphi \\ \mathbf{P}^1 & & \mathbf{P}^1 \end{array} .$$

Note that f is a fibration of (hyperelliptic) curves of genus g and φ is a fibration of conics, and hence X is a rational surface. Let $K = \mathbf{C}(\mathbf{P}^1)$ be the rational function field and considering the generic fiber X_η of f as a curve of genus g over K , we define the Jacobian variety of X_η

$$J = \text{Jac}(X_\eta).$$

From now on we restrict our attention to the case of $g = 2$. Then the Néron model of J exists

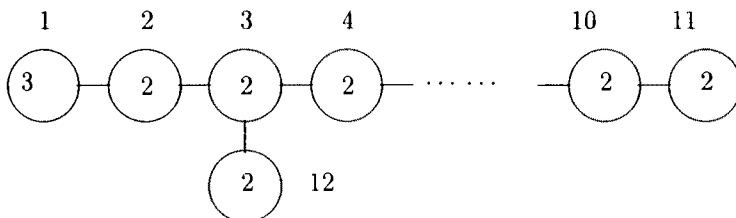
$$\mathcal{J}^0 \rightarrow \mathbf{P}^1.$$

Theorem 3.1 ([Sa3], [Sa4]) *Assume that $g = 2$ and $f : X \rightarrow \mathbf{P}^1$ is a Lefschetz fibration. Then we have the following assertions:*

1. $\mathcal{J}^0 \rightarrow \mathbf{P}^1$ has a good smooth compactification $h : \mathcal{J} \rightarrow \mathbf{P}^1$ whose total space is a Calabi-Yau 3-fold.
2. There exists a natural embedding of $X \hookrightarrow \mathcal{J}$.
3. With respect the line bundle $L = X + X^- + 2F$ where F is a class of fiber of h , the Mordell-Weil lattice structure on $MW(J/K)$ is isometric to

$$(MW(J/K), \langle, \rangle_L) = D_{12}^+$$

Here D_{12}^+ is a unimodular lattice whose dynkin diagram is given by



The existence of good smooth compactification in the Lefschetz fibration case is due to Nakamura [N]. The Hodge diamond of $\mathcal{J}$ is given by the following:

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
 & 0 & 14 & & 0 \\
 1 & & 14 & 14 & & 1 \\
 & 0 & & 14 & 0 \\
 & & 0 & & 0 \\
 & & 1 & &
 \end{array}$$

4 Theta functions of lattices and Yukawa coupling

In examples in §3, we know the structure of Mordell-Weil lattices. By virtue of result in Cor. 2.1, this let us know the number of rational curves *arising from section* whose intersection number with respect to special line bundles are fixed by the theta function of lattices.

Let us start with lattice theta functions. For any positive integral lattice L , we let

$$\theta_L(z) = \sum_{\eta \in L} q^{<\eta, \eta>}$$

be the theta function of the lattice L where $q = \exp(\pi iz)$. We have

$$\theta_L(z) = \sum_{m=1}^{\infty} N_L(m) q^m.$$

where

$$N_L(m) = \#\{\eta \in L \mid <\eta, \eta> = m\}.$$

Let us recall the Jacobi theta functions:

$$\theta_2(z) = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2,$$

$$\theta_3(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2,$$

$$\theta_4(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1})^2.$$

Then the theta function of D_{12}^+ is given by (see 7.3, Ch. 4 in [C-S]):

$$\theta_{D_{12}^+}(z) = 1/2(\theta_2^{12}(z) + \theta_3^{12}(z) + \theta_4^{12}(z))$$

On the other hand, the theta function of E_8 is Eisenstein series $E_2(z)$, while

$$\theta_{E_8 \oplus E_8}(z) = (E_2(z))^2 = E_4(z).$$

By using the expansion of these Theta function, we know the number of rational curves arising from section of fixed degree (= height) with respect to the specific line bundle.

The further calculation of Gromov-Witten invariants needs more effort, however the author expects that there should be good differential equation which was satisfied by the correlation functions associated the Calabi-Yau 3-folds in §3.

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