

# Conjectures about the differential operators in an algorithm for computing the residues.

Shinichi TAJIMA (田島 慎一) Niigata Univ.  
Yayoi NAKAMURA (中村 弥生) Ochanomizu Univ.

Let  $X = \mathbb{C}^2$  and fix a coordinate system  $z = (x, y)$  of  $X$ . We denote by  $\mathcal{O}_X$  the sheaf of holomorphic functions on  $X$ . Let  $f_1, f_2 \in \mathcal{O}_X$  and  $(f_1, f_2)$  be a regular sequence. Denote by  $I$  the sheaf of ideal of  $\mathcal{O}_X$  generated by  $f_1, f_2$ . Put  $A = \{z \in X | f_1 = f_2 = 0\}$ . Assume that at least one zero has multiplicity greater than 1. We denote by  $m$  the algebraic local cohomology class associated to the meromorphic function  $1/f_1 f_2$ .

In [4], we gave an algorithm to compute the cohomology class  $m$  and the residues. This algorithm has been constructed by the aid of the theory of  $\mathcal{D}_X$ -module and is based on the properties of the annihilators of  $m$ .

In this note, we examine the more detailed properties of annihilators which are useful for our algorithm. We use the computer algebra system Kan ([5]) and Risa/Asir ([2]).

## 1 The operators used in our algorithm

Let  $\Omega_X$  be the sheaf of holomorphic differential form on  $X$ . We assume that the set of common zeros  $A$  consists of finitely many points  $A_1, \dots, A_\nu$ . There is a pairing

$$\text{Res}_{A_\ell} : \Omega_X / I\Omega_X \otimes \mathcal{E}xt_{\mathcal{O}_X}^2(\mathcal{O}_X / I, \mathcal{O}_X) \rightarrow \mathbb{C}.$$

For  $m$ , this pairing yields a unique linear mapping  $\Omega_X / I\Omega_X \ni \phi(z)dz \mapsto \text{Res}_{A_\ell} \langle \phi(z)dz, m \rangle \in \mathbb{C}$  defined by the residue of the differential form  $\phi(z)dz/f_1 f_2$  at  $A_\ell$ .

Put  $V_K = \{\phi(z)dz \in \Omega_X / I\Omega_X \mid \text{Res}_{A_j} \langle \phi(z)dz, m \rangle = 0, j = 1, \dots, \nu\}$ . Let  $\mu_j$  be the multiplicity of  $A_j$ ,  $j = 1, \dots, \nu$  and  $\mu = \mu_1 + \dots + \mu_\nu$ . Then,  $V_K$  can be regarded as  $\mu - \nu$  dimensional vector space. Denote by  $\mathcal{A}nn$  the ideal generated by differential operators which annihilate  $m$ . Then we have the following theorem.

### Theorem 1

$$V_K = \{(R^* \psi(z))dz \mid R \in \mathcal{A}nn, \psi(z)dz \in \Omega_X / I\Omega_X\}.$$

Now we give conjectures about the properties of operators  $P_1, \dots, P_k \in \mathcal{A}nn$  which we use in our algorithm for computing the residues.

**Conjecture (A)** There exist  $P_1, \dots, P_k \in \mathcal{A}nn$  whose adjoint operators act on the vector space  $\mathbb{C}[x, y]/I$  and  $\text{Im}(P_1^*, \dots, P_k^*)$  span  $V_K$ . where  $\text{Im}(P_1^*, \dots, P_k^*)$  stands for the set of images of the adjoint operators  $P_j^*$ ,  $j = 1, \dots, k$  associated to  $\mathbb{C}[x, y]/I$ .

If there exist operators  $P_1, \dots, P_k \in \mathcal{A}nn$  which satisfy the property in the conjecture (A), we have following conjectures about construction of them.

**Conjecture (C1)**  $P_j$ 's are first-order differential operators.

Put  $P_j = c_{j1}\partial_x + c_{j2}\partial_y + c_{j0}$  where  $c_{j0}, c_{j1}, c_{j2} \in \mathbb{C}[x, y]$  and  $\partial_x := \partial/\partial x$ ,  $\partial_y := \partial/\partial y$ .

**Conjecture (C2)**  $\langle c_{11}, c_{12}, \dots, c_{k1}, c_{k2}, f_1, f_2 \rangle = \sqrt{\langle f_1, f_2 \rangle}$  as the ideal of  $\mathbb{C}[x, y]$ .

**Conjecture (C3)**  $\langle F_1, F_2, P_1, \dots, P_k \rangle = \mathcal{A}nn$ , where  $F_j = f_j$ ,  $j = 1, 2$  stands for differential operators of order 0.

**Conjecture (C4)** As for the number of first order differential operators, we have  $1 \leq k \leq 2$ .

## 2 Illustration of conjectures

We use the following procedure to investigate the annihilators  $P_j$ ,  $j = 1, \dots, k$ .

- (i) Construct annihilators of order zero and of order one.
- (ii) Take the gröbner bases  $GB$  of operators in (i).
- (iii) Find first order operators which generate  $GB$  together with 0th order operators. (we shall see the particular case in 2.2.2)
- (iv) Verify the condition (1).

These computation can be carried by computer algebra system Kan and Risa/Asir.

### 2.1 The case $A = \{(0, 0)\}$ .

#### 2.1.1 Example : $f_1 = x^5$ , $f_2 = y^2 + x^4 + x^3$

In this case,  $f_1$  and  $f_2$  have common zero only at the origin with multiplicity 10.

- (i) Computing syzygies on the ring of polynomials, we obtain

$$F_1 = x^5,$$

$$F_2 = y^2 + x^4 + x^3,$$

as annihilators of  $m$  of order zero and

- $-2yx\partial_x + (4x^4 + 3x^3)\partial_y - 10y$ ,
- $2yx\partial_x + (x^3 + 4y^2)\partial_y + 18y$ ,
- $(2x^2 + 2x)\partial_x + (4yx + 3y)\partial_y + 18x + 16$ ,
- $2yx\partial_x + (-4x^4 - 3x^3)\partial_y + 10y$ ,
- $(-2y^2x + 2y^2)\partial_x + (4yx^4 - yx^3 - 3yx^2)\partial_y - 10x^2 - 10y^2x$ ,
- $-2yx^2\partial_x + (4x^5 + 3x^4)\partial_y - 10yx$ ,
- $(-2x^2 + 6x)\partial_x + 9y\partial_y - 10x + 48$ ,

as annihilators of  $m$  of order one (see Section 3).

- (ii) The gröbner basis  $GB$  of the ideal generated by these operators with respect to the lexicographic order  $y \succ x$  is given by following 8 operators;

$$F_1 = x^5,$$

$$F_2 = y^2 + x^4 + x^3,$$

$$P_1 = (-2x^2 + 6x)\partial_x + 9y\partial_y - 10x + 48,$$

$$P_2 = x^3\partial_x + 5x^2,$$

$$P_3 = 2yx\partial_x + (-4x^4 - 3x^3)\partial_y + 10y,$$

$$P_4 = 3x^2\partial_x^2 + (-4x^2 + 24x)\partial_x - 20x + 30,$$

$$P_5 = 9x\partial_x^2 + (-16x^2 + 12x + 54)\partial_x - 9x^2\partial_y^2 - 80x + 60,$$

$$P_6 = -x\partial_x^4 - 8\partial_x^3 - 4x\partial_y^2\partial_x + (4x - 8)\partial_y^2.$$

- (iii) We find that the operators  $F_1$ ,  $F_2$  and  $P_1$  generate  $GB$ .

- (iv) The ideal generated by  $f_1$ ,  $f_2$  and the coefficients of  $\partial_x$  and  $\partial_y$  in  $P_1$  is equal to the radical of the ideal  $I$ , i.e. ,  $\langle f_1, f_2, -2x^2 + 6x, 9y \rangle = \langle x, y \rangle = \sqrt{\langle f_1, f_2 \rangle}$ .

In fact, we can see that the operator  $P_1$  satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism  $\Omega_X/I\Omega_X \cong \mathbb{C}[x, y]/I$ , these operators  $P_j$ ,  $j = 1, 2, 3$  act on the 10 dimensional vector space  $\mathbb{C}[x, y]/I$ . Using the gröbner basis with respect to the lexicographic order  $y \succ x$ , the monomial basis  $MB$  of  $\mathbb{C}[x, y]/I$  is  $MB = \{1, y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4\}$ . Then  $Im(P_1^*)$  is given by

$$\begin{aligned}
P_1^* 1 &= -6x + 33 & \text{mod } I, \\
P_1^* y &= -6yx + 24y & \text{mod } I, \\
P_1^* x &= -4x^2 + 27x & \text{mod } I, \\
P_1^* yx &= -4yx^2 + 18yx & \text{mod } I, \\
P_1^* x^2 &= -2x^3 + 21x^2 & \text{mod } I, \\
P_1^* yx^2 &= -2yx^3 + 12yx^2 & \text{mod } I, \\
P_1^* x^3 &= 15x^3 & \text{mod } I, \\
P_1^* yx^3 &= 6yx^3 & \text{mod } I, \\
P_1^* x^4 &= 9x^4 & \text{mod } I, \\
P_1^* yx^4 &= 0 & \text{mod } I.
\end{aligned}$$

From this computation, it follows that  $\dim \text{Im}(P_1^*) = 9$ . The other side,  $\dim \text{Im}(P_j^*) < 9$ ,  $j = 2, 3$ . Thus, we verify that the operator  $P_1$  enjoys (A).

The functions  $f_1$  and  $f_2$  are semiquasihomogeneous polynomials of degree 10 and 6 with weights  $wt(x) = 2$ ,  $wt(y) = 3$ . Put  $wt(\partial_x) = -2$  and  $wt(\partial_y) = -3$ . Then the operator  $P_1$  is the semiquasihomogeneous polynomial in  $\mathbb{C}[x, y, \partial_x, \partial_y]$  with the quasihomogeneous part  $3(2x\partial_x + 3y\partial_y + \underline{10} + \underline{6})$ . The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of  $f_1$  and  $f_2$  as semiquasihomogeneous polynomials.

### 2.1.2 Example : $f_1 = x^7$ , $f_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6)$

In this case,  $f_1$  and  $f_2$  have common zero only at the origin with multiplicity 14.

(i) Computing syzygies on the ring of polynomials, we obtain

$$F_1 = x^7,$$

$$F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6),$$

as annihilators of  $m$  of order zero and

- $(-2x^3 - 2yx^2)\partial_x + (-5yx^2 - 8y^2x)\partial_y - 24x^2 - 30yx$ ,
- $(37x^3 + 36yx^2)\partial_x + (94yx^2 + 144y^2x)\partial_y + 447x^2 + 540yx$ ,
- $((16y + 37)x^3 - 20yx^2 + 16x)\partial_x + (-24x^4 - 24yx^3 + (64y^2 + 94y)x^2 - 80y^2x + 40y)\partial_y - 48x^3 + (240y + 447)x^2 - 300yx + 192$ ,
- $((4y^2 - 2)x^2 - 2yx)\partial_x + ((16y^3 - 5y)x - 8y^2)\partial_y + (60y^2 - 24)x - 30y$ ,
- $yx^3\partial_x + 4y^2x^2\partial_y + 15yx^2$ ,
- $((-16y - 37)x^3 + (10y^2 + 55y)x^2 - 26x)\partial_x + (39x^4 + 39yx^3 + (-79y^2 - 94y)x^2 + (40y^3 + 220y^2)x - 65y)\partial_y + 78x^3 + (-270y - 447)x^2 + (150y^2 + 825y)x - 312$ ,
- $((16y^2 + 57y)x^3 - 10x^2 + 6yx)\partial_x + (-24yx^4 - 24y^2x^3 + (64y^3 + 174y^2)x^2 - 25yx)\partial_y - 48yx^3 + 747yx^2 + (480y^3 - 120)x + 42y$ ,
- $((4y^2 - 2)x^2 - 2yx)\partial_x + ((16y^3 - 5y)x - 8y^2)\partial_y + (60y^2 - 24)x - 30y$ ,
- $((-16y^2 - 57y)x^3 + (32y^3 + 114y^2 + 10)x^2 - 26yx + 12y^2)\partial_x + (24yx^4 - 24y^2x^3 + (-112y^3 - 174y^2)x^2 + (128y^4 + 348y^3 + 25y)x - 50y^2)\partial_y - 84x^4 - 120yx^3 - 747yx^2 + (-192y^3 + 1410y^2 + 120)x - 84y^3 - 282y$ ,
- $((-16y^2 - 57y)x^3 + 10x^2 - 6yx)\partial_x + (24yx^4 + 24y^2x^3 + (-64y^3 - 174y^2)x^2 + 25yx)\partial_y + 48yx^3 - 747yx^2 + (-480y^3 + 120)x - 42y$ ,
- $((-48y^4 - 171y^3)x^3 + (-16y^3 - 27y^2)x^2 + (-18y^3 + 10y)x - 6y^2)\partial_x + (72y^3x^4 + (72y^4 + 24y^2)x^3 + (-192y^5 - 522y^4 + 24y^3)x^2 + (-64y^4 - 99y^3)x + 25y^2)\partial_y + 42x^4 + 84yx^3 + (-144y^3 - 345y^2)x - 84y^3 + 120y$

as annihilators of  $m$  of order one.

(ii) The gröbner basis  $GB$  of the ideal generated by these operators with respect to the lexicographic order  $y \succ x$  is given by following 10 operators

$$F_1 = x^7,$$

$$F_2 = y^2 + x(x^4 + 2x^3y - 3x^5y - x^6),$$

$$P_1 = (21x^3 + 16x)\partial_x + (-24x^4 + 40y)\partial_y + 147x^2 + 192,$$

$$P_2 = x^4\partial_x + 7x^3,$$

$$P_3 = -x^3\partial_x + 4x^6\partial_y - 7x^2,$$

$$P_4 = -2yx\partial_x + 5x^5\partial_y + 36x^6 - 16x^4 - 14y,$$

$$P_5 = (-5x^3\partial_y - 24x^2)\partial_x + (96x^5 - 35x^2)\partial_y - 168x,$$

$$P_6 = 4x^2\partial_x^2 + (9x^3 + 40x)\partial_x + 16x^4\partial_y + 63x^2 + 56,$$

$$P_7 = 3x\partial_x^2 + 24\partial_x - 5x^4\partial_y^2 + (-27x^5 + 36x^3)\partial_y,$$

$$P_8 = 5x\partial_x^3 + 45\partial_x^2 - 288x^2\partial_x + 25x^3\partial_y^2 + (1152x^5 + 90x^4 - 240x^2)\partial_y - 2016x.$$

(iii) We find that the operators  $F_1$ ,  $F_2$  and  $P_1$  generate  $GB$ .

(iv) Then, the ideal generated by  $f_1$ ,  $f_2$  and the coefficients of  $\partial_x$  and  $\partial_y$  in  $P_1$  is equal to the radical of the ideal  $I$ , i.e. ,  $\langle f_1, f_2, 21x^3 + 16x, -24x^4 + 40y \rangle = \langle x, y \rangle = \sqrt{\langle f_1, f_2 \rangle}$ .

In fact, we can verify that the operator  $P_1$  satisfies the property in the conjecture (A) and the other first order operators not as follows. Under the isomorphism  $\Omega_X/I\Omega_X \cong \mathbb{C}[x, y]/I$ , these operators  $P_j$ ,  $j = 1, 2, 3, 4$  act on the 14 dimensional vector space  $\mathbb{C}[x, y]/I$ . Using the gröbner basis with respect to the lexicographic order  $y \succ x$ , we have  $MB = \{1, y, x, yx, x^2, yx^2, x^3, yx^3, x^4, yx^4, x^5, yx^5, x^6, yx^6\}$ . Then  $Im(P_1^*)$  is given by

$$\begin{aligned} P_1^*1 &= 84x^2 + 136 && \text{mod } I, \\ P_1^*y &= 24x^4 + 84yx^2 + 96y && \text{mod } I, \\ P_1^*x &= 63x^3 + 120x && \text{mod } I, \\ P_1^*yx &= 24x^5 + 63yx^3 + 80yx && \text{mod } I, \\ P_1^*x^2 &= 42x^4 + 104x^2 && \text{mod } I, \\ P_1^*yx^2 &= 24x^6 + 42yx^4 + 64yx^2 && \text{mod } I, \\ P_1^*x^3 &= 21x^5 + 88x^3 && \text{mod } I, \\ P_1^*yx^3 &= 21yx^5 + 48yx^3 && \text{mod } I, \\ P_1^*x^4 &= 72x^4 && \text{mod } I, \\ P_1^*yx^4 &= 32yx^4 && \text{mod } I, \\ P_1^*x^5 &= 56x^5 && \text{mod } I, \\ P_1^*yx^5 &= 16yx^5 && \text{mod } I, \\ P_1^*x^6 &= 40x^6 && \text{mod } I, \\ P_1^*yx^6 &= 0 && \text{mod } I. \end{aligned}$$

From this computation, it follows that  $\dim Im(P_1^*) = 13$ . The other side,  $\dim Im(P_j^*) < 13$ ,  $j = 2, 3, 4$ .

The functions  $f_1$  and  $f_2$  are semiquasihomogeneous polynomials of degree 14 and 10 with weights  $wt(x) = 2$ ,  $wt(y) = 5$ . Put  $wt(\partial_x) = -2$  and  $wt(\partial_y) = -5$ . Then the operator  $P_1$  is the semiquasihomogeneous polynomial in  $\mathbb{C}[x, y, \partial_x, \partial_y]$  with quasihomogeneous part  $8(\underline{2x\partial_x} + \underline{5y\partial_y} + \underline{14} + \underline{10})$ . The underlined parts indicate that the quasihomogeneous part of the operator is determined by the degree of  $f_1$  and  $f_2$  as semiquasihomogeneous polynomials.

## 2.2 In the case that $A$ consists of several points

### 2.2.1 Example: $f_1 = (x^2 + y^2)^2 + 3x^2y - y^3$ , $f_2 = x^2 + y^2 - 1$

In this case,  $A = \{(0, 1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)\}$  with multiplicities 2 at each points.

(i) Computing syzygies on the ring of polynomials, we obtain

$$\begin{aligned} F_1 &= 16x^6 - 24x^4 + 9x^2, \\ F_2 &= 4x^4 - 5x^2 - y + 1 \end{aligned}$$

as annihilators of  $m$  of order zero and

- $(x^2 + y^2 - 1)\partial_y + 2y$ ,
- $(x^2 + y^2 - 1)\partial_x + 2x$ ,
- $(2y^2 + y)x\partial_x + (-2y - 1)x^2\partial_y + 6y^2 + 3y - 3$ ,
- $(2y^3 - y^2 - y)\partial_x + (-2y^2 + y + 1)x\partial_y + (-6y + 3)x$ ,
- $(2yx^2 + y^2 - y)\partial_x + (-2x^3 + (-y + 1)x)\partial_y + (6y - 3)x$ ,
- $(2y^2 + y)x\partial_x + (-2y - 1)x^2\partial_y + 6y^2 + 3y - 3$ ,
- $(-2y^2 - y)x\partial_x + (2y + 1)x^2\partial_y - 6y^2 - 3y + 3$ ,
- $(2y + 1)x\partial_x + (-2x^2 - 4y^2 + y + 3)\partial_y - 6y + 5$ ,

as annihilators of  $m$  of order one.

(ii) The gröbner basis  $GB$  of these operators with respect to the lexicographic order  $y \succ x$  is given by following 6 operators;

$$\begin{aligned} F_1 &= 16x^6 - 24x^4 + 9x^2, \\ F_2 &= 4x^4 - 5x^2 - y + 1 \\ P_1 &= (4x^3 - 3x)\partial_x + (8x^4 - 6x^2)\partial_y - 16x^4 + 36x^2 - 6, \\ P_2 &= (-16x^5 + 24x^3 - 9x)\partial_x - 96x^4 + 96x^2 - 18, \\ P_3 &= (8x^4 - 6x^2)\partial_x^2 + ((12x^3 - 9x)\partial_y + 64x^3 - 12x)\partial_x + (48x^2 - 18)\partial_y + 96x^2 + 12, \\ P_4 &= (4x^3 - 3x)\partial_x^3 + (48x^2 - 12)\partial_x^2 + ((-12x^3 + 9x)\partial_y^2 + (24x^3 - 30x)\partial_y + 144x)\partial_x \\ &\quad + (-48x^2 + 18)\partial_y^2 + (96x^2 - 60)\partial_y + 96. \end{aligned}$$

(iii) We find that the operators  $F_1$ ,  $F_2$  and  $P_1$  generate  $GB$ .

(iv) The ideal generated by  $f_1, f_2$  and the coefficients of  $\partial_x$  and  $\partial_y$  in  $P_1$  is equal to the radical of the ideal  $I$ , i.e.,  $\langle f_1, f_2, 4x^3 - 3x, 8x^4 - 6x^2 \rangle = \langle 4x^3 - 3x, 2x^2 + y - 1 \rangle = \sqrt{\langle f_1, f_2 \rangle}$ .

In fact, we can verify that the operator  $P_1$  satisfies the property in the conjecture (A) and the other first order operators are not as follows. Under the isomorphism  $\Omega_X/I\Omega_X \cong \mathbb{C}[x, y]/I$ , the operators  $P_j$ ,  $j = 1, 2$  act on the 6 dimensional vector space  $\mathbb{C}[x, y]/I$ . Using the gröbner basis with respect to the lexicographic order  $y \succ x$ , we have  $MB = \{1, x, x^2, x^3, x^4, x^5\}$ . Then  $Im(P_1^*)$  is given by

$$\begin{aligned} P_1^* 1 &= -16x^4 + 24x^2 - 3 \mod I, \\ P_1^* x &= -16x^5 + 20x^3 \mod I, \\ P_1^* x^2 &= -8x^4 + 12x^2 \mod I, \\ P_1^* x^3 &= -12x^5 + 15x^3 \mod I, \\ P_1^* x^4 &= -6x^4 + 9x^2 \mod I, \\ P_1^* x^5 &= -9x^5 + 45/4x^3 \mod I. \end{aligned}$$

From this computation, it follows that  $\dim Im(P_1^*) = 3 (= 6 - 3)$ . The other side,  $\dim Im(P_2^*) = 1 < 3$ .

Put  $I_1 = \langle (4x^2 - 3)^2, 4x^2 - 4y - 5 \rangle$  and  $I_2 = \langle x^2, y - 1 \rangle$ . Then  $I = I_1 \cap I_2$ . Let  $m_1$  be the cohomology class with support at  $V(I_1)$  and  $m_2$  the cohomology class with support at  $V(I_2)$  which satisfy  $m = m_1 + m_2$ . From the ideals  $\langle (4x^2 - 3)^2, 4x^2 - 4y - 5, P_1 \rangle$  and  $\langle x^2, y - 1, P_1 \rangle$ , we obtain  $R_1 = (12xy + 6x)\partial_x + (18y + 9)\partial_y + 12y + 42$  as an annihilator of first order of  $m_1$  and  $R_2 = x\partial_x + 2$  as an annihilator of first order of  $m_2$ . These operators satisfy the localization of the property in the conjecture (A) to  $\mathcal{O}_X/I_j$ ,  $j = 1, 2$ .

**2.2.2 Example :**  $f_1 = x^6 + (y^2 - 3)x^4 + (y^4 + y^2 + 3)x^2 + y^6 - y^4 + y^2 - 1$ ,  $f_2 = x^6 + (3y^2 - 3)x^4 + (3y^4 + 3y^2 + 3)x^2 + y^6 - 3y^4 + 3y^2 - 1$

In this case,  $A$  consists of  $\{(x, y) | x^8 - x^6 + 3x^4 - x^2 + 1 = x^6 + 2x^2 - y^2 = 0\}$  with multiplicity 1,  $(0, 1)$  with multiplicity 2,  $(0, -1)$  with multiplicity 2,  $(1, 0)$  with multiplicity 6, and  $(-1, 0)$  with multiplicity 6.

(i) Computing syzygies on the ring of polynomials, we obtain

$$\begin{aligned} F_1 &= -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 - 3y^2 + 3, \\ F_2 &= x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2 \end{aligned}$$

as annihilators of  $m$  of order zero and 26 operators of order one.

(ii) The gröbner basis  $GB$  of these operators with respect to the lexicographic order  $y \succ x$  is given by following 5 operators;

$$\begin{aligned} F_1 &= -6x^{14} + 25x^{12} - 56x^{10} + 85x^8 - 82x^6 + 47x^4 - 16x^2 - 3y^2 + 3, \\ F_2 &= x^{16} - 4x^{14} + 9x^{12} - 14x^{10} + 14x^8 - 9x^6 + 4x^4 - x^2, \\ P_1 &= (-13x^{11} + 26x^9 - 52x^7 + 52x^5 - 26x^3 + 13x)\partial_x - 132x^{10} + 176x^8 - 344x^6 \\ &\quad + 152x^4 + (44y^2 - 96)x^2 + 16y^2 + 10, \\ P_2 &= (yx^{10} - yx^8 + 3yx^6 - yx^4 + yx^2)\partial_y + 2x^{10} - 2x^8 + 6x^6 - 2x^4 + 2x^2, \\ P_3 &= ((yx^9 - yx^7 + 3yx^5 - yx^3 + yx)\partial_y + 2x^9 - 2x^7 + 6x^5 - 2x^3 + 2x)\partial_x \\ &\quad + (10yx^8 - 8yx^6 + 18yx^4 - 4yx^2 + 2y)\partial_y + 20x^8 - 16x^6 + 36x^4 - 8x^2 + 4. \end{aligned}$$

(iii) In this case, we need four operators  $F_1, F_2, P_1$  and  $P_2$  to generate  $GB$ .

(iv) Then the ideal generated by  $f_1, f_2$  and the coefficients of  $\partial_x$  and  $\partial_y$  in  $P_1$  and  $P_2$  is equal to the radical of the ideal  $I$ , i.e.,  $\langle F_1, F_2, -13x^{11} + 26x^9 - 52x^7 + 52x^5 - 26x^3 + 13x, yx^{10} - yx^8 + 3yx^6 - yx^4 + yx^2 \rangle = \langle -x^{11} + 2x^9 - 4x^7 + 4x^5 - 2x^3 + x, -yx^9 + yx^7 - 3yx^5 + yx^3 - yx, -2x^{10} + 3x^8 - 6x^6 + 5x^4 - x^2 - y^2 + 1 \rangle = \sqrt{\langle f_1, f_2 \rangle}$ .

In fact, we can verify that the operators  $P_1$  and  $P_2$  satisfy the property in the conjecture (A). Under the isomorphism  $\Omega_X/I\Omega_X \cong \mathbb{C}[x, y]/I$ , the operators  $P_1$  and  $P_2$  act on the 32 dimensional vector space  $\mathbb{C}[x, y]/I$ . And it follows that the vector space  $Im(P_1^*, P_2^*)$  is 12 dimension.

Put  $I_1 = \langle x^4 + (y^2 + 1)x^2 - y^2 + 1, 2x^4 - x^2 + y^4 + 2, x^6 + 2x^2 - y^2 \rangle$ ,  $I_2 = \langle x^2, y - 1 \rangle$ ,  $I_3 = \langle x^2, y + 1 \rangle$ ,  $I_4 = \langle (x - 1)^3, y^2 \rangle$ ,  $I_5 = \langle (x + 1)^3, y^2 \rangle$ . Then  $I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5$ . Let  $m_j$  be the cohomology class with support at  $V(I_j)$ ,  $j = 1, 2, 3, 4, 5$ , which satisfy  $m = m_1 + m_2 + m_3 + m_4 + m_5$ . From the ideals generated by  $P_1, P_2$ , and  $I_j$ , we obtain the annihilators of each  $m_j$ . For  $m_2$  and  $m_3$ , we have  $x\partial_x + 2$ . Concerning to  $m_4$ , we have  $\langle (x - 1)^3, y^2, (12x - 12)\partial_x - x^2 + 44x - 7, y\partial_y + 2 \rangle$  as annihilators of  $m_4$ . Note that  $I_4$  is generated by  $(x - 1)^3$  and  $y^2$ , both are univariate polynomials. For such a case, we need two first order differential operators. In the same way, we have  $\langle (x + 1)^3, y^2, (12x + 12)\partial_x - x^2 - 44x - 7, y\partial_y + 2 \rangle$  as

annihilators of  $m_5$ . Note that since the ideal  $I_1$  is simple,  $m_1$  does not require any first order differential operators.

### 3 Construction of annihilators of first order

We can find annihilators of first order by the computations of syzygies. Put  $P = a\partial_x + b\partial_y + c$  where  $a, b, c \in \mathbb{C}[x, y]$ . If there exist  $u_{11}, u_{12}, u_{21}$  and  $u_{22}$  which satisfy  $-af_{1x} - bf_{1y} = u_{11}f_1 + u_{12}f_2$  and  $-af_{2x} - bf_{2y} = u_{21}f_1 + u_{22}f_2$ ,  $P$  annihilates the cohomology class associated to the meromorphic function  $1/f_1f_2$  with  $c = -u_{11} - u_{22}$ . In other words,  $(a, b, u_{11}, u_{12}, u_{21}, u_{22})$  is a syzygy of  $\begin{pmatrix} -f_{1x} \\ -f_{2x} \end{pmatrix}, \begin{pmatrix} -f_{1y} \\ -f_{2y} \end{pmatrix}, \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f_1 \end{pmatrix}, \begin{pmatrix} 0 \\ f_2 \end{pmatrix}$ . Thus, we can obtain the first order differential operators annihilating the cohomology class  $m$  with respect to the given meromorphic function by using Kan. This observation is due to T. Oaku ([3]) and the algorithm has been implemented by him.

If these conjectures are right, we can compute the algebraic local cohomology group as left  $\mathcal{D}_X$ -module without any information on the  $b$ -function. Then, we will be able to obtain more efficient algorithm for computing the residues.

### References

- [1] M. Kashiwara, *On the holonomic systems of linear differential equations, II*. Inventiones mathematicae **49** (1978), 121–135.
- [2] M. Noro and T. Takeshima, *Risa/Asir—a computer algebra system*, Proceedings of International Symposium on Symbolic and Algebraic Computation (ed. Paul S. Wang), 387–396, ACM, New York, 1992. (<ftp://endeavor.fujitsu.co.jp/pub/isis/asir>).
- [3] T. Oaku, *Algorithms for the b-functions, restrictions, and algebraic local cohomology groups of D-modules*, Adv. in Appl. Math. **19** (1997), 61–105.
- [4] S. Tajima and Y. Nakamura, *Computing residues of a several variables rational function*, Sûrikaiseki Kenkyûsho kôyûroku, Kyoto Univ. **1085** (1999), 71–81.
- [5] N. Takayama, *Kan: A system for computation in algebraic analysis* (1991–), (<http://www.math.s.kobe-u.ac.jp>).