

A NOTE ON GT-ADMISSIBLE VARIETIES

TOMOHIDE TERASOMA (UNIV. OF TOKYO)

1. CONTENTS

In this paper, we introduce a notion of *GT*-admissible varieties. Roughly speaking, it is a (semi-)simplicial object in the category of varieties $\mathcal{M}_{0,n}, \Delta^*, \dots$ and their products with tangential points. We introduce a cohomology theory of *GT*-admissible varieties. To obtain patching varieties, we construct higher homotopy for complexes in the sense of Hanamura. It seems very possible to reconstruct this cohomology theory using a formulation of A_∞ -category. All constructions in this paper can be done in the setting of more general Tannakian category.

2. CATEGORY (*Basic*)

2.1. Tangential points. In this subsection, we define a category (*Basic*) to define the set of associator and Grothendieck-Teichmüller group. The object of (*Basic*) consists of three objects Δ^* , $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Formally speaking, Δ^* , $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ are finite sets, called tangential points. For example, the most easiest one Δ^* consists of two tangential points $\{+, -\}$. The tangential points of $\mathcal{M}_{0,4}$ consist of $\{\vec{01}, \vec{0\infty}, \dots\}$ (all together 6 points). The tangential points of $\mathcal{M}_{0,5}$ consist of plane trivalent tree with five end points up to mirror. (Plane tree is equipped with a cyclic ordering for each vertex.) Therefore the number of tangential points is 60. Tangential points are expressed geometrically as follows.

The symbol $\mathcal{M}_{0,5}$ is usually used as a moduli space of genus zero curves with marked 5 points. Mumford and Knudsen defined the stable compactification $\overline{\mathcal{M}}_{0,5}$ of the moduli space $\mathcal{M}_{0,5}$ by adding a normal crossing divisor (=1dimensional) in $\overline{\mathcal{M}}_{0,5}$. There exist 15 crossing points in the boundary divisor. Let p be a crossing point and U_p be a small neighbourhood of p . The intersection $U_p \cap \mathcal{M}_{0,5}(\mathbb{R})$ consists of 4 connected components. Each component corresponds one to one to tangential point of $\mathcal{M}_{0,5}$. In the following context, $\mathcal{M}_{0,5}$ represents the set of these 60 tangential points and forget the geometric meaning of $\mathcal{M}_{0,5}$.

2.2. Two theories of fundamental groups. There exist two theories of fundamental groups, topological fundamental group and de Rham fundamental group. For $X \in (\textit{Basic})$ and $p \in X$, the topological fundamental group $\pi_1(X, p)$ can be defined purely combinatorially. For example, the group $\pi_1(\Delta^*, +)$ is generated by a positive generator θ . More generally, for $p, q \in X$,

we can define the set of path $\pi_1(X, p, q)$ connecting p and q up to homotopy. We can define the $\mathbb{Q}$ -linear span $\mathbb{Q}[\pi_1(X, p, q)]$ of $\pi_1(X, p, q)$, which is a left $\mathbb{Q}[\pi_1(X, q)]$ (resp. right $\mathbb{Q}[\pi_1(X, p)]$) module of rank one and the completion of $\mathbb{Q}[\pi_1(X, p, q)]$ with respect to the augmentation ideal of $\mathbb{Q}[\pi_1(X, q)]$ is denoted as $\mathcal{U}^B(X, p, q)$. There exist "multiplication"

$$\mathcal{U}^B(X, q, r) \otimes \mathcal{U}^B(X, p, q) \rightarrow \mathcal{U}^B(X, p, r)$$

which satisfies the axiom of algebroid, i.e. associative multiplication $\gamma\eta$ is defined only if the starting point of γ is equal to the ending point of η .

There is a theory of de Rham fundamental group $\pi_1^{DR}(X, p)$ of X with a base point p . For $p, q \in X$, there is a canonical isomorphism

$$\pi_1^{DR}(X, p) \simeq \pi_1^{DR}(X, q)$$

in de Rham fundamental group theory, which is a different point from the Betti theory. The group $\pi_1^{DR}(X, p)$ is defined as the set of group like elements of a Hopf algebra $\mathcal{U}^{DR}(X, p)$. There exists a canonical isomorphism

$$\mathcal{U}^{DR}(X, p) \simeq \mathcal{U}^{DR}(X, q)$$

as Hopf algebras. For example, we have $\mathcal{U}(\Delta^*, +) = \mathbb{Q}[[e]]$, where $e = \text{Res}_0$.

2.3. Functoriality. We introduce morphisms in (*Basic*). Morphisms consist of

1. An inclusion $:\Delta^* \rightarrow \mathcal{M}_{0,4}$. The tangential points $+$ and $-$ goes to $\overrightarrow{ab}$ and $\overrightarrow{ac}$, where $\{a, b, c\} = \{0, 1, \infty\}$. (Therefore altogether, 6 morphism of this type.)
2. An infinitesimal inclusion: $\mathcal{M}_{0,4} \rightarrow \mathcal{M}_{0,5}$. We will not write down precisely here. There are 12 connected components of $\mathcal{M}_{0,5}(\mathbb{R})$ and each connected component is bounded by 5 divisors, i.e. it is a pentagon. There exists a unique infinitesimal inclusion by which a connected component of $\mathcal{M}_{0,4}(\mathbb{R})$ goes to an edge of pentagon.
3. A composite of type (1) and type (2).

The category of pointed objects and two pointed objects in (*Basic*) are written as $(*\text{Basic})$ and $(**\text{Basic})$, respectively.

Let $\# = B, DR$. The correspondence $(X, p) \rightarrow \mathcal{U}^\#(X, p)$ and $(X, p, q) \rightarrow \mathcal{U}^\#(X, p, q)$ form functors

$$\mathcal{U}^\# : (*\text{Basic}) \rightarrow (\text{Vec}_{\mathbb{Q}}) \quad \text{and} \quad \mathcal{U}^\# : (**\text{Basic}) \rightarrow (\text{Vec}_{\mathbb{Q}}).$$

Moreover these functors give rise to a functor

$$\mathcal{U}^\# : (\text{Basic}) \rightarrow (\text{Hopf}_{\mathbb{Q}})$$

from the category (*Basic*) to the category of Hopf algebroids over $\mathbb{Q}$.

Theorem 2.1 (Drinfeld). *There exists a functorial isomorphism*

$$\rho : \mathcal{U}^B \otimes \mathbb{C} \simeq \mathcal{U}^{DR} \otimes \mathbb{C}$$

where

$$(2.1) \quad \rho(\log \theta) = 2\pi i e \text{ for } \rho : \mathcal{U}^B(\Delta, +) \otimes \mathbb{C} \simeq \mathcal{U}^{DR}(\Delta, +) \otimes \mathbb{C}.$$

Hodge theory gives a functorial isomorphism. Less trivial part is the compatibility for infinitesimal inclusions.

2.4. Associator, Grothendieck-Teichmüller group.

Definition 2.2. 1. The set of functorial isomorphisms of $\mathbb{C}$ -Hopf algebras

$$Ass = Isom_{Hopf_{\mathbb{C}}}(\mathcal{U}^B, \mathcal{U}^{DR})$$

is called the set of associators. We define $Ass_{2\pi i}$ as

$$Ass_{2\pi i} = \{\rho \in Ass \mid \rho \text{ satisfies the condition (2.1)}\}$$

2. Let $\# = B, DR$. The set of functorial isomorphisms of $\mathbb{Q}$ -Hopf algebras

$$GT^{\#} = Isom_{Hopf_{\mathbb{Q}}}(\mathcal{U}^{\#}, \mathcal{U}^{\#})$$

is called the $\#$ -Grothendieck-Teichmüller group. We define $GT_1^{\#}$ as

$$GT_1^{\#} = \{g \in GT^{\#} \mid g(\log \theta) = \log \theta \text{ (if } \# = B) \text{ } g(e) = e \text{ (if } \# = DR) \\ \text{for } g : \mathcal{U}^{\#}(\Delta^*, +) \rightarrow \mathcal{U}^{\#}(\Delta^*, +)\}$$

We have an exact sequence

$$1 \rightarrow GT_1^{\#} \rightarrow GT^{\#} \rightarrow G_m \rightarrow 1.$$

The set Ass and $Ass_{2\pi i}$ is a left (resp. right) principal homogeneous space under the group $GT^{DR}(\mathbb{C})$ and $GT_1^{DR}(\mathbb{C})$, $(GT^B(\mathbb{C})$ and $GT_1^B(\mathbb{C}))$, respectively. The group $GT_1^{\#}$ is a nilpotent Lie group and its Lie algebra is denoted as $\mathcal{GT}_1^{\#}$. By the definition of GT^B , $\mathcal{U}^B(\mathcal{M}_{0,4})$ and $\mathcal{U}^B(\mathcal{M}_{0,5})$ are representations of GT^B .

2.5. Category ($Fund$). We define a category ($Fund$). The object of the category ($Fund$) consists of

1. $\mathcal{M}_{0,i}$ for $i \geq 4$,
2. $\Delta^n - (\text{big diagonal})$, $(\Delta^*)^n - (\text{big diagonal})$, and
3. their products.

The morphisms consist of inclusions, infinitesimal inclusions and certain projections. We define categories $(*Fund)$ and $(**Fund)$ by pointed and two pointed objects of ($Fund$). As in ($Basic$) case, we can define two functors $\mathcal{U}^B$ and $\mathcal{U}^{DR}$. (For an object $(X, p) \in (*Fund)$, $\mathcal{U}^B(X, p)$ is the completion of the group algebra of the fundamental group with respect to the augmentation ideal.) The following theorem is due to Ihara and Lochak.

Theorem 2.3. Let $\# = B$ or DR . The action of $GT^{\#}$ on the functor $\mathcal{U}^{\#} : (**Basic) \rightarrow (Hopf_{\mathbb{Q}})$ extends uniquely to the action on the functor $\mathcal{U}^{\#} : (**Fund) \rightarrow (Hopf_{\mathbb{Q}})$.

This principle is known as MacLane coherence principle.

3. DIFFERENTIAL GRADED ALGEBRA

3.1. From Hopf algebroid to differential graded algebra. Let $\mathcal{U}$ be a $\mathbf{Q}$ -Hopf algebroid with an augmentation $\epsilon : \mathcal{U} \rightarrow \mathbf{Q}$ over a set X , which is complete for the topology defined by the augmentation ideal $I = \text{Ker}(\epsilon)$. In this paper, the coproduct Δ of the Hopf algebroid is always cocommutative and coassociative. Therefore the groupoid like element in $\mathcal{U}$ forms a pro-unipotent groupoid. We introduce linear topology on $\mathcal{U}$ and assume that $\mathcal{U}$ is compact for the topology. In particular, $\mathcal{U}/I^n$ is a finite dimensional Hopf algebroid. Functors Hom , $\otimes$ are always considered in the category of locally compact vector spaces.

Definition 3.1. 1. Let $p, q \in X$. We define a complex $K^\bullet(\mathcal{U})_{p,q}$ by

$$\begin{array}{ccccc} \cdots & \rightarrow & \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} & \rightarrow & \mathcal{U}_{p,q} \\ & & a \otimes b \otimes c & \mapsto & \begin{pmatrix} \mathcal{U}_{p,q} \otimes \mathcal{U}_{p,q} \\ \epsilon(a)b \otimes c - \epsilon(b)a \otimes c \\ + \epsilon(c)a \otimes b \\ a \otimes b \end{pmatrix} \mapsto \epsilon(a)b - \epsilon(b)a \end{array}$$

Degree of the complex is given by $K^{-i}(\mathcal{U}) = \mathcal{U}^{\otimes i+1}$.

2. We define the n -coproduct $\Delta^{(n)}$ by the composite

$$(1^{\otimes n-2} \otimes \Delta) \circ \cdots (1 \otimes \Delta) \circ \Delta : \mathcal{U}_{p,q} \rightarrow \mathcal{U}_{p,q} \otimes \cdots \otimes \mathcal{U}_{p,q}.$$

Via this coproduct, $\mathcal{U} \otimes \cdots \otimes \mathcal{U}$ is a two sided $\mathcal{U}$ module. Via this $\mathcal{U}$ structures, the differentials in the complex $K^\bullet(\mathcal{U})$ are homomorphisms of two sided $\mathcal{U}$ modules.

Proposition 3.2. The cohomologies of the complex is

$$H^{-i}(K^\bullet(\mathcal{U})_{p,q}) = \begin{cases} \mathbf{Q} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Proof. Let $p \in X$ and G_p be the group like element in $\mathcal{U}_p$. We define a simplicial complex Y as follows: the set of n -simplices is $\{(g_0, \dots, g_n) \mid g_i \in G\}$. Since G_p is a pro- $\mathbf{Q}$ -unipotent group, the completion of the chain complex of Y is canonically isomorphic to $K^\bullet(\mathcal{U})_p$ and the completion of the cohomologies is isomorphic to the cohomology of the completion for the chain complex of Y . $\square$

Definition 3.3 (Differential graded algebra). 1. Let $p, q \in X$. We define

$$\Omega^i(\mathcal{U})_p = \text{Hom}_{(\text{left})\mathcal{U}_q}(K^{-i}(\mathcal{U}_{p,q}), \mathbf{Q}).$$

Note that this complex does not depend on the choice of q . Using the transpose of the differential of $K^\bullet$, we have a complex

$$\Omega^\bullet(\mathcal{U})_p : 0 \rightarrow \Omega^0(\mathcal{U})_p \rightarrow \Omega^1(\mathcal{U})_p \rightarrow \Omega^2(\mathcal{U})_p \rightarrow \cdots$$

There exists a left action of $\mathcal{U}_p$ on $\Omega^\bullet$ arising from the right action of $\mathcal{U}_p$ on $K^\bullet(\mathcal{U})_{p,q}$.

2. We introduce a product structure on $\Omega^\bullet(\mathcal{U})_p$. Let ω and η be elements of $\Omega^i(\mathcal{U})_p$ and $\Omega^j(\mathcal{U})_p$, respectively. We define $\omega \cdot \eta \in \Omega^{i+j}(\mathcal{U})_p$ by

$$(\omega \cdot \eta)(a_0 \otimes \cdots \otimes a_{i+j}) = (\omega \otimes \eta)(a_0 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_{i+j}).$$

Then this multiplication is associative and we have $d(a \cdot b) = da \cdot b + (-1)^{\deg(a)} a \cdot db$.

Proposition 3.4. *The cohomology $H^i(\Omega^\bullet(\mathcal{U})_p)$ of $\Omega^\bullet(\mathcal{U})_p$ is equal to $H^i(G, \mathbb{Q})$. The induced right action of $\mathcal{U}_p$ is trivial. i.e. the action of $\mathcal{U}_p$ factors through $\epsilon : \mathcal{U}_p \rightarrow \mathbb{Q}$.*

Proof. Let Y be the simplicial complex defined as before. Since the group G is pro- $\mathbb{Q}$ -nipotent, the action of G on Y is fixed point free. Therefore the quotient space $G \backslash Y$ is $K(G, 1)$ space. The cochain complex of $G \backslash Y$ is equal to $\Omega^\bullet(\mathcal{U})_p$. The triviality of the action of $\mathcal{U}_p$ will be explained later. $\square$

3.2. Differential graded algebroid structure on $K^\bullet$. Let $\mathcal{U}$ be a Hopf algebroid over a set X , p, q two points in X and $K^\bullet(\mathcal{U})_{p,q}$ be the complex defined in 3.1. We introduce an associative product structure

$$K^\bullet(\mathcal{U})_{q,r} \otimes K^\bullet(\mathcal{U})_{p,q} \rightarrow K^\bullet(\mathcal{U})_{p,r},$$

for $p, q, r \in X$.

Definition 3.5. *Let a, b be elements in $\mathbb{N}$ and set $k = a + b$. Minimal path connecting $(0, 0)$ and (a, b) is a sequence of $(a_i, b_i) \in \mathbb{N}^2$ ($i=0, \dots, k$) such that*

1. *either $a_{i+1} = a_i + 1, b_{i+1} = b_i$, or $a_{i+1} = a_i, b_{i+1} = b_i + 1$, and*
2. *$a_0 = 0, b_0 = 0$ and $a_k = a, b_k = b$.*

for $0 \leq i \leq k - 1$. The set of minimal path connecting $(0, 0)$ and (a, b) is denoted by $MP_{a,b}$. For a minimal path $p = (a_i, b_i)_i \in MP_{a,b}$, the number $\sum_{i=0}^a a_i - (a(a+1)/2)$ is called the volume of the path and denoted by $\text{vol}(p)$. We define the signature $\text{sign}(p)$ of a path p by $(-1)^{\text{vol}(p)}$.

Let p, q, r be elements in X . We define a dot product

$$* \cdot * : K^a(\mathcal{U})_{q,r} \otimes K^b(\mathcal{U})_{p,q} \rightarrow K^{a+b}(\mathcal{U})_{p,r}.$$

Let $g_0, \dots, g_a, h_0, \dots, h_b$ be group(oid) like elements in $\mathcal{U}$. Using these elements the product is defined by

$$\begin{aligned} & (g_0 \otimes \cdots \otimes g_a) \cdot (h_0 \otimes \cdots \otimes h_b) \\ &= \sum_{p=(a_i, b_i) \in MP_{a,b}} \text{sign}(p) (g_{a_0} h_{b_0} \otimes g_{a_1} h_{b_1} \otimes \cdots \otimes g_{a_k} h_{b_k}) \end{aligned}$$

We can describe the above product using Δ and the multiplication of $\mathcal{U}$. Then we have

$$d(v \cdot w) = dv \cdot w + (-1)^a v \cdot dw$$

for $v \in K^a(\mathcal{U})_{q,r}, w \in K^b(\mathcal{U})_{p,q}$. The product is associative, i.e. for $x \in K^\bullet(\mathcal{U})_{r,s}, y \in K^\bullet(\mathcal{U})_{q,r}, z \in K^\bullet(\mathcal{U})_{p,q}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We define an action of $K^\bullet(\mathcal{U})_{p,q}$ from $\Omega^\bullet(\mathcal{U})_p$ to $\Omega^\bullet(\mathcal{U})_q$, using the dot product structure on $K^\bullet(\mathcal{U})$ as follows.

$$\begin{aligned} * \cdot * : K^\bullet(\mathcal{U})_{p,q} \otimes \Omega^\bullet(\mathcal{U})_p &\rightarrow \Omega^\bullet(\mathcal{U})_q \\ a \otimes \varphi &\mapsto a \cdot \varphi = (b \rightarrow \varphi(b \cdot a)) \end{aligned}$$

This homomorphism is a homomorphism of complex. The the associativity for “ $\cdot$ ” implies the associativity of the action of $K^\bullet$ on $\Omega^\bullet$.

4. PATCHING BY CECH COMPLEX OF COMPLEXES

4.1. Cubic flag and n -homotopy. In this subsection, we define an n -homotopy $h(S)$ for a finite totally ordered set S , which will be used in the next subsection to defined n -homotopies for patching complexes. Let S be a totally ordered set with finite n elements. Cube $\square^S$ over S is defined by $\{0, 1\}^S$. If $S = \{1, \dots, n\}$, the cube over S is denoted as $\square^n$. A minimal path from $(0, \dots, 0) \in \square^S$ to $(1, \dots, 1) \in \square^S$ (minimal path of S for simplicity) is defined by a permutation $\sigma = (\sigma_1, \dots, \sigma_n)$ of S . A minimal path can be regarded as a sequence $v_0, \dots, v_n$ of $\square^S$ defined by

1. $v_0 = (0, \dots, 0)$,
2. $v_i = v_{i-1} + e_{\sigma_i}$, where e_σ is the elementary unit vector for the σ component.

Let $\Sigma(S)$ be the set of minimal path of S expressed as permutations of S and $W(S)$ be the vector space spanned by $\Sigma(S)$ over $\mathbb{Q}$.

Definition 4.1 (Dot product). *Let T_1 be a subset in S and $T_2 = S - T_1$. For elements g and h in $\Sigma(T_1)$ and $\Sigma(T_2)$, $(gh) \in \Sigma(S)$ denotes the composite of words g and h . We define a product*

$$\begin{aligned} W(T_1)^{\otimes(p+1)} \otimes W(T_2)^{\otimes(q+1)} &\rightarrow W(S)^{\otimes(p+q+1)} \\ (g_0, \dots, g_p) \otimes (h_0, \dots, h_q) &\mapsto g \cdot h \end{aligned}$$

by

$$g \cdot h = \sum_{p=(a_i, b_i) \in MP(p, q)} \text{sign}(p)(g_{a_0} h_{b_0}) \otimes \dots \otimes (g_{a_{p+q}} h_{b_{p+q}}),$$

where g_i and h_j are elements in $\Sigma(T_1)$ and $\Sigma(T_2)$.

We define $d : W(S)^{\otimes n} \rightarrow W(S)^{\otimes(n-1)}$ by

$$\begin{aligned} W(S)^{\otimes n} &\rightarrow W(S)^{\otimes(n-1)} \\ (v_1 \otimes \dots \otimes v_n) &\mapsto \sum_{k=1}^n (-1)^{k-1} v_1 \otimes \dots \otimes v_{k-1} \otimes v_{k+1} \otimes \dots \otimes v_n, \end{aligned}$$

for $v_1, \dots, v_n \in \Sigma(S)$.

Definition 4.2 (n -Homotopy for an ordered set S). *Let S be a finite totally ordered set with $\#S = n \geq 1$. We set $\text{tot}(S) \in \Sigma(S)$ by $\text{tot}(S) = (g_1, \dots, g_n)$, where $S = \{g_1, \dots, g_n\}$ and $g_1 < \dots < g_n$. We define n -homotopy $h(S)$ of S by the induction on $\#S$.*

1. If $S = \{a\}$, then we define

$$h(S) = (a).$$

2. If $S = \{a, b\}$ and $a < b$, then we define

$$h(S) = (ab) \otimes (ba).$$

3. If $\#S \geq 3$, then we define

$$h(S) = \text{tot}(S) \otimes \left(\sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} \text{sign}(S, T) h(T) \cdot h(S - T) \right).$$

where $\text{sign}(S, T)$ is the signature of the permutation $\left(\begin{smallmatrix} \text{tot}(S) \\ \text{tot}(T), \text{tot}(S - T) \end{smallmatrix} \right)$.

Proposition 4.3. We have

$$d \left(\sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} \text{sign}(S, T) h(T) \cdot h(S - T) \right) = 0$$

and

$$d(h(S)) = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#T} \text{sign}(S, T) h(T) \cdot h(S - T).$$

4.2. First step for patching differential graded algebras. Let $\mathcal{K}$ be a finite simplicial complex with a total order on the set of vertices. Then $\mathcal{K}$ is a category whose objects and morphisms are given by simplices and their inclusions. Let X be a contravariant functor from the complex $\mathcal{K}$ to the category of Hopf algebroid, i.e. for a simplex σ in $\mathcal{K}$, X_σ is a finite set and $\mathcal{U}(\sigma)$ is a Hopf algebroid over the set X_σ . For a face $\tau < \sigma$ of σ , we have a map of finite set $X_\sigma \rightarrow X_\tau$ and a homomorphism of Hopf algebroid $\mathcal{U}(\sigma) \rightarrow \mathcal{U}(\tau)$ compatible with the composite for inclusions $\sigma_1 < \sigma_2 < \sigma_3$ of simplices.

Then we have a differential graded algebroid $K^\bullet(\mathcal{U}_\sigma)$ over X_σ and a family of differential graded algebra $\Omega^\bullet(\mathcal{U}_\sigma)_x$ for $x \in X_\sigma$. The action of $K^\bullet(\mathcal{U}_\sigma)$ on $\Omega^\bullet(\mathcal{U}_\sigma)$ is introduced in the last section. A set of base points $B = (b_\sigma)_\sigma$ with $b_\sigma \in X_\sigma$ is called a system of base points of X . In this subsection, we define a complex $\Omega^\bullet(X) = \Omega^\bullet(X)_B$ for a system of base points B .

Let σ be a simplex of $\mathcal{K}$. We define a full subcategory $\mathcal{K}_\sigma$ of $\mathcal{K}$ defined by

$$\text{ob}(\mathcal{K}_\sigma) = \{\tau \mid \sigma < \tau\}.$$

All the morphisms in $\mathcal{K}_\sigma$ of codimension one is denoted as $\text{Mor}(\sigma)$, i.e.

$$\text{Mor}(\sigma) = \{(\alpha < \beta) \mid \sigma < \alpha < \beta, \dim \alpha + 1 = \dim \beta\}.$$

We introduce a total order in $\text{Mor}(\sigma)$. For example lexico graphic order for (α, β) . For a morphism $(\alpha < \beta) \in \text{Mor}(\sigma)$, the complex $K^\bullet(\mathcal{U}(\alpha))_{b_\alpha, b_\beta | x_\alpha}$ is denoted by $K_{\alpha, \beta}^\bullet$ for short. Then we have a natural homomorphism $\mu_{\alpha, \beta}$ of complexes:

$$\mu_{\alpha, \beta} : K_{\alpha, \beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\alpha))_{b_\alpha} \rightarrow \Omega^\bullet(\mathcal{U}(\alpha))_{b_\beta | \alpha} \rightarrow \Omega^\bullet(\mathcal{U}(\beta))_{b_\beta}.$$

Definition 4.4. 1. For a simplex σ in $\mathcal{K}$, we define a complex $\tilde{\Omega}^\bullet(\sigma)$ by

$$\tilde{\Omega}^\bullet(\sigma) = \left(\bigotimes_{(\alpha < \beta) \in \text{Mor}(\sigma)} K_{\alpha, \beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \right).$$

The differential of $\tilde{\Omega}^\bullet(\sigma)$ is given as the differential on the tensor product. As usual, using an isomorphism

$$\begin{aligned} K^\bullet \otimes L^\bullet &\simeq L^\bullet \otimes K^\bullet \\ a \otimes b &\mapsto (-1)^{\deg a \cdot \deg b} b \otimes a, \end{aligned}$$

the differential is independent of the order of tensor product.

2. Let $\sigma < \tau$ be a homomorphism in $\mathcal{K}$ of codimension one. We define $\delta_{\sigma,\tau} : \tilde{\Omega}^\bullet(\sigma) \rightarrow \tilde{\Omega}^\bullet(\tau)$:

$$\begin{aligned} &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\sigma)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ &\quad \parallel \\ &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \bigotimes_{\substack{(\alpha < \beta) \in \text{Mor}(\sigma) - \text{Mor}(\tau) \\ (\alpha < \beta) \neq (\sigma < \tau)}} K_{\alpha,\beta}^\bullet \otimes K_{\sigma,\tau}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ &\quad \downarrow \\ &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \\ \text{by} & \\ &(\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \otimes \bigotimes_{\alpha,\beta} b_{\alpha,\beta} \otimes c_{\sigma,\tau} \otimes \omega_{b_\sigma}) \\ &\quad \downarrow \\ &\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \cdot \prod_{\alpha,\beta} \epsilon(b_{\alpha,\beta}) \cdot (c_{\sigma,\tau} \cdot \omega_{b_\sigma}). \end{aligned}$$

The following proposition is a direct consequence from the definition of $\delta_{\sigma,\tau}$.

Proposition 4.5. Let $\sigma < \gamma_1 < \tau$ and $\sigma < \gamma_2 < \tau$ be distinct sequence of codimension one simplices in $\mathcal{K}$. We define ∂ as

$$\begin{aligned} \partial : K_{\gamma_1,\tau}^\bullet \otimes K_{\sigma,\gamma_1}^\bullet \otimes K_{\gamma_2,\tau}^\bullet \otimes K_{\sigma,\gamma_2}^\bullet &\rightarrow K_{\sigma,\tau}^\bullet \\ x \otimes y \otimes z \otimes w &\mapsto \epsilon(z)\epsilon(w)x \cdot y - \epsilon(x)\epsilon(y)z \cdot w \end{aligned}$$

Then $\delta_{\gamma_1,\tau}\delta_{\sigma,\gamma_1} - \delta_{\gamma_2,\tau}\delta_{\sigma,\gamma_2}$ is equal to

$$\begin{aligned} &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\sigma)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ &\quad \parallel \\ &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \bigotimes_{\substack{(\alpha < \beta) \in \text{Mor}(\sigma) - \text{Mor}(\tau) \\ (\alpha < \beta) \neq (\sigma < \gamma_1), (\gamma_1 < \tau), \\ (\sigma < \gamma_2), (\gamma_2 < \tau)}} K_{\alpha,\beta}^\bullet \\ &\quad \otimes K_{\gamma_1,\tau}^\bullet \otimes K_{\sigma,\gamma_1}^\bullet \otimes K_{\gamma_2,\tau}^\bullet \otimes K_{\sigma,\gamma_2}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\ &\quad \downarrow \\ &\bigotimes_{(\alpha < \beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \end{aligned}$$

is given by

$$\begin{aligned} &(\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \otimes \bigotimes_{\alpha,\beta} b_{\alpha,\beta} \otimes x \otimes y \otimes z \otimes w \otimes \omega_{b_\sigma}) \\ &\quad \downarrow \\ &\bigotimes_{\alpha,\beta} a_{\alpha,\beta} \cdot \prod_{\alpha,\beta} \epsilon(b_{\alpha,\beta}) \cdot (\partial(x \otimes y \otimes z \otimes w) \cdot \omega_{b_\sigma}). \end{aligned}$$

This proposition shows that the following diagram does not commute in general.

$$\begin{array}{ccc} \tilde{\Omega}^\bullet(\sigma) & \xrightarrow{\delta_{\sigma,\gamma_1}} & \tilde{\Omega}^\bullet(\gamma_1) \\ \delta_{\sigma,\gamma_2} \downarrow & & \downarrow \delta_{\gamma_1,\tau} \\ \tilde{\Omega}^\bullet(\gamma_2) & \xrightarrow{\delta_{\gamma_2,\tau}} & \tilde{\Omega}^\bullet(\tau) \end{array}$$

In the next subsection, we show that this diagram commutes up to homotopy. Moreover we show the existence of higher homotopy to define a total complex of $\{\Omega^\bullet(\sigma)\}_{\sigma \in \mathcal{K}}$.

4.3. Higher homotopy for differential graded algebras. Let $\mathcal{K}$ be a simplicial complex and X a contravariant functor from $\mathcal{K}$ to the category of Hopf algebroids. We use the same notations as in the last section. For a Hopf algebroid $\mathcal{U}$ on X and $x, y \in X$, we define a linear homomorphism ten of degree -1 by

$$\begin{aligned} ten : K^\bullet(\mathcal{U})_{x,y} \otimes K^\bullet(\mathcal{U})_{x,y} &\rightarrow K^\bullet(\mathcal{U})_{x,y} \\ (a_0 \otimes \cdots \otimes a_i) \otimes (b_0 \otimes \cdots \otimes b_j) &\mapsto a_0 \otimes \cdots \otimes a_i \otimes b_0 \otimes \cdots \otimes b_j. \end{aligned}$$

Note that this homogeneous linear map is not a homomorphism of complexes.

Let $n \geq 2$ be a natural number and $\sigma < \tau$ be simplices in $\mathcal{K}$ of codimension n . Then $\tau - \sigma$ is a totally ordered set S with $\#S = n$. There is a one to one correspondence between the cube $\square^S$ and the set of simplices γ contained in τ containing σ , i.e. $\sigma < \gamma < \tau$. By this correspondence, the set $\square^S$ is regarded as a subset of simplices in $\mathcal{K}$.

The pair of simplices $(\alpha < \beta)$ of codimension one in $\square^S$ is denoted as $Mor(\sigma, \tau)$. We put

$$Chain_{\sigma, \tau}^\bullet := \bigotimes_{(\alpha < \beta) \in Mor(\sigma, \tau)} K_{\alpha, \beta}^\bullet$$

Let $\kappa \in \Sigma(S)$ be a minimal path in $\square^S$. This corresponds to a sequence of simplices $\kappa_0 < \cdots < \kappa_n$ in $\square^S$. We define a homomorphism $c(\kappa) : Chain_{\sigma, \tau}^\bullet \rightarrow K_{\sigma, \tau}^\bullet$ of complexes by

$$\begin{aligned} \bigotimes_{(\alpha, \beta) \in Mor(\sigma, \tau)} K_{\alpha, \beta}^\bullet &\simeq \bigotimes_{(\alpha, \beta) \notin \kappa} K_{\alpha, \beta}^\bullet \otimes \bigotimes_{(\alpha, \beta) \in \kappa} K_{\alpha, \beta}^\bullet &\ni (a_{\alpha, \beta}, b_{\kappa_i, \kappa_{i+1}}) \\ \downarrow & & \\ \bigotimes_{(\alpha, \beta) \in \kappa} K_{\alpha, \beta}^\bullet &&\ni \prod \epsilon(a_{\alpha, \beta}) \cdot (b_{\kappa_i, \kappa_{i+1}}) \\ \downarrow & & \\ K_{\sigma, \tau}^\bullet &&\ni \prod \epsilon(a_{\alpha, \beta}) \cdot b_{\kappa_{n-1}, \kappa_n} \cdots \cdots b_{\kappa_0, \kappa_1} \end{aligned}$$

Here we used the dot product defined in the last section.

We define a homogeneous linear map (not a homomorphism of complex in general)

$$h(S) : Chain_{\sigma, \tau}^\bullet \rightarrow K_{\sigma, \tau}^\bullet$$

of degree $-n + 1$ by the induction of n as follows.

1. If $n = 2$, we may assume that $S = \{1, 2\}$.

$$h(S) = ten \circ (c(12) \otimes c(21)) \circ \Delta :$$

$$Chain_{\sigma, \tau}^\bullet \rightarrow Chain_{\sigma, \tau}^\bullet \otimes Chain_{\sigma, \tau}^\bullet \rightarrow K_{\sigma, \tau}^\bullet \otimes K_{\sigma, \tau}^\bullet \rightarrow K_{\sigma, \tau}^\bullet$$

where

$$\Delta : Chain_{\sigma, \tau}^\bullet \rightarrow Chain_{\sigma, \tau}^\bullet \otimes Chain_{\sigma, \tau}^\bullet$$

is the coproduct homomorphism obtained from that of $K_{\alpha, \beta}^\bullet$. (See Definition 3.3. 2.)

2. For a simplex γ such that $\sigma < \gamma < \tau$, we define

$$split_\gamma : Chain_{\sigma,\tau}^\bullet \rightarrow Chain_{\gamma,\tau}^\bullet \otimes Chain_{\sigma,\gamma}^\bullet$$

by operating the augmentation homomorphism ϵ for a component (α, β) satisfying neither $\sigma < \alpha < \beta < \gamma$ nor $\gamma < \alpha < \beta < \tau$. The subset of S corresponding to the simplex γ is denoted as T . For $T \subset S$, we define a linear homomorphism $h(S - T) \cdot h(T) : Chain_{\sigma,\tau}^\bullet \rightarrow K_{\sigma,\tau}^\bullet$ by the composite

$$\begin{array}{ccc} h(S - T) \cdot h(T) : Chain_{\sigma,\tau}^\bullet & \xrightarrow{split_\gamma} & Chain_{\gamma,\tau}^\bullet \otimes Chain_{\sigma,\gamma}^\bullet \\ \downarrow h(S-T) \otimes h(T) & & \downarrow \text{dot product} \\ & K_{\gamma,\tau}^\bullet \otimes K_{\sigma,\gamma}^\bullet & \longrightarrow K_{\sigma,\tau}^\bullet. \end{array}$$

3. We consider a homogeneous linear map

$$\begin{array}{ccc} ten \circ (c(tot(S)) \otimes (h(S - T) \cdot h(T))) \circ \Delta : & & \\ Chain_{\sigma,\tau}^\bullet & \xrightarrow{\Delta} & Chain_{\sigma,\tau}^\bullet \otimes Chain_{\sigma,\tau}^\bullet \\ \downarrow c(tot(S)) \otimes (h(S-T) \cdot h(T)) & & \downarrow ten \\ & K_{\sigma,\tau}^\bullet \otimes K_{\sigma,\tau}^\bullet & \rightarrow K_{\sigma,\tau}^\bullet. \end{array}$$

of degree $-n + 1$. We define $h(S) : Chain_{\sigma,\tau}^\bullet \rightarrow K_{\sigma,\tau}^\bullet$ by

$$h(S) = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#(S-T)} sign(S, S - T) \cdot ten \circ (c(tot(S)) \otimes (h(S - T) \cdot h(T))) \circ \Delta$$

The following proposition is a direct consequence of Proposition 4.3.

Proposition 4.6. 1. Under the notation as above, we have

$$(4.1) \quad dh(S) - h(S)d = \sum_{\emptyset \neq T \subsetneq S} (-1)^{\#(S-T)} sign(S, S - T) h(S - T) \cdot h(T).$$

2. The righthand side of (4.1) is a homomorphism of complexes.

4.4. Patching differential graded algebra. Let $\sigma < \tau$ be simplices of $\mathcal{K}$ of codimension $n \geq 2$. We define a homogeneous linear map $h(\sigma, \tau) : \tilde{\Omega}^\bullet(\sigma) \rightarrow \tilde{\Omega}^\bullet(\tau)$ of degree $-n + 1$ by

$$\begin{aligned}
\tilde{\Omega}^\bullet(\sigma) &= \bigotimes_{(\alpha,\beta) \in \text{Mor}(\sigma)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\simeq \bigotimes_{(\alpha,\beta) \in \text{Mor}(\sigma) - \text{Mor}(\sigma,\tau)} K_{\alpha,\beta}^\bullet \otimes \text{Chain}_{\sigma,\tau}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\xrightarrow{1 \otimes h(S)} \bigotimes_{(\alpha,\beta) \in \text{Mor}(\sigma) - \text{Mor}(\sigma,\tau)} K_{\alpha,\beta}^\bullet \otimes K_{\sigma,\tau}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\sigma))_{b_\sigma} \\
&\rightarrow \bigotimes_{(\alpha,\beta) \in \text{Mor}(\sigma) - \text{Mor}(\sigma,\tau)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \\
&\simeq \bigotimes_{(\alpha,\beta) \in \text{Mor}(\sigma) - \text{Mor}(\sigma,\tau) - \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \bigotimes_{(\alpha,\beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} \\
&\xrightarrow{\epsilon \otimes 1 \otimes 1} \bigotimes_{(\alpha,\beta) \in \text{Mor}(\tau)} K_{\alpha,\beta}^\bullet \otimes \Omega^\bullet(\mathcal{U}(\tau))_{b_\tau} = \tilde{\Omega}^\bullet(\tau)
\end{aligned}$$

Using $\delta_{\sigma,\tau}$ and $h(\sigma,\tau)$, we define a complex $\Omega^\bullet(X)$ and a homogeneous linear map $d_X : \Omega^\bullet(X) \rightarrow \Omega^\bullet(X)$ as follows:

$$\begin{aligned}
\Omega^i(X) &\simeq \bigoplus_{\sigma \in \mathcal{K}} \tilde{\Omega}^{i-\dim(\sigma)}(\sigma), \\
d_X &= (-1)^{\deg} \cdot \left(\sum_{\sigma < \tau \text{ is codimension one}} \text{sign}(\sigma) \delta_{\sigma,\tau} \right. \\
&\quad \left. + \sum_{\text{codimension of } (\sigma < \tau) \geq 2} \text{sign}(\sigma) h(\sigma,\tau) \right),
\end{aligned}$$

where $\text{sing}(\sigma) = \binom{\text{tot}(K)}{\text{tot}(\sigma), \text{tot}(K-\sigma)}$. Now we can state the main theorem

Theorem 4.7. *We have*

$$d_X \circ d_X = 0.$$

This is a direct consequence of Proposition 4.6.

4.5. GT-admissible varieties.

Definition 4.8. *Let $\mathcal{K}$ be a simplicial complex. A contravariant functor Y from $\mathcal{K}$ to the category (Fund) is called a GT-admissible variety.*

Let $\# = B$ or DR . By attaching fundamental algebroid, we defined a functor $\mathcal{U}^\#$ from the category (Fund) to the category $(\text{Hopf}_{\mathbf{Q}})$ of Hopf algebroid space over $\mathbf{Q}$. Then the composite $\mathcal{U}^\# \circ Y$ is functor from $\mathcal{K}$ to the category of Hopf algebroids. We apply the construction of the last section to the functor $X = \mathcal{U}^\# \circ Y$ and get a complex $\Omega^\bullet(X)$.

Definition 4.9. *The cohomology $H_\#^i(Y)$ of Y is defined by the cohomology $H^i(\Omega^\bullet(\mathcal{U}^\# \circ Y))$ of $\Omega^\bullet(\mathcal{U}^\# \circ Y)$.*

Theorem 4.10. 1. *For a GT variety Y , the cohomology $H_\#^i(Y)$ is a $GT^\#$ -module.*

2. Let Φ be an associator. Then there exists an object

$$H^i(Y) = (H_B^i(Y), H_{DR}^i(Y), \text{comp}) \in \text{Vec}_{\mathbf{Q}} \times_{\text{Vec}_{\mathbf{C}}} \text{Vec}_{\mathbf{Q}}$$

such that the first and the second factors are functorially isomorphic to $H_B^i(Y)$ and $H_{DR}^i(Y)$, respectively.

3. Moreover if the associator Φ is Drinfeld associator, the third factor in 2 is equal to the comparison map.

Remark 4.11. We fix an inclusion $\overline{\mathbf{Q}} \rightarrow \mathbf{C}$. If a GT-admissible variety Y comes from a covering of an algebraic variety $\mathcal{Y}$, $H_{DR}^i(Y)$ (resp. $H_B^i(Y)$, $H_B^i(Y) \otimes \mathbf{Q}_l$) is canonically isomorphic to the classical cohomology $H_{DR}^i(\mathcal{Y}, \mathbf{Q})$ (resp. $H_B^i(\mathcal{Y}/\mathbf{Q})$, $H_{et}^i(\mathcal{Y}, \mathbf{Q}_l)$). Moreover the action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ factors through the natural homomorphism $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{GT}^B(\mathbf{Q}_l)$.