

On a property of fuzzy stopping times

Y. YOSHIDA[§], M. YASUDA[†], J. NAKAGAMI[†] and M. KURANO[‡]

[§]Faculty of Economics and Business Administration, Kitakyushu University, Kokuraminami, Kitakyushu 802, Japan. [†]Faculty of Science, Chiba University, [‡]Faculty of Education, Yayoi-cho, Inage-ku, Chiba 263, Japan.

Abstract

This note is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times is introduced and constructed by subsets of α -cut for fuzzy states. The results are applied to the optimization of a corresponding problem with an additive weighting function.

Keywords: Fuzzy stopping times; Markov property; α -cuts of fuzzy sets; optimality.

1 Introduction and notations

The stopping time with fuzziness, which is called a fuzzy stopping time, is considered by our previous paper [11] in which optimization of a corresponding fuzzy problem is pursued by the constructive method.

In this note, we introduce a new class of fuzzy stopping times defined by subsets of the α -cuts of fuzzy states and we apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards. As related works, refer to [1, 5, 6, 7, 15].

In the remainder of this section, a fuzzy stopping time for a fuzzy dynamic system is defined explicitly. A new class of fuzzy stopping time is introduced in Section 2 and its construction is discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, a example is given to illustrate the results.

Let E, E_1, E_2 be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. A fuzzy set $\tilde{u} : E \mapsto [0, 1]$ is called convex if

$$\tilde{u}(\lambda x + (1 - \lambda)y) \geq \tilde{u}(x) \wedge \tilde{u}(y), \quad x, y \in E, \lambda \in [0, 1],$$

where $a \wedge b := \min\{a, b\}$ for real numbers a, b (c.f. Chen-wei Xu [2]). Also, a fuzzy relation $\tilde{h} : E_1 \times E_2 \mapsto [0, 1]$ is called convex if

$$\tilde{h}(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geq \tilde{h}(x_1, y_1) \wedge \tilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1, y_1, y_2 \in E_2$ and $\lambda \in [0, 1]$.

Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\tilde{u}$, on E whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x \in E} \tilde{u}(x) = 1$. The α -cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{u}$ is defined by

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\} \quad (\alpha > 0) \quad \text{and} \quad \tilde{u}_0 := \text{cl}\{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of E . Clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0, 1]$.

Let $\mathbf{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbf{R})$ is the set of all bounded closed intervals in $\mathbf{R}$. The elements of $\mathcal{F}(\mathbf{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbf{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$,

$$(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbf{R}: x_1 + x_2 = x} \{\tilde{m}(x_1) \wedge \tilde{n}(x_2)\} \quad (x \in \mathbf{R}) \quad (1.1)$$

and

$$(\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbf{R}). \quad (1.2)$$

And hence

$$(\tilde{m} + \tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]),$$

where $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A + \emptyset = \emptyset + A := A$ and $\lambda \emptyset := \emptyset$ for any non-empty closed intervals $A, B \in \mathcal{C}(\mathbf{R})$. We use the following lemma.

Lemma 1.1 (Chen-wei Xu [2]).

- (i) For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \tilde{n} \in \mathcal{F}(\mathbf{R})$.
- (ii) Let $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$. Then $\sup_{x \in E_1} \{\tilde{u}(x) \wedge \tilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.

We consider the dynamic fuzzy system([9]), which is denoted by the elements $(S, \tilde{q})$ as follows.

Definition 1.

- (i) The state space S is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by an element of $\mathcal{F}(S)$.
- (ii) The law of motion for the system is denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0, 1]$, and assume that $\tilde{q} \in \mathcal{F}(S \times S)$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, the state is moved to a new fuzzy state $Q(\tilde{s})$ after unit time, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined by

$$Q(\tilde{s})(y) := \sup_{x \in S} \{\tilde{s}(x) \wedge \tilde{q}(x, y)\} \quad (y \in S). \quad (1.3)$$

Note that the map Q is well-defined by Lemma 1.1.

For the dynamic fuzzy system $(S, \tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^{\infty}$ by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \quad (1.4)$$

A fuzzy stopping time for this sequence $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in the next section. In order to define a fuzzy stopping time, we need the following preliminaries.

Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_{\alpha} : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ ($\alpha \in [0, 1]$) are defined as follows: For $D \in \mathcal{C}(S)$,

$$Q_{\alpha}(D) := \begin{cases} \{y \in S \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \text{cl}\{y \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \quad (1.5)$$

From the assumption on $\tilde{q}$, the maps Q_{α} is well-defined. The iterates Q_{α}^t ($t \geq 0$) are defined by setting $Q_{\alpha}^0 := I(\text{identity})$ and iteratively,

$$Q_{\alpha}^{t+1} := Q_{\alpha} Q_{\alpha}^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the α -cuts of $Q(\tilde{s})$ defined by (1.3) is specified using the maps Q_{α} .

Lemma 1.2 ([9, 10]). For any $\alpha \in [0, 1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:

- (i) $Q(\tilde{s})_{\alpha} = Q_{\alpha}(\tilde{s}_{\alpha})$;
- (ii) $\tilde{s}_{t,\alpha} = Q_{\alpha}^{t-1}(\tilde{s}_{\alpha})$. ($t \geq 1$),

where $\tilde{s}_{t,\alpha} := (\tilde{s}_t)_{\alpha}$ and $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here. And a new class of fuzzy stopping times is introduced, which is constructed thorough subsets of α -cuts of fuzzy states.

For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$. Let $\mathbf{N} = \{1, 2, \dots\}$ and $\mathcal{F}'$ a subset of $\mathcal{F}$.

Definition 2 (cf.[11]). A *fuzzy stopping time(FST)* on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma} : \mathcal{F}' \times \mathbf{N} \mapsto [0, 1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s}, t)$ is non-increasing in t and there exists a natural number $t(\tilde{s}) \in \mathbf{N}$ with $\tilde{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t(\tilde{s})$.

We note here that 0 represents ‘stop’ and 1 represents ‘continue’ in the grade of membership (cf.[11]). An FST $\tilde{\sigma}(\tilde{s}, \cdot)$ means the degree of ‘continue’ at time t starting at a fuzzy state $\tilde{s} \in \mathcal{F}'$. The set of all FSTs on $\mathcal{F}'$ is denoted by $\Sigma(\mathcal{F}')$. Assuming $Q(\mathcal{F}') \subset \mathcal{F}'$, an FST $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ is called *Markov* if there exist a mapping $\delta : \mathcal{F}' \mapsto [0, 1]$ satisfying

- (i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and
- (ii) $\tilde{\sigma}(\tilde{s}, t) = \delta(\tilde{s}_t)$ for all $\tilde{s} \in \mathcal{F}'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

The above δ is called a *support* of $\tilde{\sigma}$. We consider ourselves with the construction of Markov FSTs. For this purpose, we assume the following condition holds.

Condition A1. For each $\alpha \in [0, 1]$, there exists a non-empty subset $\mathcal{K}_\alpha$ of $\mathcal{C}(S)$ satisfying

$$Q_\alpha(\mathcal{K}_\alpha) \subset \mathcal{K}_\alpha. \quad (2.1)$$

Using this subset $\mathcal{K}_\alpha$, we define a sequence of subsets $\{\mathcal{K}_\alpha^t\}_{t=1}^{\infty}$ inductively by

$$\mathcal{K}_\alpha^1 := \mathcal{K}_\alpha \quad (2.2)$$

and for each $t \geq 2$,

$$\mathcal{K}_\alpha^t := \{c \in \mathcal{C}(S) \mid Q_\alpha(c) \in \mathcal{K}_\alpha^{t-1}\}. \quad (2.3)$$

Clearly, $\mathcal{K}_\alpha^t = Q_\alpha^{-1}(\mathcal{K}_\alpha^{t-1}) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha)$. Also, it holds from (2.1) that $\mathcal{K}_\alpha^t \subset \mathcal{K}_\alpha^{t+1}$ ($t \geq 1$).

To simplify our discussion, we assume the following condition holds henceforth.

Condition A2. For all $\alpha \in [0, 1]$, it holds that

$$\mathcal{C}(S) = \bigcup_{t=1}^{\infty} \mathcal{K}_\alpha^t.$$

For $c \in \mathcal{C}(S)$ and $\alpha \in [0, 1]$, define $\hat{\sigma}_\alpha(c)$ by

$$\hat{\sigma}_\alpha(c) := \min\{t \geq 1 \mid c \in \mathcal{K}_\alpha^t\}. \quad (2.4)$$

That is, it is the first entry time of $c \in \mathcal{C}(S)$ with the grade α . We define a restricted class $\hat{\mathcal{F}} \subset \mathcal{F}$ by

$$\hat{\mathcal{F}} := \{\tilde{s} \in \mathcal{F} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0, 1]\}. \quad (2.5)$$

Using the class $\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0, 1]\}$, for the restricted element $\tilde{s} \in \hat{\mathcal{F}}$, let us construct

$$\hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0, 1]} \{\alpha \wedge 1_{D_\alpha}(t)\} \quad (t \geq 1), \quad (2.6)$$

where 1_{D_α} is the indicator of a set $D_\alpha = \{t \in \mathbb{N} \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique of constructing a corresponding fuzzy number from the class of level sets. Now let

$$\hat{\sigma}(\tilde{s}, \cdot)_\alpha := \min\{t \in \mathbb{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}. \quad (2.7)$$

Then we obtain the following theorem.

Theorem 2.1.

- (i) $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \hat{\sigma}_\alpha(\tilde{s}_\alpha)$, $\tilde{s} \in \hat{\mathcal{F}}$, $\alpha \in [0, 1]$;
- (ii) $\hat{\sigma}$ is an FST on $\hat{\mathcal{F}}$.

Proof. By (2.6) and (2.7), we have that $\hat{\sigma}(\tilde{s}, \cdot)_\alpha \leq t$ is equivalent to $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq t$ for all $t \geq 1$. This fact shows (i). From Condition A2, there exists $t^* \in \mathbb{N}$ with $\tilde{s}_0 \in \mathcal{K}_0^{t^*}$. So, $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq \tilde{s}_0(\tilde{s}_0) \leq t^*$ for all $\alpha \in [0, 1]$, which shows by (2.5) that $\hat{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t^*$. Since $\hat{\sigma}(\tilde{s}, t+1) \leq \hat{\sigma}(\tilde{s}, t)$ holds clearly for $t \geq 1$ from the definition (2.6), we also obtain (ii). *q.e.d.*

In order to show the Markov property of $\hat{\sigma}$, we need the following lemma.

Lemma 2.1. *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then*

(i) $\hat{\sigma}(\tilde{s}, t) = \alpha$ if and only if, for any $\epsilon > 0$,

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^t \quad \text{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^t;$$

(ii) $\tilde{s}_t \in \hat{\mathcal{F}}$ ($t \geq 1$).

Proof. By (2.6), $\hat{\sigma}(\tilde{s}, t) = \sup\{\alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. So, (i) follows from (2.4). From Lemma 1.2(ii), for $l \geq 1$, $\hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_{l,\alpha}) = \hat{\sigma}_\alpha(Q_\alpha^{l-1}(\tilde{s}_\alpha))$. So, by (2.3) and (2.4),

$$\begin{aligned} \hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) &= \min\{t \geq 1 \mid Q_\alpha^{l-1}(\tilde{s}_\alpha) \in \mathcal{K}_\alpha^t\} \\ &= \min\{t \geq 1 \mid \tilde{s}_\alpha \in \mathcal{K}_{\alpha+\epsilon}^{t+l-1}\} \\ &= \max\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l-1), 1\}, \end{aligned}$$

and it is non-increasing in $\alpha \in [0, 1]$ since $\tilde{s} \in \hat{\mathcal{F}}$. Therefore we obtain (ii). *q.e.d.*

Theorem 2.2. *Let $\tilde{s} \in \hat{\mathcal{F}}$. Then, $\hat{\sigma}$ is a Markov FST with $\tilde{s}$.*

Proof. Let $\{\tilde{s}_t\}_{t=1}^\infty$ be defined by (1.4) with $\tilde{s}_1 = \tilde{s}$. First, we prove

$$\hat{\sigma}(\tilde{s}, t+r) = \hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) \quad \text{for } t, r \in \mathbb{N}. \quad (2.8)$$

Note that $\hat{\sigma}(\tilde{s}_{t+1}, r)$ is well-defined from Lemma 2.1(ii). Let $\alpha = \hat{\sigma}(\tilde{s}, t+r)$. From Lemma 2.1(i), we have

$$\tilde{s}_{\alpha+\epsilon} \in \mathcal{K}_{\alpha+\epsilon}^{t+r} \quad \text{and} \quad \tilde{s}_{\alpha-\epsilon} \notin \mathcal{K}_{\alpha-\epsilon}^{t+r} \quad \text{for any } \epsilon > 0.$$

Noting $Q_\alpha^t(\mathcal{K}_\alpha^l) = \mathcal{K}_\alpha^{l-t}$ ($1 \leq t < l$) and Lemma 1.2(ii), we obtain

$$\tilde{s}_{t+1, \alpha+\epsilon} = Q_{\alpha+\epsilon}^t(\tilde{s}_{\alpha+\epsilon}) \in Q_{\alpha+\epsilon}^t(\mathcal{K}_{\alpha+\epsilon}^{t+r}) = \mathcal{K}_{\alpha+\epsilon}^r \quad (2.9)$$

and

$$\tilde{s}_{t+1, \alpha-\epsilon} = Q_{\alpha-\epsilon}^t(\tilde{s}_{\alpha-\epsilon}) \notin Q_{\alpha-\epsilon}^t(\mathcal{K}_{\alpha-\epsilon}^{t+r}) = \mathcal{K}_{\alpha-\epsilon}^r. \quad (2.10)$$

Therefore, we get $\hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha$ from Lemma 2.1(i). Namely, $\hat{\sigma}(\tilde{s}, t+r) = \hat{\sigma}(\tilde{s}_{t+1}, r)$. Since $\hat{\sigma}(\tilde{s}, t+r) \leq \hat{\sigma}(\tilde{s}, t)$ from Theorem 2.1(ii), we obtain $\hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha$, and so (2.8) holds.

Next, we put $\delta(\tilde{s}) = \hat{\sigma}(\tilde{s}, 1)$ for $\tilde{s} \in \hat{\mathcal{F}}$. From (2.8), we get

$$\begin{aligned} \hat{\sigma}(\tilde{s}, t) &= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, t-1) \\ &= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, 1) \wedge \hat{\sigma}(\tilde{s}_3, t-2) \\ &= \dots \\ &= \bigwedge_{l=1}^t \hat{\sigma}(\tilde{s}_l, 1) \\ &= \bigwedge_{l=1}^t \delta(\tilde{s}_l) \\ &= \delta(\tilde{s}_t) \quad \text{for } t \in \mathbb{N}. \end{aligned}$$

Since we also have $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$ from Theorem 2.1(ii), $\hat{\sigma}$ is a Markov FST with $\tilde{s}$. *q.e.d.*

3 Applications to fuzzy stopping problem

In this section, applying the results in the previous section, we obtain the optimal FST for a fuzzy dynamic system with fuzzy rewards ([10]) when the weighting function is additive.

Firstly, we will formulate the stopping problem to be considered here. Let $\tilde{r} : S \times \mathbf{R} \mapsto [0, 1]$ be a fuzzy relation satisfying $\tilde{r} \in \mathcal{F}(S \times \mathbf{R})$. If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}$, the following fuzzy reward is earned:

$$R(\tilde{s})(z) := \sup_{x \in S} \{\tilde{s}(x) \vee \tilde{r}(x, z)\}, \quad z \in \mathbf{R}.$$

Then we can define a sequence of fuzzy rewards $\{R(\tilde{s}_t)\}_{t=1}^{\infty}$, where $\{\tilde{s}_t\}_{t=1}^{\infty}$ is defined in (1.4) with the initial fuzzy state $\tilde{s}_1 = \tilde{s}$. Let

$$\varphi(\tilde{s}, t) := \sum_{l=1}^{t-1} R(\tilde{s}_l) \quad \text{for } t \in \mathbf{N}. \quad (3.1)$$

We need the following lemma, which is proved in [9].

Lemma 3.1 ([9, 10]). For $t \in \mathbf{N}$ and $\alpha \geq 0$,

$$\varphi(\tilde{s}, t)_{\alpha} = \sum_{l=1}^{t-1} R_{\alpha}(\tilde{s}_l, \alpha)$$

holds, where

$$R_{\alpha}(\tilde{s}_l, \alpha) := \begin{cases} \{z \in \mathbf{R} \mid \tilde{r}(x, z) \geq \alpha \text{ for some } x \in \tilde{s}_l, \alpha\} & \text{for } \alpha > 0 \\ cl\{z \in \mathbf{R} \mid \tilde{r}(x, z) > 0 \text{ for some } x \in \tilde{s}_l, \alpha\} & \text{for } \alpha = 0. \end{cases} \quad (3.2)$$

Let $g : C(\mathbf{R}) \mapsto \mathbf{R}$ be any additive map with $g(\phi) = 0$, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S).$$

Adapting this g for a weighting function (see [4]), when an FST $\hat{\sigma} \in \Sigma(\hat{\mathcal{F}})$ and an initial fuzzy state $\tilde{s} \in \hat{\mathcal{F}}$ are used, the scalarization of the total fuzzy reward is given by

$$\begin{aligned} G(\tilde{s}, \hat{\sigma}) &= \int_0^1 g(\varphi(\tilde{s}, \hat{\sigma}_{\alpha})_{\alpha}) d\alpha \\ &= \int_0^1 g\left(\sum_{t=1}^{\hat{\sigma}_{\alpha}-1} R_{\alpha}(\tilde{s}_t, \alpha)\right) d\alpha, \end{aligned} \quad (3.3)$$

where $\sum_{t=1}^0 R_{\alpha}(\tilde{s}_t, \alpha) = \phi$ and $\hat{\sigma}_{\alpha}$ means $\hat{\sigma}(\tilde{s}, \cdot)_{\alpha} = \min\{t \in \mathbf{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}$ for simplicity. Since $\varphi(\tilde{s}, \hat{\sigma}_{\alpha}) \in C(\mathbf{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \hat{\sigma}_{\alpha})_{\alpha})$ is left-continuous in $\alpha \in (0, 1]$, therefore the right-hand integral of (3.3) is well-defined. For a given $\mathcal{F}' \subset \mathcal{F}$, our objective is to maximize (3.3) over all FSTs $\hat{\sigma} \in \Sigma(\mathcal{F}')$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}'$.

Definition 3. An FST $\hat{\sigma}^*$ with $\tilde{s} \in \mathcal{F}'$ is called an $\tilde{s}$ -optimal if

$$G(\tilde{s}, \hat{\sigma}) \leq G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}').$$

If $\hat{\sigma}^*$ is $\tilde{s}$ -optimal for all $\tilde{s} \in \mathcal{F}'$, $\hat{\sigma}^*$ is called *optimal* in $\mathcal{F}'$.

Now we will seek a $\tilde{s}$ -optimal or an optimal FST by using the results in the previous sections. For each $\alpha \in [0, 1]$, let

$$\mathcal{K}_{\alpha}(g) := \{c \in C(S) \mid g(R_{\alpha}(c)) \leq 0\}. \quad (3.4)$$

Here we need the following Assumptions B1 and B2, which are assumed to hold henceforth.

Assumption B1 (Closedness).

$$Q_{\alpha}(\mathcal{K}_{\alpha}(g)) \subset \mathcal{K}_{\alpha}(g) \quad \text{for all } \alpha \in [0, 1]$$

Now we define the sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$ by (2.2) – (2.3), that is,

$$\mathcal{K}_\alpha^t(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) \quad \text{for } t \geq 1. \quad (3.5)$$

Assumption B2. For all $\alpha \in [0, 1]$, it holds that

$$C(S) = \bigcup_{t=1}^\infty \mathcal{K}_\alpha^t(g).$$

Using the sequence $\{\mathcal{K}_\alpha^t(g)\}_{t=1}^\infty$ given in (3.5), we define $\hat{\sigma}_\alpha$, $\hat{\mathcal{F}}$, $\hat{\sigma}$ and $\hat{\sigma}(\tilde{s}, \cdot)_\alpha$, respectively, by (2.4), (2.5), (2.6) and (2.7). Then, from Theorems 2.1 and 2.2, $\hat{\sigma}$ is a Markov FST on $\hat{\mathcal{F}}$.

The following theorem will be proved by applying the idea of the one-step look ahead (OLA) policy ([3, 8, 14]) for stochastic stopping problems.

Theorem 3.1. Under Assumptions B1 and B2, $\hat{\sigma}$ is optimal in $\hat{\mathcal{F}}$.

Proof. Firstly, consider the deterministic stopping problem which maximizes $g(\varphi(\tilde{s}, t)_\alpha)$ over $t \geq 1$. As g is additive, $g(\varphi(\tilde{s}, t)_\alpha) = \sum_{l=1}^{t-1} g(R_\alpha(\tilde{s}_l, \alpha))$. Therefore $g(\varphi(\tilde{s}, t)_\alpha) \geq g(\varphi(\tilde{s}, t+1)_\alpha)$ if and only if $\tilde{s}_{t,\alpha} \in K_\alpha(g)$. By the assumption B1, $\tilde{s}_{t,\alpha} \in K_\alpha(g)$ implies $g(\varphi(\tilde{s}, t)_\alpha) \geq g(\varphi(\tilde{s}, l)_\alpha)$ for all $l \geq t$. Thus, since $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \hat{\sigma}(\tilde{s}, \cdot)_\alpha$ by Theorem 2.1, we can show

$$g(\varphi(\tilde{s}, \hat{\sigma}(\tilde{s}, \cdot)_\alpha)) \geq g(\varphi(\tilde{s}, \tilde{\sigma}(\tilde{s}, \cdot)_\alpha))$$

for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ and $\alpha \in [0, 1]$. This implies that $G(\tilde{s}, \hat{\sigma}) \geq G(\tilde{s}, \tilde{\sigma})$ for all $\tilde{\sigma} \in \Sigma(\mathcal{F}')$ by using (3.3). This complete the proof. *q.e.d.*

4 A numerical example

An example is given to illustrate the previous results of fuzzy stopping problem in this section.

Let $S := [0, 1]$. The fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by

$$\tilde{q}(x, y) = \begin{cases} 1 & \text{if } y = \beta x \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{r}(x, z) = \begin{cases} 1 & \text{if } z = x - \lambda \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is an observation cost and $0 < \beta < 1$ for $x, y, z \in [0, 1]$ and $z \in \mathbf{R}$. Then, Q_α and R_α defined by (1.5) and (3.2) are independent of α and are calculated as follows:

$$Q_\alpha([a, b]) = \beta[a, b] \quad \text{and} \quad R_\alpha([a, b]) = [a - \lambda, b - \lambda]$$

for $0 \leq a \leq b \leq 1$.

Let $g([a, b]) := (a + 2b)/3$ for $0 \leq a \leq b \leq 1$, which is additive. Then, $\mathcal{K}_\alpha(g)$ is given as

$$\mathcal{K}_\alpha(g) = \{[a, b] \in C(S) \mid a + 2b \leq 0\},$$

So $\mathcal{K}_\alpha^t(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) = \{[a, b] \in C(S) \mid a + 2b \leq 3\lambda\beta^{1-t}\}$. Since $\mathcal{K}_\alpha^t(g)$ is independent of α , we see that $Q_\alpha(\mathcal{K}_\alpha(g)) = \{\beta[a, b] \mid [a, b] \in \mathcal{K}_\alpha(g)\} \subset \mathcal{K}_\alpha(g)$ and $\bigcup_{t=1}^\infty \mathcal{K}_\alpha^t(g) = C(S)$. Thus Assumptions B1 and B2 in Section 3 are satisfied in this example.

Let the initial fuzzy state be

$$\tilde{s}(x) := (1 - |8x - 4|) \vee 0 \quad \text{for } x \in [0, 1].$$

For the stopping time $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ given in (2.4), we easily obtain that $\tilde{s}_\alpha = [(3 + \alpha)/8, (5 - \alpha)/8]$ and $\hat{\sigma}_\alpha(\tilde{s}_\alpha) = \min\{t \geq 1 \mid 13 - \alpha \leq 24\lambda\beta^{1-t}\}$. Thus, as $\hat{\sigma}_\alpha(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0, 1]$, we have $\tilde{s} \in \hat{\mathcal{F}}$.

Since $\hat{\sigma}_\alpha(\hat{s}_\alpha) \in \mathcal{K}^t(g)$ means $13 - \alpha \leq 24\lambda\beta^{1-t}$, then

$$\hat{\sigma}(\tilde{s}, t) = 1 \wedge ((13 - 24\lambda)\beta^{1-t} \vee 0).$$

The numerical value of $\hat{\sigma}$ is given in Table 1.

Table 1. An $\tilde{s}$ -optimal FST ($\lambda = 0.48, \beta = 0.98$).

t	1	2	3	4	5	6	7	8	...
$\hat{\sigma}(\tilde{s}, t)$	1	1	1	.7603	.5108	.2552	.00	.00	...

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