

ON AN ABSTRACT RADIATION CONDITION

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INTRODUCTION

We shall present an abstract radiation condition in terms of the Mourre theory of conjugate operator method.

Let $\mathcal{H}$ be a Hilbert space and A be a self-adjoint operator in $\mathcal{H}$. For $s \geq 0$ consider the Hilbert space $\mathcal{A}^s = D(\langle A \rangle^s)$ with the graph norm, and if $s < 0$, $\mathcal{A}^s = (\mathcal{A}^{-s})^*$. Then, if $s \geq 0$, $\mathcal{A}^s \subseteq \mathcal{H} \subseteq \mathcal{A}^{-s}$ continuously and densely, and the scalar product of $\mathcal{H}$ extends to a natural duality $(\cdot, \cdot)_{s, -s} : \mathcal{A}^s \times \mathcal{A}^{-s} \rightarrow \mathbb{C}$ for all $s \in \mathbb{R}$. We denote by $P_{\pm}$ the spectral projectors of A associated to the half-lines $[0, +\infty)$ and $(-\infty, 0]$, respectively.

We recall now some (Besov) spaces of operators (see [ABG]). Let S be a bounded operator on $\mathcal{H}$. We say that $S \in C^k(A)$, k positive integer, if the application $\mathbb{R} \ni \tau \rightarrow \mathcal{W}(\tau)[S] = e^{i\tau A} S e^{-i\tau A} \in \mathcal{B}(\mathcal{H})$ is strongly C^k ; in this case $ad_A^k S$ can be extended as a bounded operator on $\mathcal{H}$. Consider $\theta \in (0, 1]$, $p \in [1, \infty]$; we say that $S \in C^{\theta, p}(A)$ if $(\tau \rightarrow (\mathcal{W}(\tau) - I)^m [S] \| |\tau|^{\theta+1/p} \in L^p((0, \infty))$, where $m = 1$ if $\theta < 1$, and $m = 2$ if $\theta = 1$. (If $p = \infty$, this condition should be read as $\sup_{\tau > 0} \|(\mathcal{W}(\tau) - I)^m [S]\| |\tau|^{\theta} < \infty$.) For general $\theta > 0$, we say that $S \in C^{\theta, p}(A)$ if $S \in C^l(A)$ and $ad_A^l S \in C^{\theta-l, p}(A)$, where l is the largest integer $l < \theta$.

Let L be a self-adjoint operator in $\mathcal{H}$. Then $L \in C^{\theta, p}(A)$ (or $C^k(A)$) if $(L - z)^{-1} \in C^{\theta, p}(A)$ (or $C^k(A)$) for some (and hence all) $z \in \mathbb{C} \setminus \sigma(L)$.

If L is a self-adjoint operator of class $C^1(A)$, then the commutator $i[L, A]$ is defined as a continuous form on the domain of L . Then one can define the strict Mourre set $\mu^A(L)$ of L with respect to A as the set of $\lambda \in \mathbb{R}$ with the property that there exists $J = (\lambda - \delta, \lambda + \delta) \neq \emptyset$ and $d > 0$ such that

$$E_L(J) i[L, A] E_L(J) \geq d E_L(J).$$

We recall that if L has a spectral gap and $L \in C^{1,1}(A)$, then there exist $R_L(\lambda \pm i0) = \lim_{\epsilon \rightarrow 0} (L - \lambda \mp i\epsilon)^{-1}$ uniformly in $\mathcal{B}(\mathcal{A}^s, \mathcal{A}^{-s})$, whenever $s > 1/2$.

The following theorem was given in [BGS1] (for the proof see [BGS2]; see also [J] for some earlier results).

THEOREM 1. *Let $s > 1/2$ be a real number and L be a self-adjoint operator with a spectral gap and of class $C^{s+1/2,1}(A)$. Then we have $P_{\mp} R_L(\lambda \pm i0) \mathcal{A}^s \subseteq \mathcal{A}^{s-1}$ for each $\lambda \in \mu^A(L)$.*

It turns out that in some stronger hypotheses this condition characterizes $R_L(\lambda \pm i0)$. Namely, we prove the following theorem, extending some results of [B2], [M].

THEOREM 2. *Let $1 \geq \theta > 1/2$ be a real number, $L \geq -M$ be a bounded from below self-adjoint operator of class $C^{1+\theta, \infty}(A)$ such that $i[L, A] \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, where $\mathcal{G}$ is the form domain of L , and $\lambda \in \mu^A(L)$. Suppose $u \in \mathcal{A}^{-s}$, $s \in (1/2, \theta)$ satisfies:*

- a) $(u, (L - \lambda)\varphi)_{-s, s} = 0$ for all $\varphi \in (L + M)^{-1} \mathcal{A}^s$,
- b) there exists $\alpha < \theta/2$ such that $\langle A \rangle^{-\alpha} P_{-}(A) u \in \mathcal{H}$ (or $\langle A \rangle^{-\alpha} P_{+}(A) u \in \mathcal{H}$).

Then $u = 0$.

The proof follows Isozaki's proof of some type of radiation conditions which are strongly related to those presented here. (See [I1], [I2], [I3].) We only remark here that Theorem 2 provides some useful results in the study of the layered media.

One of the tools needed here is the functional calculus using almost analytic extensions of symbols. Let $m \in \mathbf{R}$. We denote by S^m the set of symbols $f \in C^\infty(\mathbf{R})$ that satisfy

$$p_k(f) = \sup_{x \in \mathbf{R}} \langle x \rangle^{m-k} |f^{(k)}(x)| < \infty.$$

Then S^m , endowed with the seminorms p_k is a Fréchet space. The following result can be found in [B2], [M] (see also [DG] for the main idea).

PROPOSITION 3. *Consider a bounded family of symbols $\{f_\epsilon\} \subset S^m$. Then there exists a family of functions (the almost analytic extensions) $\{\tilde{f}_\epsilon\} \subset C^\infty(\mathbf{C})$ such that:*

- i. $|\operatorname{Im} z| \leq \langle \operatorname{Re} z \rangle$ on $\operatorname{supp} \tilde{f}_\epsilon$,
- ii. $|\bar{\partial} \tilde{f}_\epsilon(z)| \leq C_N \langle z \rangle^{m-N-1} |\operatorname{Im} z|^{N-1}$ for all $N \geq 0$ and all $z \in \mathbf{C}$, where the constants C_N do not depend on z and ϵ .

This construction provides an useful representation for the functional calculus of a self-adjoint operator, due to Helffer- Sjöstrand ([HS]): Let A be a self-adjoint operator on $\mathcal{H}$ and $f \in S^{-\delta}$, $\delta > 0$. Then

$$f(A) = \frac{1}{\pi} \int_{\mathbf{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-1} dx dy,$$

where $z = x + iy$ and $\tilde{f}$ is an almost analytic extension of f . If B is a bounded operator with $ad_A n$ is a bounded form on the domain of A , and $ad_A k f^{(k)}(A)$ (respectively $f^{(k)}(A) ad_A k$) $k = 1, \dots, n-1$, are bounded operators, then

$$\begin{aligned} [B, f(A)] &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} ad_A^k(B) f^{(k)}(A) + R_n^r(A, B) \\ &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} f^{(k)}(A) ad_A^k(B) + R_n^l(A, B), \end{aligned}$$

where

$$\begin{aligned} R_n^r(A, B) &= -\frac{1}{\pi} \int_{\mathbf{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-1} ad_A^n(B) (A - z)^{-n} dx dy, \\ R_n^l(A, B) &= \frac{(-1)^n}{\pi} \int_{\mathbf{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-n} ad_A^n(B) (A - z)^{-1} dx dy. \end{aligned}$$

For a proof, see for instance [M].

1. COMMUTATORS

LEMMA 1.1. *Let $B \in \mathcal{C}^{\theta, \infty}(A)$, $0 < \theta < 1$, be a bounded operator and α_1, α_2 positive numbers such that $0 < \alpha_1 + \alpha_2 < \theta$. Then*

$$(1.1) \quad \|\langle A \rangle^{\alpha_1} [B, (A - z)^{-1}] \langle A \rangle^{\alpha_2}\| \leq C(|\operatorname{Im} z|^{-\theta-1} + |\operatorname{Im} z|^{-1} + \langle z \rangle |\operatorname{Im} z|^{-2} + \langle z \rangle^2 |\operatorname{Im} z|^{-3})$$

whenever $|Imz| \neq 0$.

Proof. Consider $0 < \alpha < \theta$. We consider first the operator $\langle A \rangle^\alpha [B, (A - z)^{-1}]$. Suppose $Imz > 0$; the case $Imz < 0$ is similar.

(i) We have (weakly)

$$[B, (A - z)^{-1}] = \int_{-\infty}^0 e^{\mu t} e^{i\lambda t} [B, e^{itA}] dt,$$

where $z = \lambda + i\mu$. Using that $B \in \mathcal{C}^{\theta, \infty}(A)$, we get

$$||[B, (A - z)^{-1}]|| \leq \int_{-\infty}^0 e^{\mu t} |t|^\theta dt \leq C\mu^{-\theta-1} \int_{-\infty}^0 e^t |t|^\theta dt,$$

hence

$$(1.2) \quad ||[B, (A - z)^{-1}]|| \leq C\mu^{-\theta-1}.$$

(ii) Denote $\nu(\lambda) = \langle \lambda \rangle^\alpha$. Helffer-Sjöstrand formula gives (first as bounded operators between $\mathcal{A}^\alpha$ and $\mathcal{A}^{-\alpha}$)

$$(1.3) \quad [B, \langle A \rangle^\alpha] = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\nu}(z) [B, (A - z)^{-1}] dx dy.$$

The norm of the integrand in (3) can be bounded by

$$||\bar{\partial} \tilde{\nu}(z) [B, (A - z)^{-1}]|| \leq C \langle z \rangle^{\alpha-1-N} |Imz|^{N-\theta-1}.$$

If one takes $N = \theta + 1$ to avoid the singularities, we get

$$||\bar{\partial} \tilde{\nu}(z) [B, (A - z)^{-1}]|| \leq C \langle z \rangle^{\alpha-2-\theta},$$

which is integrable if $\alpha < \theta$. Hence

$$[B, \langle A \rangle^\alpha] \in \mathbf{B}(\mathcal{H}).$$

(iii) We can write then

$$(1.4) \quad \langle A \rangle^\alpha [B, (A - z)^{-1}] = [B, \langle A \rangle^\alpha (A - z)^{-1}] - [\langle A \rangle^\alpha, B] (A - z)^{-1}.$$

The norm of the second hand in the rhs of (4) is bounded by $C|Imz|^{-1}$.

(iv) We estimate now the first term in the rhs of (4). Let g be a smooth function on $\mathbf{R}$, $g(t) = 1$ if $|t| \geq 1$ and $g(t) = 0$ if $|t| < 1/2$. Then

$$(1.5) \quad [B, \langle A \rangle^\alpha (A - z)^{-1}] = [B, g(A) \langle A \rangle^\alpha (A - z)^{-1}] + [B, (1 - g(A)) \langle A \rangle^\alpha (A - z)^{-1}].$$

The second term of the rhs of (1.5) equals

$$[B, \langle A \rangle^\alpha] (A - z)^{-1} + [B, (A - z)^{-1}] (1 - g(A)) \langle A \rangle^\alpha,$$

and has the norm less than (using (2))

$$(1.6) \quad C(|Imz|^{-1} + |Imz|^{\theta+1}).$$

We denote $g_z(\lambda) = g(\lambda)\langle\lambda\rangle^\alpha(\lambda - z)^{-1}$. We shall use the following form of the Helffer-Sjöstrand form (see [BGS2], section 4):

$$(1.7) \quad [B, g_z(A)] = \frac{1}{\pi} \int_{\mathbf{R}} ((g_z(\lambda) - \lambda g'_z(\lambda)) [B, ImR_A(\lambda + i\lambda)] - \partial_\lambda(\lambda g_z(\lambda)) [B, ImiR_A(\lambda + i\lambda)]) d\lambda \\ - \frac{1}{\pi} \int_{\mathbf{R}} \int_0^\lambda g_z^{(2)}(\lambda) [B, ImR_A(\lambda + i\mu)] \mu d\mu d\lambda.$$

The norm of the integrand in the first term of (1.7) can be estimated by (using (2) and on supp g)

$$C \left(\frac{\langle\lambda\rangle^\alpha}{|\lambda - z|} + \frac{\langle\lambda\rangle^{\alpha+1}}{|\lambda - z|^2} \right) \langle\lambda\rangle^{-\theta-1} \leq C \langle\lambda\rangle^{-\theta-1} (|Imz|^{-1} + \langle z \rangle |Imz|^{-2}).$$

Hence the first integral in (7) can be bounded as follows

$$(1.8) \quad \left\| \int_{\mathbf{R}} (g_z(\lambda) - \lambda g'_z(\lambda) + 2(i+1)^{-1} \partial_\lambda(\lambda g_z(\lambda)) [B, R_A(\lambda + i\lambda)]) d\lambda \right\| \leq C(|Imz|^{-1} + \langle z \rangle |Imz|^{-2}).$$

To estimate the second integral we note first that

$$(1.9) \quad \left\| \int_0^\lambda g_z^{(2)}(\lambda) [B, R_A(\lambda + i\mu)] \mu d\mu d\lambda \right\| \leq C \langle\lambda\rangle^{1-\theta}$$

on supp g . Then

$$(1.10) \quad \left\| \int_0^\lambda g_z^{(2)}(\lambda) [B, R_A(\lambda + i\mu)] \mu d\mu d\lambda \right\| \leq C \langle\lambda\rangle^{1-\theta} \left(\frac{\langle\lambda\rangle^\alpha}{|\lambda - z|^3} + \frac{\langle\lambda\rangle^{\alpha-1}}{|\lambda - z|^2} + \frac{\langle\lambda\rangle^{\alpha-2}}{|\lambda - z|} \right) \\ \leq C \langle\lambda\rangle^{-1+\alpha-\theta} (|Imz|^{-1} + \langle z \rangle |Imz|^{-2} + \langle z \rangle^2 |Imz|^{-3})$$

Summing up:

$$|[B, g_z(A)]| \leq C(\langle z \rangle^2 |Imz|^{-3} + \langle z \rangle |Imz|^{-2} + \lambda)^{-1+\alpha-\theta} |Imz|^{-1}.$$

Then one gets

$$(1.11) \quad \|\langle A \rangle^\alpha [B, (A - z)^{-1}]\| \leq C(\langle z \rangle^2 |Imz|^{-3} + \langle z \rangle |Imz|^{-2} + \lambda)^{-1+\alpha-\theta} |Imz|^{-1} + |Imz|^{-\theta-1}.$$

In the same way

$$(1.12) \quad \|[B, (A - z)^{-1} \langle A \rangle^\alpha]\| \leq C(\langle z \rangle^2 |Imz|^{-3} + \langle z \rangle |Imz|^{-2} + \lambda)^{-1+\alpha-\theta} |Imz|^{-1} + |Imz|^{-\theta-1}.$$

The general result follows by interpolation. ■

LEMMA 1.2. Let $\{\chi_t\} \in S^a$, $a < 1$ be a bounded family of symbols, and $B \in C^{1+\theta, \infty}(A)$ a bounded operator. Then

$$\begin{aligned} i[B, \chi_t(A)] &= i[B, A]\chi'_t(A) + R_{1,t}, \\ i[B, \chi_t(A)] &= \chi'_t(A)i[B, A] + R_{2,t}, \end{aligned}$$

where

$$\begin{aligned} \langle A \rangle^{\alpha_1} R_{1,t} \langle A \rangle^{\alpha_2} &\in \mathbf{B}(\mathcal{H}) \quad \text{and} \quad \|\langle A \rangle^{\alpha_1} R_{1,t} \langle A \rangle^{\alpha_2}\| \leq C \\ \langle A \rangle^{\alpha_2} R_{2,t} \langle A \rangle^{\alpha_1} &\in \mathbf{B}(\mathcal{H}) \quad \text{and} \quad \|\langle A \rangle^{\alpha_2} R_{2,t} \langle A \rangle^{\alpha_1}\| \leq C, \end{aligned}$$

whenever $\alpha_1 + \alpha_2 + a < 1 + \theta$, $\alpha_1 + \alpha_2 < 1 + \theta$, $\alpha_1 < \theta$. Here C stands for constants not depending on t .

Proof. We have $i[B, \chi_t(A)] = i[B, A]\chi'_t(A) + R_{1,t}$ where

$$R_{1,t} = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\chi}_t i[D, (A - z)^{-1}] (A - z)^{-1} dx dy,$$

with $D = i[B, A] \in C^{\theta, \infty}(A)$, bounded.

We take $\delta = \theta - \alpha_1 - \epsilon$ with ϵ sufficiently small such that $\alpha_2 - \delta < 1$ and $a + \alpha_2 - \delta < 1$. (This is possible by hypothesis.) Then, by Lemma 1,

$$\begin{aligned} &\|\partial \tilde{\chi}_t \langle A \rangle^{\alpha_1} i[D, (A - z)^{-1}] \langle A \rangle^{\delta} (A - z)^{-1} \langle A \rangle^{\alpha_2 - \delta}\| \\ &\leq C_N \langle z \rangle^{a-1-N} |Imz|^N (|Imz|^{-1} + |Imz|^{-\theta-1}) \langle z \rangle^{\alpha_2 - \delta} |Imz|^{-1} \end{aligned}$$

on $\text{supp} \partial \tilde{\chi}_t$. We take $N = \theta + 2$ and thus obtain that

$$\|\partial \tilde{\chi}_t \langle A \rangle^{\alpha_1} i[D, (A - z)^{-1}] (A - z)^{-1} \langle A \rangle^{\alpha_2}\| \leq C \langle z \rangle^{a-3+\alpha-\delta}$$

which is integrable and C does not depend on t . Hence $\langle A \rangle^{\alpha_1} R_{1,t} \langle A \rangle^{\alpha_2}$ extends to a bounded operator and the estimate in the statement holds. One proceed similarly to get the second assertion. ■

LEMMA 1.3. Let B be a bounded operator of class $C^{\theta, \infty}(A)$, $0 < \theta \leq 1$ and α_1, α_2 positive numbers such that $\alpha_1 + \alpha_2 < \theta$. Then $\langle A \rangle^{\alpha_1} [B, \langle A \rangle^{\alpha_2}]$ extends to a bounded operator on $\mathcal{H}$.

Proof. Recall that in the proof of Lemma 1 we proved that $[B, \langle A \rangle^{\delta}] \in \mathbf{B}(\mathcal{H})$ whenever $\delta = \alpha_1 + \alpha_2 + \epsilon < \theta$. We denote $\theta_i = \alpha_i / \delta$, $i = 1, 2$ and set $A_{\delta} = \langle A \rangle^{\delta}$; this is a self-adjoint operator $A_{\delta} \geq 1$. We have than to control $A_{\delta}^{\theta_1} [B, h(A_{\delta})]$, where $h \in S^{\theta_2}$, $h(s) = s^{\theta_2}$ if $s \geq 1/2$ and $h(s) = 0$ if $s \leq 1/4$. We have (first in form sense)

$$A_{\delta}^{\theta_1} i[B, h(A_{\delta})] = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{h}(z) A_{\delta}^{\theta_1} (A_{\delta} - z)^{-1} i[B, A_{\delta}] (A_{\delta} - z)^{-1} dx dy.$$

On the support of $\bar{\partial} \tilde{h}$ the norm of the integrand can be estimated as

$$\|\bar{\partial} \tilde{h}(z) A_{\delta}^{\theta_1} (A_{\delta} - z)^{-1} i[B, A_{\delta}] (A_{\delta} - z)^{-1}\| \leq C \langle z \rangle^{\theta_1 + \theta_2 - 1 - 2}.$$

The rhs is an integrable function, since $\theta_1 + \theta_2 < 1$. Therefore $A_{\delta}^{\theta_1} i[B, h(A_{\delta})]$ extends to a bounded operator on $\mathcal{H}$. ■

LEMMA 1.4. Let B be a bounded operator of class $C^{\theta, \infty}(A)$, $0 < \theta \leq 1$ and α_1, α_2 positive numbers such that $\alpha_1 + \alpha_2 < \theta$, and $\{g_t\} \subset S^a$, $a \leq 0$, a bounded family of symbols. Then:

$$\|\langle A \rangle^{\alpha_1} i[B, g_t(A)] \langle A \rangle^{\alpha_2}\| \leq C,$$

where C does not depend on t .

Proof. (i) Consider first the case where $a < 0$. Then

$$\langle A \rangle^{\alpha_1} i[B, g_t(A)] \langle A \rangle^{\alpha_2} = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{g}_t(z) \langle A \rangle^{\alpha_1} i[B, (A - z)^{-1}] \langle A \rangle^{\alpha_2} dx dy.$$

Using Lemma 1 the norm of the integrand can be majorized by $C\langle z \rangle^{a-2}$.

(ii) If $a = 0$, let $\epsilon > 0$ be such that $\alpha_1 + \alpha_2 + \epsilon < \theta$ and write

$$\langle A \rangle^{\alpha_1 + \epsilon} \langle A \rangle^{-\epsilon} i[B, g_t(A)] \langle A \rangle^{\alpha_2} = \langle A \rangle^{\alpha_1 + \epsilon} i[\langle A \rangle^{-\epsilon}, B] \langle A \rangle^{\alpha_2} g_t(A) + \langle A \rangle^{\alpha_1 + \epsilon} [B, g_t(A)] \langle A \rangle^{-\epsilon} \langle A \rangle^{\alpha_2}.$$

We use the proof of the previous lemma to show that the first term is a bounded operator and its norm can be bounded by a constant not depending on t . For the second term we use (i). ■

2. THE PROOF OF THEOREM 2

We can suppose, without restricting the generality, that in Theorem 2 we have $M = 1$ and $\lambda = 0$.

LEMMA 2.1. If $\Phi \in C_0^\infty(\mathbb{R})$ is a real function, $\Phi = 1$ on a neighborhood of 0, then

$$(2.1) \quad (u, \Phi(L)\varphi)_{-s, s} = (u, \varphi)_{-s, s}, \quad \text{for all } \varphi \in \mathcal{A}^s.$$

Proof. We have, for $\varphi \in \mathcal{A}^1$,

$$(2.2) \quad (u, (1 - \Phi(L))\varphi)_{-s, s} = (u, L\Psi(L)\varphi)_{-s, s},$$

where $\Psi(t) = (1 - \Phi(t))t^{-1}$. Therefore, to have (2) for $\varphi \in \mathcal{A}^1$ it suffices to prove that $\Psi(L) = (L+1)^{-1}\varphi_1$ with $\varphi_1 \in \mathcal{A}^1$. We can write $(L+1)\Psi(L) = (1 - \Phi(L)) + \Psi(L)$. Thus, since $(1 - \Phi(L))\varphi \in \mathcal{A}^1$, it remains to show that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. We have

$$\begin{aligned} i[\Psi(L), A] &= \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\Psi} i[(L - z)^{-1}, A] dx dy \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\Psi} (L - z)^{-1} (L + 1)^{1/2} (L + 1)^{-1/2} i[L, A] (L + 1)^{-1/2} (L + 1)^{1/2} (L - z)^{-1} dx dy \end{aligned}$$

The norm of the integrand can be bounded by $C\langle z \rangle^{-2-2}|Imz|^2\langle z \rangle|Imz|^{-2} = C\langle z \rangle^{-3}$. We get that $i[\Psi(L), A]$ is a bounded operator and we obtain easily that $\Psi(L)\varphi \in \mathcal{A}^1$ if $\varphi \in \mathcal{A}^1$. Thus equation (1) holds for $\varphi \in \mathcal{A}^1$; the general result follows by density using the fact that $\Phi(L) \in \mathcal{B}(\mathcal{A}^s)$. ■

Remark. In fact the previous Lemma says that $\Phi(L)u = u$ for all $\Phi \in C_0^\infty(\mathbf{R})$, $\Phi = 1$ on a neighborhood of 0; this fact can be easily seen using that $\Phi(L) \in \mathbf{B}(\mathcal{A}^s) \cap \mathbf{B}(\mathcal{A}^{-s})$ and it is symmetric with respect to the duality $(\cdot, \cdot)_{s, -s}$.

LEMMA 2.2. Let $\chi \in C_0^\infty(\mathbf{R})$ such that $0 \leq \chi(s) \leq 1$, $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$. We consider the $C_0^\infty(\mathbf{R})$ function

$$\chi_t(y) = \int_{\langle y \rangle}^{\infty} s^{-2\beta} \chi^2(s/t) ds,$$

where $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Then

$$(2.3) \quad (L\phi^2(L)\chi_t(A)u, u)_{s, -s} = 0.$$

Proof. The Lemma follows by hypothesis as $\Phi^2(L)\chi_t(A)u \in (L+1)^{-1}\mathcal{A}^s$. ■

We shall set T for different bounded operators with norm independent on t .

Remark. We have

$$2 \operatorname{Re} i((L\phi^2(L)\chi_t(A)u, u)_{s, -s}) = 0.$$

We shall give to this relation the form and the meaning

$$i([L\Phi^2(L), \chi_t(A)]u, u)_{s, -s} = 0.$$

Set $L_1 = \Phi^2(L)L$. Then L_1 is a bounded operator of class $\mathcal{C}^{\theta+1, \infty}(A)$ (Thm. 6.2.5 [ABG]).

LEMMA 2.3. We have

$$i[L_1, \chi_t(A)] = i[L_1, A]A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t) + \langle A \rangle^{-s}T\langle A \rangle^{-s}.$$

Proof. One applies Lemma 1.2 for $R_{1,t}$, $\alpha_1 = \alpha_2 = s$, $a = 1-2\beta$. (Here $\alpha_1 + \alpha_2 + a = 2s+1-2\beta < \theta+1$ since $\beta > \theta/2$, and $s < \theta$.) ■

As a direct consequence we get the next Lemma.

LEMMA 2.3'. $\sup_{t \geq 1} |i[L_1, A]A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{s, -s}| < \infty$.

LEMMA 2.4. If $\tilde{\Phi} \in C_0^\infty(\mathbf{R})$, $\tilde{\Phi} = 1$ on a small enough neighborhood of 0, then

$$(2.4) \quad \sup_{t \geq 1} |(\tilde{\Phi}(\tilde{L})i[L, A]\tilde{\Phi}(\tilde{L})A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t)u, u)_{s, -s}| < \infty.$$

Proof. We know that $\tilde{\Phi}(\tilde{L})u = u$. We have

$$A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t)\tilde{\Phi}(\tilde{L}) = \tilde{\Phi}(\tilde{L})A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t) + [\tilde{\Phi}(\tilde{L}), A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t)].$$

Set $g_t(A) = A\langle A \rangle^{-2\beta-1}\chi^2(\langle A \rangle/t)$. Here $\{g_t\} \in S_{-2\beta}$ is a bounded family of symbols. One applies Lemma 1.2 to get

$$[\tilde{\Phi}(\tilde{L}), g_t(A)] = [\tilde{\Phi}(\tilde{L}), A]g'_t(A) + R_{1,t}.$$

(In this case $\alpha_1 = \alpha_2 = s$, $a = -\beta$.) Thus, since $2\beta + 1 > s + 1 > 2s$,

$$(2.5) \quad A\langle A \rangle^{-2\beta-1} \chi^2(\langle A \rangle/t) \tilde{\Phi}(L) = \tilde{\Phi}(L) A\langle A \rangle^{-2\beta-1} \chi^2(\langle A \rangle/t) + \langle A \rangle^{-s} T\langle A \rangle^{-s}.$$

Hence

$$(2.6) \quad \begin{aligned} & (i[L_1, A] A\langle A \rangle^{-2\beta-1} \chi^2(\langle A \rangle/t) \tilde{\Phi}(L) u, \tilde{\Phi}(L) u)_{s,-s} \\ &= (i[L_1, A] \tilde{\Phi}(L) A\langle A \rangle^{-2\beta-1} \chi^2(\langle A \rangle/t) u, \tilde{\Phi}(L) u)_{s,-s} + (i[L_1, A] \langle A \rangle^{-s} T\langle A \rangle^{-s} u, u)_{s,-s}. \end{aligned}$$

Since $i[L_1, A] \in \mathbf{B}(\mathcal{H}) \cap \mathcal{C}^{\theta, \infty}(A)$ (Prop. 5.2.2 [ABG]), $i[L_1, A]$ is a bounded operator on $\mathcal{A}^s$, $s < \theta$ (Thm 5.3.3, Lemma 5.3.2 [ABG]). Therefore the second term in (2.6) is bounded by a constant independent on t . But, in form sense,

$$\tilde{\Phi}(L) i[L_1, A] \tilde{\Phi}(L) = \tilde{\Phi}(L) \Phi(L) i[L, A] \Phi(L) \tilde{\Phi}(L) + L \tilde{\Phi}(L) i[A, 1 - \Phi(L)] \tilde{\Phi}(L) + \tilde{\Phi}(L) i[A, 1 - \Phi(L)] L \tilde{\Phi}(L).$$

We take $\text{supp } \tilde{\Phi}$ to be in the set where $\Phi = 1$; then

$$\tilde{\Phi}(L) i[A, 1 - \Phi(L)] \tilde{\Phi}(L) = 0 \quad \text{on } \mathcal{A}^1 \times \mathcal{A}^1.$$

Therefore the bounded operator given by this form on $\mathcal{H}$ ($L \in C^1(A)$) is zero. Similarly we get that $\tilde{\Phi}(L) i[A, 1 - \Phi(L)] L \tilde{\Phi}(L) = 0$. Summing up

$$\tilde{\Phi}(L) i[L_1, A] \tilde{\Phi}(L) = \tilde{\Phi}(L) i[L, A] \tilde{\Phi}(L).$$

The lemma follows by (2.6), (2.5) and the previous relation. ■

We can denote $\tilde{\Phi}$ also by Φ .

LEMMA 2.5. $\sup_{t \geq 1} |(\Phi(L) i[L, A] \Phi(L) A\langle A \rangle^{-\beta} \chi(\langle A \rangle/t) u, \langle A \rangle^{-\beta} \chi(\langle A \rangle/t) u)| \leq \infty$.

Proof. Set $B = \Phi(L) i[L, A] \Phi(L)$. Then B is a bounded operator of class $\mathcal{C}^{\theta, \infty}(A)$. Denote $f_t(x) = \langle x \rangle^{-\beta} \chi(\langle x \rangle/t)$, $x \in \mathbf{R}$. We take $\beta_0 < \beta$, but still $\beta_0 > s/2$, $\beta_0 < \theta/2$. We write

$$\begin{aligned} \langle A \rangle^s [B, f_t(A)] \langle A \rangle^{s-\beta} &= \langle A \rangle^{s-\beta_0} \langle A \rangle^{\beta_0} [B, f_t(A)] \langle A \rangle^{s-\beta} \\ &= \langle A \rangle^{s-\beta_0} [B, f_t(A) \langle A \rangle^{\beta_0}] \langle A \rangle^{s-\beta} - \langle A \rangle^{s-\beta_0} [B, \langle A \rangle^{\beta_0}] \langle A \rangle^{s-2\beta} \chi(\langle A \rangle/t). \end{aligned}$$

But $2s - 2\beta_0 < 2s - \theta < \theta$, so the first term is a bounded operator and its norm does not depend on t (Lemma 1.4). The second term is bounded since $s - 2\beta < 0$ and $\langle A \rangle^{s-\beta_0} [B, \langle A \rangle^{\beta_0}]$ is bounded by Lemma 1.3. Now the lemma follows easily. ■

LEMMA 2.6. For all $\beta > \alpha$ we have $\langle A \rangle^{-\beta} u \in \mathcal{H}$.

Proof. (i) Consider first $\beta > \max(\alpha, s/2)$, $\beta < \theta/2$. Let F_+ be a smooth bounded real function, $F_+ = 1$ on $[1, \infty)$, $F_+ = 0$ on $(-\infty, 1/2]$. We shall show first that

$$(2.7) \quad \sup_{t \geq 1} |(\Phi(L) i[L, A] \Phi(L) A\langle A \rangle^{-\beta-1} F_+(A) \chi(\langle A \rangle/t) u, \langle A \rangle^{-\beta} F_+(A) \chi(\langle A \rangle/t) u)| < \infty.$$

We use again the notation $B = \Phi(L) i[L, A] \Phi(L)$. If $F_- = 1 - F_+$ then

$$\begin{aligned} & (BA\langle A \rangle^{-\beta-1} \chi(\langle A \rangle/t)(A)u, \langle A \rangle^{-\beta} \chi(\langle A \rangle/t)u) \\ &= (BA\langle A \rangle^{-\beta-1} \chi(\langle A \rangle/t)(F_+ + F_-)(A)u, \langle A \rangle^{-\beta} (F_+ + F_-)(A) \chi(\langle A \rangle/t)u). \end{aligned}$$

Here $\langle A \rangle^{-\beta} u \in \mathcal{H}$. Moreover, by Thm 3.10 [BGS2], the fact that B is of class $\mathcal{C}^{s,2}$ for all $s < \theta$ ensure that $F_+(A)BF_-(A) \in \mathbf{B}(\mathcal{A}^{\beta-s}, \mathcal{A}^{\beta-s})$. Hence $\langle A \rangle^\beta F_+(A)BF_-(A)\langle A \rangle^\beta = T \in \mathbf{B}(\mathcal{H})$, and this gives

$$\begin{aligned} & (BA\langle A \rangle^{-\beta-1}\chi(\langle A \rangle/t)F_+(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)F_-(A)) \\ & = (T\langle A \rangle^{\beta-s}\langle A \rangle^{-\beta-1}\chi(\langle A \rangle/t)Au, \langle A \rangle^{-2\beta}\chi(\langle A \rangle/t)F_-(A)u). \end{aligned}$$

Therefore

$$(2.8) \quad \sup_{t \geq 1} |(BA\langle A \rangle^{-\beta-1}\chi(\langle A \rangle/t)F_+(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)F_-(A))| < \infty.$$

Similarly one gets

$$(2.9) \quad \sup_{t \geq 1} |(BA\langle A \rangle^{-\beta-1}\chi(\langle A \rangle/t)F_-(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)F_+(A))| < \infty.$$

Now (2.7) follows by (2.8), (2.9) and the previous lemma.

We can write $A\langle A \rangle^{-1}F_+(A) = g^2(A)F_+(A)$ with $g \in S^0$. But $\langle A \rangle^{s-\beta}[B, g(A)]\langle A \rangle^{s-\beta}$ is bounded by Lemma 1.4 ($2s - 2\beta < 2s - \theta < 2\theta - \theta = \theta$). Hence

$$\sup_{t \geq 1} |(B\langle A \rangle^{-\beta}\chi(\langle A \rangle/t)g(A)F_+(A)u, \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)g(A)F_+(A))| < \infty.$$

Using now the Mourre estimate we get

$$\sup_{t \geq 1} \|\Phi(L)\langle A \rangle^{-\beta}\chi(\langle A \rangle/t)g(A)F_+(A)u\| \leq \infty.$$

As $[\Phi(L), \langle A \rangle^{-\beta}\chi(\langle A \rangle/t)g(A)F_+(A)]\langle A \rangle^s$ is a bounded operator with norm independent on t (by Lemma 1.4) it follows

$$\sup_{t \geq 1} \|\langle A \rangle^{-\beta}\chi(\langle A \rangle/t)g(A)F_+(A)u\| \leq \infty.$$

This provide, using Beppo-Levi Theorem,

$$(2.10) \quad \langle A \rangle^{-\beta}g(A)F_+(A)u \in \mathcal{H}.$$

If we take $\tilde{F}_+$ to be a smooth bounded real function on $\mathbf{R}$, $\tilde{F}_+ = 1$ on $[2, \infty)$ and $\text{supp } \tilde{F}_+ \subset [1, \infty)$, we can write

$$\tilde{F}_+(A)\langle A \rangle^{-\beta} = (\tilde{F}_+/gF_+)(A)(gF_+)(A)\langle A \rangle^{-\beta},$$

and $(\tilde{F}_+/gF_+)(A)(gF_+)(A)$ is a bounded operator. Then (2.10) gives that $\tilde{F}_+(A)\langle A \rangle^{-\beta}u \in \mathcal{H}$. Thus the lemma follows in this case since $(1 - \tilde{F}_+(A)\langle A \rangle^{-\beta})u \in \mathcal{H}$ by hypothesis (b) of Thm 2.

(ii) Now we can repeat the argument with s replaced by 2α and see that $\langle A \rangle^{-\beta}u \in \mathcal{H}$ for all $\beta < \theta/2$, $\beta > \alpha$. ■

LEMMA 2.7. *In the conditions of Thm. 2, $u \equiv 0$.*

Proof. Denote $u_\epsilon = \langle \epsilon A \rangle^{-\beta}u$. We shall show that $\|u_\epsilon\| \leq C$, where C does not depend on ϵ . This implies that $u \in \mathcal{H}$. Since $u = \Phi(L)u$, u is in the domain of L ; and, as $Lu = 0$, it follows that either 0 is a eigenvalue of L , or $u \equiv 0$. The first case is imposible due to the Mourre estimate.

Recall that $L_1 = L\Phi^2(L)$. We shall denote by T different bounded operators with norm independent on t and ϵ . We begin by computing

$$\begin{aligned}
(i[L_1, A]u_\epsilon, u_\epsilon) &= \lim_{t \rightarrow \infty} (i[L_1, A(A + itA)^{-1}it]u_\epsilon, u_\epsilon) \\
&= \lim_{t \rightarrow \infty} i(L_1 A(A + itA)^{-1}it \langle \epsilon A \rangle^{-\beta} u, \langle \epsilon A \rangle^{-\beta} u) \\
&\quad - \lim_{t \rightarrow \infty} i(\langle \epsilon A \rangle^{-\beta} u, L_1 A(A - itA)^{-1}it \langle \epsilon A \rangle^{-\beta} u) \\
&= - \lim_{t \rightarrow \infty} i([\langle \epsilon A \rangle^{-\beta}, L_1] A(A + itA)^{-1}it \langle \epsilon A \rangle^{-\beta} u, u)_{\beta, -\beta} \\
&\quad - \lim_{t \rightarrow \infty} i(u, L_1 A(A - itA)^{-1}it \langle \epsilon A \rangle^{-2\beta} u)_{-\beta, +\beta} \\
&\quad + \lim_{t \rightarrow \infty} i(L_1 A(A + itA)^{-1}it \langle \epsilon A \rangle^{-2\beta} u, u)_{\beta, -\beta} \\
&\quad + \lim_{t \rightarrow \infty} i(u, [\langle \epsilon A \rangle^{-\beta}, L_1] A(A - itA)^{-1}it \langle \epsilon A \rangle^{-\beta} u)_{-\beta, +\beta} \\
&= - \lim_{t \rightarrow \infty} i([\langle \epsilon A \rangle^{-\beta}, L_1] A(A + itA)^{-1}it \langle \epsilon A \rangle u, u)_{\beta, -\beta} \\
&\quad + \lim_{t \rightarrow \infty} i(u, [\langle \epsilon A \rangle^{-\beta}, L_1] A(A - itA)^{-1}it \langle \epsilon A \rangle u)_{-\beta, +\beta}
\end{aligned}$$

We have

$$i[L_1, \langle \epsilon A \rangle^{-\beta}] = -\beta i[L_1, A] \epsilon^2 A \langle \epsilon A \rangle^{-\beta-2} + \langle A \rangle^{-\beta} T \langle A \rangle^{-\beta-1}$$

(by Lemma 2.1, with $1 + 2\beta < 1 + \theta$, $\beta < \theta$, $a = 1$) and also:

$$i[L_1, \epsilon A]^{-\beta} = -\beta \epsilon^2 A \langle \epsilon A \rangle^{-\beta-1} i[L_1, A] + \langle A \rangle^{-\beta-1} T \langle A \rangle^{-\beta}.$$

It follows then

$$\begin{aligned}
&- i([\langle \epsilon A \rangle^{-\beta}, L_1] it A(A + itA)^{-1} \langle \epsilon A \rangle^{-\beta} u, u)_{\beta, -\beta} \\
&= (-\beta i[L_1, A] \epsilon^2 A^2 it(A + itA)^{-1} it \langle \epsilon A \rangle^{-2\beta-2} u, u)_{\beta, -\beta} + (T \langle A \rangle^{-\beta-1} it A(A + itA)^{-1} u_\epsilon, \langle A \rangle^{-\beta} u_\epsilon \rightarrow \\
&\rightarrow (-\beta i[L_1, A] \epsilon^2 A^2 \langle \epsilon A \rangle^{-2\beta-2} u, u)_{\beta, -\beta} + (T \langle A \rangle^{-\beta-1} u_\epsilon, \langle A \rangle^{-\beta} u_\epsilon
\end{aligned}$$

Similarly for the second commutator. We get thus

$$(i[L_1, A]u_\epsilon, u_\epsilon) = 2\beta(\epsilon^2 A^2 \langle \epsilon A \rangle^{-2-\beta} B_1 u, u_\epsilon) + (T \langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u),$$

where $B_1 = i[L_1, A]$. We write $\epsilon^2 A^2 \langle \epsilon A \rangle^{-2-\beta} = \langle \epsilon A \rangle^\beta - \langle \epsilon A \rangle^{-1-\beta}$. Lemma 1.4 gives

$$(\langle \epsilon A \rangle^\beta B_1 u, u_\epsilon) = (B_1 u_\epsilon, u_\epsilon) + (T \langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u)$$

and

$$(\langle \epsilon A \rangle^{-2} B_1 u_\epsilon, u_\epsilon) = (B_1 \langle \epsilon A \rangle^{-1} u_\epsilon, \langle \epsilon A \rangle^{-1} u_\epsilon) + (T \langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

Hence

$$(1 - 2\beta)(B_1 u_\epsilon, u_\epsilon) = -2\beta(B_1 \langle \epsilon A \rangle^{-1} u_\epsilon, \langle \epsilon A \rangle^{-1} u_\epsilon) + (T \langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

We use that $u = \Phi(L)u$ as in Lemma 2.4 to get

$$(B_1 \langle \epsilon A \rangle^{-1} u_\epsilon, \langle \epsilon A \rangle^{-1} u_\epsilon) = (\Phi(L) i[L, A] \Phi(L) \langle \epsilon A \rangle^{-1} u_\epsilon, \langle \epsilon A \rangle^{-1} u_\epsilon) + (T \langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

The Mourre inequality (suppose $\text{supp}\Phi$ small enough) provide

$$(1 - 2\beta)(B_1 u_\epsilon, u_\epsilon) \leq (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

Again:

$$(B_1 u_\epsilon, u_\epsilon) = (\Phi(L)i[L, A]\Phi(L)u_\epsilon, u_\epsilon) + (T\langle A \rangle^{-\beta} u, \langle A \rangle^{-\beta} u).$$

Then the Mourre inequality gives

$$\|\Phi(L)u_\epsilon\| \leq C.$$

Commuting $\Phi(L)$ and $\langle \epsilon A \rangle^{-\beta}$ (by Lemma 1.4), we get

$$\|\langle \epsilon A \rangle^{-\beta} u\| \leq C,$$

which gives $u \in \mathcal{H}$ and thus finishes the proof. ■

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