

we obtain that if $A \succeq B \succeq O$ and $A' \succeq B' \succeq O$, then $AA' \succeq BB' \succeq O$. This yields immediately $A^m \succeq B^m \succeq O$ for $m = 1, 2, \dots$. On the contrary, this property does not hold in the case of usual order ' $\geq$ ' for operators. Namely, for two bounded hermitian operators A, B on $\mathcal{H}$, when $A - B$ is a positive semi-definite operator, we write $A \geq B$. It is well-known as the Löwner-Heinz inequality [1, bf 3] that if $A \geq B \geq O$, then $A^x \geq B^x \geq O$ for all $x \in [0, 1]$.

2. THE CASE OF GENERAL SELF-DUAL CONES IN A HILBERT SPACE

The set of all bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $L(\mathcal{H})$.

Theorem 2.1. *Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and A, B in $L(\mathcal{H})$ with $A \geq O, B \geq O$ and $A \supseteq B \supseteq O$ satisfying the following conditions:*

- (i) *A and B are compact.*
- (ii) *$\overline{A(\mathcal{S})} \subset (\mathcal{H}^+)^\circ$ and $\overline{(A-B)(\mathcal{S})} \subset (\mathcal{H}^+)^\circ$, where $\mathcal{S}$ denotes the set of all unit vectors in $\mathcal{H}^+$.*

Then there exists a number $s > 0$ such that $A^x \supseteq B^x \supseteq O$ for all $x \in [s, \infty)$.

Proof. We shall prove the second inequality. Since B is a compact operator, $\overline{B(\mathcal{S})}$ is compact. By the condition (ii) there exists a number $\varepsilon \in (0, 1)$ such that for every $v \in \mathcal{S}$ an ε -neighborhood $U(Bv; \varepsilon)$ of Bv is contained in $\mathcal{H}^+$. Indeed, if such a neighborhood is not contained in $\mathcal{H}^+$, then for every natural number n there exists $v_n \in \mathcal{S}$ such that $U(Bv_n; \frac{1}{n}) \cap (\mathcal{H}^+)^\circ \neq \emptyset$. Since B is compact, there exists a subsequence $\{Bv_{n_k}\}$ converging to some $w_0 \in \overline{B(\mathcal{S})}$. This implies $w_0 \in (\mathcal{H}^+)^\circ$, a contradiction.

Now, consider a map: $x \mapsto A^x, x \in \mathbb{R}$. By the norm continuity of the map, there exists a number $\mu \in (0, 1)$ such that

$$\|A - A^x\| < \varepsilon$$

holds for all $x \in (1 - \mu, 1 + \mu)$, hence

$$\|Bv - B^x v\| \leq \|B - B^x\| \|v\| = \|B - B^x\| < \varepsilon.$$

This means $B^x v \in \mathcal{H}^+$, i.e., $B^x \supseteq O$. Hence $B^{mx} \supseteq O$ for $m = 1, 2, \dots$. Setting $m_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$, we have $B^x \supseteq O$ for all $x \geq m_0(1 - \varepsilon)$.

The first inequality in the statement can be proved similarly by considering a map: $x \mapsto A^x - B^x$.

Theorem 2.2. *Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and A in $L(\mathcal{H})$ with $A \geq O$ and $A \supseteq O$ satisfying the following conditions:*

- (i) *A is invertible.*
- (ii) *$A(\mathcal{H}^+) \subsetneq \mathcal{H}^+$.*

Then $A^{-\lambda} \not\supseteq O$ for all $\lambda > 0$.

Proof. Suppose that $A^{-\lambda_0} \supseteq O$ for some $\lambda_0 > 0$. In the case where λ_0 is a rational number, we choose $m, n \in \mathbb{N}$ with $m - n\lambda_0 = -1$. It follows by assumption that

$$A^{-1} = A^m A^{-n\lambda_0} \supseteq O.$$

This means that A is an order isomorphism, i.e., $A(\mathcal{H}^+) = \mathcal{H}^+$, a contradiction. It is known that if λ_0 is an irrational number then the set $\{m - n\lambda_0 | m, n \in \mathbb{N}\}$

is dense in $\mathbb{R}$. We choose a sequence $\{r_n\}$ from the dense set converging to -1 . Then

$$A^{-1} = \lim_{n \rightarrow \infty} A^{r_n} \geq O.$$

Similarly, we get the contradiction.

Remark. (cf. [4]) For a facial homogeneous cone $\mathcal{H}^+$, $A(\mathcal{H}^+) = \mathcal{H}^+$ implies $A^x \geq O$ for all $x \in \mathbb{R}$.

3. THE CASE OF FINETELY GENERATED SELF-DUAL CONES

From the rest of this manuscript we deal with a finite dimensional Hilbert space. In this section we consider the case of finitely generated self-dual cones. We first prove the following lemma:

Lemma 3.1. *Let a_i , be real numbers and λ_i be positive numbers with $1 \leq i \leq n$. Put*

$$f(x) = a_1 \lambda_1^x + \cdots + a_n \lambda_n^x$$

for $x > 0$. Suppose that there exists an unbounded increasing sequence $\{x_m\}$ such that

$$f(x_m) \geq 0, \quad m = 1, 2, \dots.$$

Then $f(x)$ is identically 0, or there exists $s > 0$ such that

$$f(x) > 0, \quad x \in [s, \infty).$$

If, in particular,

$$f(x_m) = 0, \quad m = 1, 2, \dots,$$

then $f(x)$ is identically 0.

Proof. Let $f(x_m) \geq 0$ for all $m \in \mathbb{N}$. Suppose that $f(x)$ is not identically 0. We may assume $\lambda_1 > \cdots > \lambda_n > 0$ and $a_1 \neq 0$. Since

$$\frac{f(x)}{\lambda_1^x} = a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1} \right)^x + \cdots + a_n \left(\frac{\lambda_n}{\lambda_1} \right)^x,$$

it follows that

$$a_1 = \lim_{x \rightarrow \infty} \frac{f(x)}{\lambda_1^x}.$$

Hence we have $a_1 \geq 0$, and so $a_1 > 0$ by the assumption. By the continuity of the function, we obtain the desired results. It is now immediate that the latter statement holds.

Theorem 3.2. Let $\mathcal{H}^+$ be a self-dual cone generated by a finite set in an n -dimensional Euclidean space $\mathcal{H}$. If $A, B \in L(\mathcal{H})$ satisfy $A \geq O, B \geq O$ and $A \succeq B \succeq O$, then there exists a number $s > 0$ such that $A^x \succeq B^x \succeq O$ for all $x \in [s, \infty)$.

Proof. Suppose that $\mathcal{H}^+$ is a self-dual cone and

$$\mathcal{H}^+ = \{c_1 v_1 + \cdots + c_m v_m \mid c_1, \dots, c_m \geq 0, v_1, \dots, v_m \in \mathcal{H}^+\}$$

where $\{v_1, \dots, v_m\}$ is linearly independent. By the assumption we have $A^k - B^k \succeq O$ for $k = 1, 2, \dots$. Then $((A^k - B^k)v_i, v_j) \geq 0$ for all i, j . Put

$$f_{ij}(x) = ((A^x - B^x)v_i, v_j).$$

We write

$$A^x = U \begin{pmatrix} \alpha_1^x & & 0 \\ & \ddots & \\ 0 & & \alpha_n^x \end{pmatrix} U^{-1}, B^x = V \begin{pmatrix} \alpha_{n+1}^x & & 0 \\ & \ddots & \\ 0 & & \alpha_{2n}^x \end{pmatrix} V^{-1},$$

for $\alpha_i \geq 0$ and unitaries U, V . Let $\{\beta_1, \dots, \beta_\ell\}$ be the set of all distinct elements of $\{\alpha_1, \dots, \alpha_{2n}\}$. Then we can write as $f_{ij}(x) = a_1 \beta_1^x + \cdots + a_\ell \beta_\ell^x$. Since $f_{ij}(m) \geq 0$ for all $m = 1, 2, \dots$, it follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s', \infty)$. Hence there exists a number $s > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s, \infty)$ and all i, j . Choose any elements $v, v' \in \mathcal{H}^+$, which are expressed by

$$v = \sum_{i=1}^m c_i v_i, \quad v' = \sum_{i=1}^m c'_i v_i$$

for some $c_i, c'_i \geq 0$. It follows that

$$\begin{aligned} ((A^x - B^x)v, v') &= \sum_{i,j=1}^m ((A^x - B^x)c_i v_i, c'_j v_j) \\ &= \sum_{i,j=1}^m c_i c'_j f_{ij}(x) \geq 0 \end{aligned}$$

for all $x \in [s, \infty)$. This completes the proof.

4. THE CASE OF $M_n(\mathbb{C})^+$

let $M_n(\mathbb{R})$ (resp. $M_n(\mathbb{R})_s$) denote the set of all real (resp. real symmetric) $n \times n$ -matrices. The set of all real positive semi-definite matrices is denoted by $M_n(\mathbb{R})^+$, which is one of the most important self-dual cones in the operator theory or in the theory of operator algebras. We know many operators preserving $M_n(\mathbb{R})^+$ such as

$$\hat{A}: X \mapsto \sum_{k=1}^m {}^t A_k X A_k$$

for $X, A_1, \dots, A_m \in M_n(\mathbb{R})$, i.e., $\hat{A} \geq O$.

We first introduce a notation. We identify $M_n(\mathbb{C})$ with an n^2 -dimensional Euclidean space $\mathbb{C}^{n^2}$ by a bijective linear map

$$\nu : \begin{pmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{pmatrix} \in M_n(\mathbb{C}) \mapsto \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1n} \\ \vdots \\ \vdots \\ \xi_{n1} \\ \vdots \\ \xi_{nn} \end{pmatrix} \in \mathbb{C}^{n^2}.$$

Given a diagonal matrix A , we shall give a characterization for $A \geq O$ with respect to the cone $M_n(\mathbb{C})^+$.

Lemma 4.1. *Let A be an $n^2 \times n^2$ diagonal matrix with entries $\lambda = \{\lambda_{11}, \dots, \lambda_{1n}, \dots, \lambda_{n1}, \dots, \lambda_{nn}\}$ with $\lambda_{ij} \geq 0$, $i, j = 1, \dots, n$. Then the following conditions are equivalent:*

- (1) $A(\nu(M_n(\mathbb{C})^+)) \subset \nu(M_n(\mathbb{C})^+)$.
- (2) $\nu^{-1}(\lambda) \in M_n(\mathbb{C})^+$.

Proof. Let A be a diagonal matrix in the assumption. Choose an arbitrary element $\Xi = (\xi_{ij}) \in M_n(\mathbb{C})^+$. Then

$$A\nu(\Xi) = \begin{pmatrix} \lambda_{11} & & & \mathbf{0} \\ & \lambda_{12} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{11}\xi_{11} \\ \lambda_{12}\xi_{12} \\ \vdots \\ \lambda_{nn}\xi_{nn} \end{pmatrix}.$$

If $A\nu(\Xi) \in \nu(M_n(\mathbb{C})^+)$ for all $\Xi \in M_n(\mathbb{C})^+$, that is,

$$\nu^{-1}(A\nu(\Xi)) = \begin{pmatrix} \lambda_{11}\xi_{11} & \cdots & \lambda_{1n}\xi_{1n} \\ \vdots & & \vdots \\ \lambda_{n1}\xi_{n1} & \cdots & \lambda_{nn}\xi_{nn} \end{pmatrix} \in M_n(\mathbb{C})^+,$$

then

$$(\nu(\Xi), \lambda) = \sum_{i,j=1}^n \lambda_{ij}\xi_{ij} \geq 0.$$

This yields $\lambda \in \nu(M_n(\mathbb{C})^+)$ from the self-duality of $\nu(M_n(\mathbb{C})^+)$.

Conversely, let $\nu^{-1}(\lambda) \in M_n(\mathbb{C})^+$. Then for $\Xi \in M_n(\mathbb{C})^+$ the Shur product

$$\nu^{-1}(\lambda) \circ \Xi = \begin{pmatrix} \lambda_{11}\xi_{11} & \cdots & \lambda_{1n}\xi_{1n} \\ \vdots & & \vdots \\ \lambda_{n1}\xi_{n1} & \cdots & \lambda_{nn}\xi_{nn} \end{pmatrix}$$

belongs to $M_n(\mathbb{C})^+$. Hence $\nu^{-1}(A\nu(\Xi)) \in M_n(\mathbb{C})^+$. This completes the proof.

Theorem 4.2. Under the order with respect to the cone $\nu(M_n(\mathbb{C})^+)$, let A, B be $n^2 \times n^2$ matrices with $A \geq O, B \geq O$ and $A \supseteq B \supseteq O$. Suppose that both A and B are diagonalizable by a unitary U , and $U\nu(M_n(\mathbb{C})^+) = \nu(M_n(\mathbb{C})^+)$. Then there exists a number $s > 0$ such that $A^x \supseteq B^x \supseteq O$ for all $x \in [s, \infty)$.

Proof. Let C, D be diagonal matrices with $A = UCU^{-1}, B = UDU^{-1}$. Since for any elements $v, w \in \nu(M_n(\mathbb{C})^+)$

$$((A - B)v, w) = (U(C - D)U^{-1}v, w) = ((C - D)v, w),$$

we may assume that A, B are diagonal matrices. Put

$$A = \begin{pmatrix} \lambda_{1_1} & & & 0 \\ & \lambda_{1_2} & & \\ & & \ddots & \\ 0 & & & \lambda_{n_n} \end{pmatrix}, B = \begin{pmatrix} \mu_{1_1} & & & 0 \\ & \mu_{1_2} & & \\ & & \ddots & \\ 0 & & & \mu_{n_n} \end{pmatrix},$$

with $\lambda_{i_j} \geq 0, \mu_{i_j} \geq 0, 1 \leq i, j \leq n$. Then

$$\nu^{-1}(A^x - B^x) = \begin{pmatrix} \lambda_{1_1}^x - \mu_{1_1}^x & \cdots & \lambda_{1_n}^x - \mu_{1_n}^x \\ \vdots & & \vdots \\ \lambda_{n_1}^x - \mu_{n_1}^x & \cdots & \lambda_{n_n}^x - \mu_{n_n}^x \end{pmatrix}.$$

By Lemma 4.1 we obtain that $A^x - B^x \supseteq O$ holds if and only if $\nu^{-1}(A^x - B^x)$ is positive semi-definite. We shall here denote $f(x)$ by an arbitrary principal minor of $\nu^{-1}(A^x - B^x)$. Then $f(x)$ is expressed by a finite linear combination of x -th powers of positive numbers. By the assumption $A \supseteq B \supseteq O$, we have $A^m \supseteq B^m \supseteq O$ for $m = 1, 2, \dots$. This implies that $f(m) \geq 0$. It follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f(x) \geq 0$ for all $x \in [s', \infty)$. Consequently, all principal minors of $\nu^{-1}(A^x - B^x)$ are non-negative for all x more that a sufficiently large number. This completes the proof.

It is immediate in the above theorem that, when $B = O$, $A \supseteq O$ implies $A^x \supseteq O$ for all $x \geq 0$ in the case of $M_2(\mathbb{C})^+$. In the next remark we give the example that for a diagonal matrix $A \geq O$ $A^x \not\supseteq O$ for $0 \leq x < 1$, and $A^x \supseteq O$ for $x \geq 1$.

Remark. Put

$$A = \begin{pmatrix} 1 & & & & & & 0 \\ & \frac{1}{\sqrt{2}} & & & & & \\ & & \frac{1}{\sqrt{2}} & & & & \\ & & & \frac{1}{\sqrt{2}} & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & \frac{1}{\sqrt{2}} \\ 0 & & & & & & & 0 \\ & & & & & & & & 1 \end{pmatrix}.$$

Then

$$\nu^{-1}(A) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$

is positive semi-definite, so $A \succeq O$. We have

$$\begin{vmatrix} 1 & \left(\frac{1}{\sqrt{2}}\right)^x & \left(\frac{1}{\sqrt{2}}\right)^x \\ \left(\frac{1}{\sqrt{2}}\right)^x & 1 & 0 \\ \left(\frac{1}{\sqrt{2}}\right)^x & 0 & 1 \end{vmatrix} = 1 - 2\left(\frac{1}{2}\right)^x.$$

This implies immediately that $A^x \not\succeq O$ for $1 \leq x < 1$, and $A^x \succeq O$ for $x \geq 1$.

5. THE CASE OF $M_2(\mathbb{R})^+$

Theorem 5.1. (cf. [2]) *Let A and B be matricial representations of linear transformations on $\nu(M_2(\mathbb{R})_s)$ with a self-dual cone $\nu(M_2(\mathbb{R})^+)$. If A and B are positive semi-definite and $A \succeq B \succeq O$, then there exists a positive number s such that $A^x \succeq B^x \succeq O$ for all $x \in [s, \infty)$.*

Proof. We may use in the proof the notation ' $\succeq$ ' as follows: For $A, B \in M_4(\mathbb{R})$, $A \succeq B \succeq O$ means $(A - B)(\nu(M_2(\mathbb{R})^+)) \subset \nu(M_2(\mathbb{R})^+)$, though this relation in $M_4(\mathbb{R})$ does not satisfy the symmetric law of the axiom of an order. Suppose $A, B \in M_4(\mathbb{R})^+$ and $A \succeq B \succeq O$. Let $\{\alpha_1, \dots, \alpha_4\}$ be the eigenvalues of A and $\{\alpha_5, \dots, \alpha_8\}$ the eigenvalues of B . We then have

$$A^x = U \begin{pmatrix} \alpha_1^x & & 0 \\ & \ddots & \\ 0 & & \alpha_4^x \end{pmatrix} U^{-1}, B^x = V \begin{pmatrix} \alpha_5^x & & 0 \\ & \ddots & \\ 0 & & \alpha_8^x \end{pmatrix} V^{-1},$$

for some real orthogonal matrices U, V and $x > 0$. Put $C(x) = A^x - B^x$ for $x > 0$. Any element of $M_2(\mathbb{R})^+$ can be expressed as a convex combination of elements of the boundary of $M_2(\mathbb{R})^+$ in the subspace $M_2(\mathbb{R})^+ - M_2(\mathbb{R})^+$ of all real symmetric matrices. Hence, in order to prove $C(x)(\nu(M_2(\mathbb{R})^+)) \subset \nu(M_2(\mathbb{R})^+)$, it suffices to show that $C(x)\xi \in \nu(M_2(\mathbb{R})^+)$ and $C(x)\eta \in \nu(M_2(\mathbb{R})^+)$ for

$$\xi = \begin{pmatrix} 1 \\ b \\ b \\ b^2 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $b \in \mathbb{R}$.

Step (i): In this part, we shall show that $\nu^{-1}(C(x)\xi)$ is symmetric for all $x > 0$. We choose distinct eigenvalues $\{\beta_i\}$ of A and B such that $\beta_1 > \dots > \beta_\ell \geq 0$

($1 \leq \ell \leq 8$). Then

$$\nu^{-1}(C(x)\xi) = \begin{pmatrix} \sum_{k=1}^{\ell} \mu_k^{(1,1)}(b)\beta_k^x & \sum_{k=1}^{\ell} \mu_k^{(1,2)}(b)\beta_k^x \\ \sum_{k=1}^{\ell} \mu_k^{(2,1)}(b)\beta_k^x & \sum_{k=1}^{\ell} \mu_k^{(2,2)}(b)\beta_k^x \end{pmatrix}.$$

Since $C(1) \succeq O$, we have $C(m) \succeq O$ for all $m \in \mathbb{N}$. Hence $\nu^{-1}(C(m)\xi)$ is positive semi-definite. Hence a (1, 2)-component and a (2, 1)-component of this matrix are equal, i.e.,

$$\mu_1^{(1,2)}(b)\beta_1^m + \cdots + \mu_{\ell}^{(1,2)}(b)\beta_{\ell}^m = \mu_1^{(2,1)}(b)\beta_1^m + \cdots + \mu_{\ell}^{(2,1)}(b)\beta_{\ell}^m$$

for $m = 1, 2, \dots$. Since β_k are distinct, we have $\mu_k^{(1,2)} = \mu_k^{(2,1)}$ for all k . This yields immediately that the off-diagonal components of $\nu^{-1}(C(x)\xi)$ are equal.

Step (ii): Let

$$C(x) = [f_{ij}(x)]_{i,j=1}^4.$$

Here $f_{ij}(x)$ is expressed as a finite linear combination of β_k^x . Then

$$\begin{aligned} \nu^{-1}(C(x)\xi) &= \\ &= \begin{pmatrix} f_{11}(x) + (f_{12}(x) + f_{13}(x))b + f_{14}(x)b^2 & f_{21}(x) + (f_{22}(x) + f_{23}(x))b + f_{24}(x)b^2 \\ f_{31}(x) + (f_{32}(x) + f_{33}(x))b + f_{34}(x)b^2 & f_{41}(x) + (f_{42}(x) + f_{43}(x))b + f_{44}(x)b^2 \end{pmatrix}. \end{aligned}$$

In this part we first show that all diagonal components of $\nu^{-1}(C(x)\xi)$ are non-negative for all real numbers b , and all x more than a sufficiently large number (which is independent on b). Since by the assumption every diagonal component of $\nu^{-1}(C(m)\xi)$, $m = 1, 2, \dots$, is non-negative for all $b \in \mathbb{R}$, we have

$$f_{i4}(m) \geq 0, \quad m = 1, 2, \dots, i = 1, 4.$$

Suppose that $f_{i4}(x)$ is not identically 0. We then obtain from Lemma that $f_{i4}(x) > 0$ for all x more than a sufficiently large number. Hence every diagonal component of $\nu^{-1}(C(x)\xi)$ is expressed by

$$f_{i4}(x) \left(b + \frac{f_{i2}(x) + f_{i3}(x)}{2f_{i4}(x)} \right)^2 + \frac{g_i(x)}{4f_{i4}(x)}.$$

Here

$$g_i(x) = 4f_{i4}(x)f_{i1}(x) - (f_{i2}(x) + f_{i3}(x))^2.$$

Similarly, since $g_i(m) \geq 0$ for $m = 1, 2, \dots$, we obtain that $g_i(x) \geq 0$ for all x more than a sufficiently large number. The above inequality is valid in the case where $f_{i4}(x)$ is identically 0. Indeed, if $f_{i4}(x)$ is identically 0, then $f_{i2}(m) + f_{i3}(m) = 0$ holds for $m \in \mathbb{N}$. For, if $f_{i_0 2}(m_0) + f_{i_0 3}(m_0) \neq 0$ for some i_0 and m_0 , then the

infimum of a diagonal component of $\nu^{-1}(C(m_0)\xi)$ is $-\infty$. This contradicts the condition that $C(m) \succeq O$ holds for all $m \in \mathbb{N}$. By Lemma, $f_{i2}(x) + f_{i3}(x)$ is identically 0. In this case, it suffices to consider the function $f_{i1}(x)$ in the same way.

Next, we examine the determinant of $\nu^{-1}(C(x)\xi)$. Put

$$G = \det \nu^{-1}(C(x)\xi).$$

Then G is expressed as

$$G = G(b, x) = a_0(x)b^4 + a_1(x)b^3 + \cdots + a_4(x).$$

Here $a_i(x)$ is a finite linear combination of x -th powers of some positive numbers. By the assumption we have

$$G(b, m) \geq 0, -\infty < b < \infty, m = 1, 2, \dots.$$

Suppose that $a_0(x)$ is not identically 0. Then $a_0(x) > 0$ for all x for a sufficiently large number, since $a_0(x)$ satisfies the hypothesis in Lemma. Put

$$L(x) = \inf_{b \in \mathbb{R}} G(b, x).$$

Then $L(x)$ is given by the following formula:

$$L(x) = \min_{1 \leq j \leq 3} L_j(x),$$

where $L_j(x) = G(\operatorname{Re} b_j(x), x)$ and $b_j(x)$ are all roots of the cubic equation $G_b(b, x) = 0$ of b . Note that $b_j(x)$ are algebraically expressed by $a_i(x)$, and are continuous for all x more than a sufficiently large number. We must show the existence of a number $s > 0$ satisfying $L(x) \geq 0$ for all $x \in [s, \infty)$. Assume that there does not exist such a number s . Then for every natural number m there exists y_m with $y_m \geq m$ such that $L_{j_0}(y_m) < 0$ for some j_0 . Since $L_{j_0}(m) \geq 0$ for all $m \in \mathbb{N}$, there exists by the intermediate value theorem a positive sequence $\{x_m\}$ with $\lim_{m \rightarrow \infty} x_m = \infty$ such that

$$L_{j_0}(x_m) = 0, \quad m = 1, 2, \dots.$$

Let $\hat{L}_{j_0}(x)$ be a polynomial of $a_i(x)$ such that the set of all zeros of $\hat{L}_{j_0}(x)$ includes the set of all zeros of $L_{j_0}(x)$. Hence there exists an unbounded increasing sequence of zeros of $\hat{L}_{j_0}(x)$. Since $\hat{L}_{j_0}(x)$ is a finite linear combination of x -th powers of some positive numbers, it follows from Lemma that $\hat{L}_{j_0}(x)$ is identically 0. This is a contradiction. On the other hand, if $a_0(x)$ is identically 0, then $a_1(x)$ is also identically 0 by the argument in the former part of Step (ii). In this case we also obtain the same result.

Similarly, we obtain the desired properties for η . This completes the proof.

The next remark shows that Theorem does not always hold for all positive number x . We shall give the example that there exists a 4×4 positive semi-definite matrix A with $A \succeq O$ satisfying $A^x \not\succeq O$ for $x \in [0, 1)$.

Remark. Note that $\nu(M_2(\mathbb{R})^+)$ is isometrically isomorphic to a circular cone

$$\mathcal{H}^+ = \left\{ \xi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid F(\xi) = x^2 + y^2 - z^2 \leq 0, z \geq 0 \right\}.$$

Consider the following positive semi-definite matrix A :

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Then $A \succeq O$. In fact, we have $A^\alpha \succeq O$ for all $\alpha \geq 1$. To see it, it suffices to examine that for $\eta(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$ we have $F(A^\alpha \eta(\theta)) \leq 0$ as follows:

$$\begin{aligned} 6A^\alpha \eta(\theta) &= \begin{pmatrix} 3^\alpha + 3 & 3^\alpha - 3 & 2 \cdot 3^\alpha \\ 3^\alpha - 3 & 3^\alpha + 3 & 2 \cdot 3^\alpha \\ 2 \cdot 3^\alpha & 2 \cdot 3^\alpha & 4 \cdot 3^\alpha \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} (3^\alpha + 3) \cos \theta + (3^\alpha - 3) \sin \theta + 2 \cdot 3^\alpha \\ (3^\alpha - 3) \cos \theta + (3^\alpha + 3) \sin \theta + 2 \cdot 3^\alpha \\ (2 \cdot 3^\alpha) \cos \theta + (2 \cdot 3^\alpha) \sin \theta + 4 \cdot 3^\alpha \end{pmatrix} \end{aligned}$$

and so

$$\begin{aligned} F(6A^\alpha \eta(\theta)) &= 18 - 10 \cdot 3^{2\alpha} - (4 \cdot 3^{2\alpha} + 36) \cos \theta \sin \theta - 8 \cdot 3^{2\alpha} (\cos \theta + \sin \theta) \\ &= -(36 + 4 \cdot 3^{2\alpha}) (\cos \theta + 1) (\sin \theta + 1) - (4 \cdot 3^{2\alpha} - 36) (2(\cos \theta + \sin \theta) + 3) \\ &\leq 0 \end{aligned}$$

for $\alpha \geq 1$ and $0 \leq \theta \leq 2\pi$. On the other hand, we shall show that for every $\alpha \in [0, 1)$ there exists a real number θ_0 such that $F(6A^\alpha \eta(\theta_0)) > 0$. Indeed, one can choose θ_0 satisfying

$$\cos \theta_0 + \sin \theta_0 = \sqrt{2} \sin \left(\theta_0 + \frac{\pi}{4} \right) = -\frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}},$$

since

$$0 < \frac{2 \cdot 3^{2\alpha}}{\sqrt{2}(9 + 3^{2\alpha})} \leq \frac{1}{\sqrt{2}}$$

for $0 \leq \alpha \leq 1$. Taking a square of both sides of the above equalities, we have

$$1 + 2 \cos \theta_0 \sin \theta_0 = \frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2}.$$

It follows that for $0 \leq \alpha < 1$

$$\begin{aligned} F(6A^\alpha \eta(\theta_0)) &= 18 - 10 \cdot 3^{2\alpha} - 2(9 + 3^{2\alpha}) \left(\frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2} - 1 \right) + 8 \cdot 3^{2\alpha} \cdot \frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}} \\ &= \frac{36}{9 + 3^{2\alpha}} (9 - 3^{2\alpha}) > 0, \end{aligned}$$

from which we have $A^\alpha \not\geq O$.

Finally, we obtain immediately the following theorem, which is understood to be a matrix version of Lemma, reviewing the proof of Theorem 5.1:

Theorem 5.2. *Let $A_1, \dots, A_n$ be positive semi-definite matricial representations of linear transformations on $\nu(M_2(\mathbb{R})_s)$ with a self-dual cone $\nu(M_2(\mathbb{R})^+)$, and $a_1, \dots, a_n$ be real numbers. Suppose that*

$$a_1 A_1^m + \dots + a_n A_n^m \geq O$$

holds for $m = 1, 2, \dots$. Then there exists $s > 0$ such that

$$a_1 A_1^x + \dots + a_n A_n^x \geq O$$

for all $x \in [s, \infty)$.

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