

ON MIXED HODGE STRUCTURE OF CHARACTER VARIETIES

Arata Komyo

Department of Mathematics, Graduate School of Science, Kobe University, D2

Abstract

In this poster, we study a variety, called character variety $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}$, whose points parametrize representation of the fundamental group of k -punctured Riemann surface of genus g into $\mathrm{GL}(n, \mathbb{C})$. In particular, we investigate the mixed Hodge structure of character varieties. There exists very interesting conjectures. For example,

Conjecture 1.2.1 of [1] —
We have the following
(i) The compactly supported mixed Hodge polynomial $H_{\mathrm{c}}^*(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}; x, y, t)$ is a polynomial in xy and t , and is independent of the choice of generic eigenvalues of multiplicities μ .
(ii) Moreover,
$$H_{\mathrm{c}}^*(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}; x, y, t) = (t\sqrt{q})^{\dim(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g})} \mathbb{H}\mu(-\frac{1}{\sqrt{q}} t\sqrt{q}),$$
where $q := xy$, and $\mathbb{H}\mu(z, w)$ is the rational function defined in [1].

Here
$$H_{\mathrm{c}}^*(X; x, y, t) := \sum \dim_{\mathbb{C}}(\mathrm{Gr}_p^F \mathrm{Gr}_W^{p+q} H_{\mathrm{c}}^j(X) \otimes x^p y^q t^j.$$
The main result
Conjecture 1.2.1 (i) is true.

λ -connection

Definition

Let Σ be a compact Riemann surface of genus g .
Fix
We fix
• k -distinct points $p_1, \dots, p_k$ in Σ ,
• $\mu = (\mu^1, \dots, \mu^k)$, where $\mu^i = (\mu_{r_i}^1, \dots, \mu_{r_i}^{i_i})$ such that $\mu_1^i \geq \mu_2^i \geq \dots$ are non-negative integers, $\sum_j \mu_j^i = n$ for each i ,
• integers d, n with $n > 0$.

We put $\Sigma_0 = \Sigma \setminus \{p_1, \dots, p_k\}$, and
$$\Xi_n^{\mu, k}(d) := \left\{ \left(\lambda, (\xi_j^i)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq r_i}} \right) \in \mathbb{C} \times \mathbb{C}^{\sum r_i} \mid \lambda d + \sum_{i,j} \mu_j^i \xi_j^i = 0 \right\}.$$

We take a member $(\lambda, \xi) \in \Xi_n^{\mu, k}(d)$, where $\xi = (\xi_j^i)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq r_i}}$.
Parabolic λ -connection
We say $(E, \nabla, \{l_{*}^{(i)}\}_{1 \leq i \leq k})$ a ξ -parabolic λ -connection of rank n and degree d of type μ if
(1) E is an algebraic vector bundle on Σ of rank n and degree d ,
(2) $\nabla : E \rightarrow E \otimes \Omega_{\Sigma}^1(p_1 + \dots + p_k)$ is a λ -connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa) = \lambda a \otimes df + f \nabla(a)$ for $f \in \mathcal{O}_{\Sigma}$ and $a \in E$, and
(3) for each p_i , $l_{*}^{(i)}$ is a filtration $E|_{p_i} = l_1^{(i)} \supset l_2^{(i)} \supset \dots \supset l_{r_i}^{(i)} \supset l_{r_i+1}^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = \mu_j^i$ and $(\mathrm{Res}_{p_i}(\nabla) - \xi_j^i \mathrm{id}_{E|_{p_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 1, \dots, r_i$.

Moduli space

Moduli space
There exists a relative coarse moduli scheme
$$\pi : \mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d} \longrightarrow \Xi_n^{\mu, k}(d)$$
$$(E, \nabla, \{l_{*}^{(i)}\}_{1 \leq i \leq k}) \mapsto (\lambda, \xi)$$
of α -stable ξ -parabolic λ -connections of rank r and degree d of type μ . Moreover, π is smooth and $\mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d}$ is nonsingular.

We denote the fiber of (λ, ξ) by $\mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, \xi, g, d}$. In particular, we put $\mathcal{M}_{\mathrm{DR}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d}$ and $\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, \xi, g, d}$ in the case of $\lambda = 1$ and $\lambda = 0$ respectively.

Compactification of the moduli space

There exists a natural $\mathbb{C}^{\times}$ -action on $\mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d}$
$$t \cdot (E, \nabla, \{l_{*}^{(i)}\}) = (E, t\nabla, \{l_{*}^{(i)}\}).$$
The following $\mathbb{C}^{\times}$ -action on $\Xi_n^{\mu, k}(d)$ is well-defined,
$$t \cdot (\lambda, \xi) = (t\lambda, t\xi).$$
Then, $\pi : \mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d} \rightarrow \Xi_n^{\mu, k}(d)$ is a $\mathbb{C}^{\times}$ -equivariant morphism.
Let $\mathcal{M}'$ be the base change of $\mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, g, d}$ via $\mathbb{C} \times \Xi_n^{\mu, k}(d) \rightarrow \Xi_n^{\mu, k}(d)$,

given by $(x, (\lambda, \xi)) \mapsto (x\lambda, x\xi)$. Here the $\mathbb{C}^{\times}$ -action on $\mathbb{C} \times \Xi_n^{\mu, k}(d)$ is given by $t \cdot (x, (\lambda, \xi)) = (tx, (\lambda, \xi))$. Then, the set $U \subset \mathcal{M}'$ of points $u \in U$ such that $\lim_{t \rightarrow \infty} t \cdot (E, \nabla, \{l_{*}^{(i)}\})$ does not exist is open, and there exists a geometric quotient $\overline{\mathcal{M}} := U/\mathbb{C}^{\times}$ which is proper over $\Xi_n^{\mu, k}(d)$ via the induce map $\pi : \overline{\mathcal{M}} \rightarrow \Xi_n^{\mu, k}(d)$.

Theorem 1
The induce map $\pi : \overline{\mathcal{M}} \rightarrow \Xi_n^{\mu, k}(d)$ is topologically trivial. Moreover any two fibers of π have isomorphic cohomology, and the mixed Hodge structure of the cohomology is pure.

Sketch of proof.

$$\begin{array}{ccc} U/\mathbb{R}^{\times} & \xrightarrow{(ii)} & \Xi_n^{\mu, k}(d) \\ (i) \downarrow & & \parallel \\ \overline{\mathcal{M}} & \xrightarrow{\pi} & \Xi_n^{\mu, k}(d), \end{array}$$

where (i) is a $U(1)$ -bundle, and (ii) is topologically trivial. Then, we obtain π is topologically trivial.
Note that

$$\overline{\mathcal{M}} \setminus \mathcal{M} = \{ \mathbb{C}^{\times}((E, \nabla, \{l_{*}^{(i)}\}), 0, (\lambda, \xi)) \mid \lim_{t \rightarrow \infty} t \cdot (E, \nabla, \{l_{*}^{(i)}\}) \text{ exists} \}$$

is trivial over $\Xi_n^{\mu, k}(d)$. Then we can show that any fibers of π have isomorphic cohomology; in particular, $H^*(\mathcal{M}_{\mathrm{Hod}, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, \xi, g, d}) \cong H^*(\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, 0, g, d})$ for all $(\lambda, \xi) \in \Xi_n^{\mu, k}(d)$. Since $\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, 0, g, d}$ is a proper orbifold (in particular a rational homology manifold), its cohomology has pure mixed Hodge structure. By standard Morse theory arguments, $H^*(\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, 0, g, d}) \rightarrow H^*(\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, 0, g, d})$ is surjective. Thus, $H^*(\mathcal{M}_{\mathrm{Dol}, \mathrm{GL}(n, \mathbb{C})}^{\mu, 0, g, d})$ also has pure mixed Hodge structure. $\square$

Character variety

We now construct a variety, called character variety, whose points parametrize representation of the fundamental group of k -punctured Riemann surface of genus g into $\mathrm{GL}(n, \mathbb{C})$ with prescribed images in semi-simple conjugacy classes $C_1, \dots, C_k$ at the puncture. Assume that

$$\prod_{i=1}^k \det C_i = 1$$

and that $(C_1, \dots, C_k)$ has type $\mu = (\mu^1, \dots, \mu^k)$; that is, C_i has type μ^i for each $i = 1, \dots, k$, where the type of a semi-simple conjugacy class $C_i \subset \mathrm{GL}(n, \mathbb{C})$ is defined as the partition $\mu^i = (\mu_1^i, \dots, \mu_{r_i}^i)$ describing the multiplicities of the eigenvalues of any matrix in C_i .

Character Variety
For a k -tuple of conjugacy classes $(C_1, \dots, C_k)$ of type μ ,
$$\mathcal{U}_{\mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g} := \{ (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathrm{GL}(n, \mathbb{C})^{2g} \times \prod_{i=1}^k C_i \mid [A_1, B_1] \cdots [A_g, B_g] X_1 \cdots X_k = Id \}.$$
We call the affine GIT quotient by conjugation
$$\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g} := \mathcal{U}_{\mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g} / \mathrm{PGL}(n, \mathbb{C}) = \mathrm{Spec}(\mathbb{C}[\mathcal{U}_{\mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}])^{\mathrm{PGL}(n, \mathbb{C})}$$
a character variety of type μ .

If $(C_1, \dots, C_k)$ is a generic k -tuple of semi-simple conjugacy classes in $\mathrm{GL}(n, \mathbb{C})$ of type μ , then the quotient $\pi_{\mu} : \mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g} \rightarrow \mathcal{U}_{\mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}$ is a principal $\mathrm{PGL}(n, \mathbb{C})$ -bundle. Consequently, when nonempty, the affine variety $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}$ is nonsingular.

Riemann-Hilbert correspondence

For each member $(E, \nabla, \{l_{*}^{(i)}\}) \in \mathcal{M}_{\mathrm{DR}, \mathrm{GL}(n, \mathbb{C})}^{\mu, \xi, g, d}$, $\mathrm{Ker}(\nabla^n|_{\Sigma_0})$ becomes a local system on Σ_0 , where ∇^n means the analytic connection corresponding to ∇ . The local system $\mathrm{Ker}(\nabla^n|_{\Sigma_0})$ corresponds to a representation of $\pi_1(\Sigma_0)$. Then, we obtain an analytic isomorphism,

$$\mathrm{RH}_{\xi} : \mathcal{M}_{\mathrm{DR}, \mathrm{GL}(n, \mathbb{C})}^{\mu, \xi, g, d} \rightarrow \mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g, d},$$

where $\lambda_j^i = \exp(-2\pi\sqrt{-1}\xi_j^i)$ and we take ξ generic, that is, parabolic connections and local systems are irreducible. We consider the family $\{\mathrm{RH}_{\xi}\}$ over the locus of generic elements in $\Xi_n^{\mu, \lambda=1, k}(d)$, then we obtain the following theorem by Theorem 1,

Theorem 2
The mixed Hodge structure of the cohomology of $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}$ is independent of the choice of generic eigenvalues of multiplicities μ .

Mixed Hodge structure of character varieties

We consider the cohomology of $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}$. Since $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g} \cong (\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g} \times (\mathbb{C}^{\times})^{2g}) / \nu_{\mu}^{2g}$, where $\nu_{\mu} = \{(e^{2\pi\sqrt{-1}d/n} I_n, e^{-2\pi\sqrt{-1}d/n}) \mid d = 1, \dots, n\}$, we obtain that

$$H^*(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}) \cong H^*(\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g})^{\nu_{\mu}^{2g}} \otimes H^*((\mathbb{C}^{\times})^{2g}).$$

Construction of generators of $H^*(\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g})^{\nu_{\mu}^{2g}}$

(I) $\mathrm{PGL}(n, \mathbb{C})$ -principal bundle on $\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g} \times \Sigma_0$:
$$\mathbb{U} := (\mathrm{PGL}(n, \mathbb{C}) \times \mathcal{U}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g} \times \Sigma_0) / (\pi_1(\Sigma_0) \times \mathrm{GL}(n, \mathbb{C})),$$
where Σ_0 is the universal covering of Σ_0 and the action is $(p, g) \cdot (h, \rho, x) = (g\rho(p)h, g^{-1}\rho g, p \cdot x)$. The characteristic classes:

$$\tilde{c}_i(\mathbb{U}) = \beta_i + \sum_{j=1}^{2g} \gamma_{i,j} e_j + \sum_{k=1}^n \delta_{i,k} f_k \quad (i = 2, \dots, n),$$

$\beta_i \in H^{2i}(\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g})$, $\gamma_{i,j}, \delta_{i,k} \in H^{2i-1}(\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g})$.
(II) Let $b_i \in C^{2i-1}(G \times_G BG)$ be a singular cochain complex ($G = \mathrm{SL}(n, \mathbb{C})$) such that $j^*(b_i) = 0$, where j is the inclusion $BG \rightarrow G \times_G EG$ as $\{e\} \times_G EG$, and generate the cohomology of the fiber G of filtration $G \times_G EG \rightarrow G$ as a ring. For

$$\Phi^G : G^{2g} \times \prod_i C_i \rightarrow G$$
$$(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \mapsto [A_1, B_1] \cdots [A_g, B_g] X_1 \cdots X_k,$$

we can show that $\Phi^G(b_i) = 0$ in $H_G^{2i-1}(G^{2g} \times \prod_i C_i)$. We take $a_i \in C^{2i-2}((G^{2g} \times \prod_i C_i) \times_G EG)$ such that $da_i = \Phi^G(b_i)$. Then,

$$\alpha_i := [a_i]_{H_G^{2i-2}} \in H_G^{2i-2}(\mathcal{U}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g}) \cong H^{2i-2}(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, g}).$$

(III) We fix diagonal matrices $D_1, \dots, D_k$ in each conjugacy classes $C_1, \dots, C_k$.
$$\mathrm{GL}(n, \mathbb{C})/H_1 \rightarrow C_i \quad [g] \mapsto g^{-1} D_i g,$$
where $H_1 = \mathrm{GL}(\mu_1^1, \mathbb{C}) \oplus \dots \oplus \mathrm{GL}(\mu_{r_1}^1, \mathbb{C})$. ($\mu_1^1 + \dots + \mu_{r_1}^1 = n$: the multiplicities of the eigenvalues of any matrix in C_1). For a conjugacy class C_i ,

$$\begin{aligned} \mathcal{U}_{\mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g, l} := & \{ (A_1, B_1, \dots, A_g, B_g, X_1, \dots, M_l, \dots, X_k) \\ & \in \mathrm{SL}(n, \mathbb{C})^{2g} \times \prod_{j=1}^l C_j \times \mathrm{GL}(n, \mathbb{C}) \times \prod_{j=l+1}^k C_j \\ & \mid \prod_{i=1}^g (A_i, B_i) X_1 \cdots M_l^{-1} D_l M_l \cdots X_k = I_n \}. \end{aligned}$$

A natural map

$$\mathcal{U}_{\mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g, l} \rightarrow \mathcal{U}_{\mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, k} : (\dots, M_l, \dots) \mapsto (\dots, M_l^{-1} D_l M_l, \dots).$$

We put $\mathcal{U}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g, l} := \mathcal{U}_{\mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g, l} / \mathrm{PGL}(n, \mathbb{C})$ by a natural $\mathrm{PGL}(n, \mathbb{C})$ -action, which is a H_l -principal bundle on $\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g}$. we consider the classifying map,

$$\begin{aligned} \mathcal{U}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g, l} \\ \downarrow \\ \mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g} \xrightarrow{f} BH_l \cong B\mathrm{GL}(\mu_1^l, \mathbb{C}) \times \dots \times B\mathrm{GL}(\mu_{r_l}^l, \mathbb{C}). \end{aligned}$$

The characteristic classes

$$c_{k_1, \dots, k_r}^l := f^*(c_{k_1} \otimes \dots \otimes c_{k_r}) \quad (0 \leq k_j \leq \mu_{r_j}^l),$$

where $c_{k_j} \in H^{2k_j}(B\mathrm{GL}(\mu_{r_j}^l, \mathbb{C}))$ and $a_0 := 1$.

Theorem 3
The classes $\alpha_i, \beta_i, \gamma_{i,j}$ and $c_{k_1, \dots, k_{r_j}}^l$ generate $H^*(\mathcal{M}_{B, \mathrm{SL}(n, \mathbb{C})}^{\mu, \lambda, g})^{\nu_{\mu}^{2g}}$.

The generators $\alpha_i, \beta_i, \gamma_{i,j}$ (resp. $c_{k_1, \dots, k_{r_j}}^l$) have homogeneous weight i (resp. $k_1 + \dots + k_{r_j}$). On the other hand, generators of $H^*((\mathbb{C}^{\times})^{2g})$ have also homogeneous weight. Then,

Theorem 4
The cohomology of $\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, k}$ is type (p, p)
i.e. $\dim_{\mathbb{C}} \mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H^j(\mathcal{M}_{B, \mathrm{GL}(n, \mathbb{C})}^{\mu, \lambda, k}) = 0$ unless $p = q$.

References

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