

The complex Ginzburg-Landau equation on general domain

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded or “unbounded” domain with boundary $\partial\Omega$. This paper is concerned with the *smoothing effect* (i.e., the existence of unique global strong solutions for L^2 -initial data) of the following initial-boundary value problem for the complex Ginzburg-Landau equation:

$$(CGL) \quad \begin{cases} \frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

Here $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $q \geq 2$ are constants, and u is a complex-valued unknown function. We assume for simplicity that Ω is of class C^2 and $\partial\Omega$ is bounded (or $\Omega = \mathbb{R}_+^N$) to characterize the domain of the Dirichlet Laplacian. There are many mathematical studies on the problem (CGL) (for the existence and uniqueness of solutions see, e.g., Temam [9], Yang [10] and Ginibre-Velo [1], [2]; for the large time behavior of solutions see, e.g., Hayashi-Kaikina-Naumkin [3]; for the inviscid limiting problem as $\lambda \downarrow 0$ and $\kappa \downarrow 0$ see, e.g., Machihara-Nakamura [4] and Ogawa-Yokota [5]).

In a previous paper [6, Theorem 1.3 with $p = 2$] we established the smoothing effect of (CGL) on the initial data without any restriction on $q \geq 2$ under the condition

$$(1.1) \quad \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{q-1}}{q-2}.$$

This condition implies that the mapping $u \mapsto (\kappa + i\beta)|u|^{q-2}u$ is accretive (see [6, Lemma 2.1]). Recently, we reported in [7, Theorem 1.1] that under the condition

$$(1.2) \quad 2 \leq q \leq 2 + \frac{4}{N},$$

the smoothing effect of (CGL) on the initial data can be obtained even if condition (1.1) breaks down. However, it was additionally assumed in [7] that Ω is a “bounded” domain.

The purpose of this paper is to remove the boundedness assumption on Ω . For that purpose we develop an abstract theory formulated in terms of subdifferential operators in the same way as in [6] and [7]. However, we should remove the compactness condition which was effectively used in [7]. To this end we introduce a new type of condition using the Yosida approximation (see condition (A5) in Section 2).

Before stating our result, we define a strong solution to (CGL) as follows:

Definition 1.1. A function $u(\cdot) \in C([0, \infty); L^2(\Omega))$ is said to be a *strong solution* to (CGL) if $u(\cdot)$ has the following properties:

- (a) $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ a.a. $t > 0$;
- (b) $u(\cdot)$ is locally absolutely continuous (so that strongly differentiable a.e.) on $\mathbb{R}_+$;
- (c) $u(\cdot)$ satisfies the equation in (CGL) a.e. on $\mathbb{R}_+$ as well as the initial condition.

Now we state the main theorem in this paper.

Theorem 1.1. Let Ω be a bounded or “unbounded” domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$). Assume that Ω is of class C^2 and $\partial\Omega$ is bounded (or $\Omega = \mathbb{R}_+^N$). Let $N \in \mathbb{N}$, $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $2 \leq q \leq 2 + 4/N$. Then for any $u_0 \in L^2(\Omega)$ there exists a unique global strong solution $u(\cdot) \in C([0, \infty); L^2(\Omega))$ to (CGL) such that

$$\begin{aligned} u(\cdot) &\in C_{\text{loc}}^{0,1/2}(\mathbb{R}_+; L^2(\Omega)) \cap C(\mathbb{R}_+; H_0^1(\Omega)), \\ \frac{du}{dt}(\cdot), \Delta u(\cdot), |u|^{q-2}u &\in L_{\text{loc}}^2(\mathbb{R}_+; L^2(\Omega)), \\ \|u(t)\|_{L^2} &\leq e^{\gamma t} \|u_0\|_{L^2} \quad \forall t \geq 0, \\ \|u(t) - v(t)\|_{L^2} &\leq e^{K_1 t + K_2 e^{2\gamma_+ t} (\|u_0\|_{L^2}^2 \vee \|v_0\|_{L^2}^2)} \|u_0 - v_0\|_{L^2} \quad \forall t \geq 0, \end{aligned}$$

where $v(\cdot)$ is a unique strong solution to (CGL) with $v(0) = v_0 \in L^2(\Omega)$, $\gamma_+ := \max\{\gamma, 0\}$, and K_1 and K_2 are positive constants depending only on $\lambda, \kappa, \beta, \gamma, q, N$.

Remark 1.1. In this paper we ignore the accretivity of the nonlinear term under condition (1.1) effectively used in [6]. However, taking account of the usefulness of the accretivity, we can unify [6, Theorem 1.3 with $p = 2$] and Theorem 1.1 (see [8]).

2. Abstract theory

Let X be a complex Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let S be a nonnegative selfadjoint operator with domain $D(S)$ in X . Let $\psi : X \rightarrow (-\infty, \infty]$ be a proper lower semi-continuous convex function, where “proper” means that $D(\psi) := \{u \in X; \psi(u) < \infty\} \neq \emptyset$. Then the subdifferential $\partial\psi(u)$ of ψ at $u \in D(\psi)$ is defined as the set $\{f \in X; \operatorname{Re}(f, v - u) \leq \psi(v) - \psi(u) \text{ for every } v \in X\}$. Here we assume for simplicity that $\psi \geq 0$ and $\partial\psi$ is single-valued. As is well-known, S is also represented by a subdifferential: $S = \partial\varphi$, where φ is given by

$$\varphi(u) := \begin{cases} \frac{1}{2} \|S^{1/2}u\|^2 & \text{if } u \in D(\varphi) := D(S^{1/2}), \\ \infty & \text{otherwise.} \end{cases}$$

Then we consider the following abstract Cauchy problem in X :

$$(ACP) \quad \begin{cases} \frac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constants. To solve (ACP) we use the Moreau-Yosida approximation ψ_ε of ψ defined as

$$\psi_\varepsilon(v) := \min_{w \in X} \left\{ \psi(w) + \frac{1}{2\varepsilon} \|w - v\|^2 \right\}, \quad v \in X, \quad \varepsilon > 0.$$

It is well-known that ψ_ε is Fréchet differentiable on X and the derivative $\psi'_\varepsilon = \partial(\psi_\varepsilon)$ coincides with the Yosida approximation $(\partial\psi)_\varepsilon$ of $\partial\psi$:

$$(\partial\psi)_\varepsilon := \frac{1}{\varepsilon}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon\partial\psi)^{-1}, \quad \varepsilon > 0$$

(see Showalter [11, Proposition IV.1.8]), and so we can use the simplified notation $\partial\psi_\varepsilon$:

$$\partial\psi_\varepsilon := \partial(\psi_\varepsilon) = (\partial\psi)_\varepsilon.$$

We introduce the following five conditions on S and ψ ; note that the compactness condition used in [7] is replaced with a new type of condition (A5).

(A1) $\exists q \in [2, \infty)$ such that $\psi(\zeta u) = |\zeta|^q \psi(u)$ for $u \in D(\psi)$ and $\zeta \in \mathbb{C}$ with $\operatorname{Re} \zeta > 0$.

(A2) $D(S) \subset D(\partial\psi)$ and $\exists C_1 > 0$ such that $\|\partial\psi(u)\| \leq C_1(\|u\| + \|Su\|)$ for $u \in D(S)$.

(A3) $\forall \eta > 0 \exists C_2 = C_2(\eta) > 0$ such that for $u \in D(S)$ and $\varepsilon > 0$,

$$|(Su, \partial\psi_\varepsilon(u))| \leq \eta \|Su\|^2 + C_2 \psi(J_\varepsilon u)^\theta \varphi(u),$$

where $\theta \in [0, 1]$ is a constant.

(A4) $\forall \eta > 0 \exists C_3 = C_3(\eta) > 0$ such that for $u, v \in D(\varphi) \cap D(\psi)$ and $\varepsilon > 0$,

$$|(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \leq \eta \varphi(u - v) + C_3 \left(\frac{\psi(J_\varepsilon u) + \psi(J_\varepsilon v)}{2} \right)^\theta \|u - v\|^2,$$

where $\theta \in [0, 1]$ is the same constant as in (A3).

(A5) $\exists C_4 > 0$ such that for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,

$$|(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)| \leq C_4 |\nu - \mu| (\sigma \|\partial\psi(u)\|^2 + \tau \|\partial\psi(v)\|^2),$$

where $\sigma, \tau > 0$ are constants satisfying $\sigma + \tau = 1$.

To state our abstract result we define a strong solution to (ACP) as follows:

Definition 2.1. A function $u(\cdot) \in C([0, \infty); X)$ is said to be a *strong solution* to (ACP) if $u(\cdot)$ has the following properties:

- (a) $u(t) \in D(S) \cap D(\partial\psi)$ a.a. $t > 0$;
- (b) $u(\cdot)$ is locally absolutely continuous (so that strongly differentiable a.e.) on $\mathbb{R}_+$;
- (c) $u(\cdot)$ satisfies the equation in (ACP) a.e. on $\mathbb{R}_+$ as well as the initial condition.

Now we state the main result in this section.

Theorem 2.1. Let $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume that conditions (A1)–(A5) are satisfied. Then for any $u_0 \in X$ there exists a unique strong solution $u(\cdot) \in C([0, \infty); X)$ to (ACP). Also, $u(\cdot)$ has the following properties:

- (a) $u(\cdot) \in C_{\text{loc}}^{0,1/2}(\mathbb{R}_+; X)$, with $\|u(t)\| \leq e^{\gamma t} \|u_0\| \quad \forall t \geq 0$;
- (b) $Su(\cdot), \partial\psi(u(\cdot)), (du/dt)(\cdot) \in L_{\text{loc}}^2(\mathbb{R}_+; X)$;
- (c) $\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are locally absolutely continuous on $\mathbb{R}_+$.

Furthermore, let $v(\cdot)$ be a unique strong solution to (ACP) with $v(0) = v_0 \in X$. Then

$$(2.1) \quad \|u(t) - v(t)\| \leq e^{K_1 t + K_2 e^{2\gamma t} (\|u_0\| \vee \|v_0\|)^2} \|u_0 - v_0\| \quad \forall t \geq 0,$$

where $K_1 := \gamma + (1 - \theta)C_3\sqrt{\kappa^2 + \beta^2}$ and $K_2 := \theta C_3\sqrt{\kappa^2 + \beta^2}/(2q\kappa)$.

Now we shall prove Theorem 2.1. To this end we first take $u_0 \in D(\varphi) \cap D(\psi)$. In what follows we assume that $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ and conditions (A1)–(A5) are satisfied. Given $\varepsilon > 0$, we consider the following problem approximate to (ACP):

$$(ACP)_\varepsilon \quad \begin{cases} \frac{du_\varepsilon}{dt} + (\lambda + i\alpha)Su_\varepsilon + (\kappa + i\beta)\partial\psi_\varepsilon(u_\varepsilon) - \gamma u_\varepsilon = 0, & t > 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Since $\partial\psi_\varepsilon$ is Lipschitz continuous on X , it follows from [6, Proposition 3.1 (i)] that $(ACP)_\varepsilon$ has a unique strong solution $u_\varepsilon(\cdot) \in C([0, \infty); X)$ such that $u_\varepsilon(\cdot) \in C^{0,1/2}([0, T]; X)$ and $(du_\varepsilon/dt)(\cdot), Su_\varepsilon(\cdot) \in L^2(0, T; X)$ for every $T > 0$.

The following lemma was obtained in [7, Lemma 2.3] by using conditions (A1) and (A3) with $\eta := \lambda/(2\sqrt{\kappa^2 + \beta^2})$.

Lemma 2.2. Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family of unique strong solutions to $(ACP)_\varepsilon$ with $u_0 \in D(\varphi) \cap D(\psi)$ as stated above. Then

$$(2.2) \quad \|u_\varepsilon(t)\| \leq e^{\gamma t} \|u_0\| \quad \forall t \geq 0,$$

$$(2.3) \quad 2\lambda \int_0^t \varphi(u_\varepsilon(s)) ds + q\kappa \int_0^t \psi(J_\varepsilon u_\varepsilon(s)) ds \leq \frac{1}{2} e^{2\gamma t} \|u_0\|^2 \quad \forall t \geq 0,$$

$$(2.4) \quad \varphi(u_\varepsilon(t)) \leq e^{K(t, \|u_0\|)} \varphi(u_0) \quad \forall t \geq 0,$$

$$(2.5) \quad \int_0^t \|Su_\varepsilon(s)\|^2 ds \leq \frac{2}{\lambda} e^{K(t, \|u_0\|)} \varphi(u_0) \quad \forall t \geq 0,$$

where $K(t, \|u_0\|) := k_1 t + k_2 e^{2\gamma t} \|u_0\|^2$ and $k_1 := 2\gamma + (1 - \theta)C_2\sqrt{\kappa^2 + \beta^2}$, $k_2 := \theta C_2\sqrt{\kappa^2 + \beta^2}/(2q\kappa)$.

Next we shall state the following *key lemma*, in which a new type of condition (A5) plays an important role. For a proof see [8, Lemma 2.5].

Lemma 2.3. *Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family of unique strong solutions to $(ACP)_\varepsilon$ with $u_0 \in D(\varphi) \cap D(\psi)$ as stated above. Then there exists a function $u(\cdot) \in C([0, \infty); X)$ such that $u(0) = u_0$ and*

$$(2.6) \quad u_\varepsilon(\cdot) \rightarrow u(\cdot) \quad (\varepsilon \downarrow 0) \text{ in } C([0, T]; X) \quad \forall T > 0,$$

$$(2.7) \quad J_\varepsilon u_\varepsilon(\cdot) \rightarrow u(\cdot) \quad (\varepsilon \downarrow 0) \text{ in } L^2(0, T; X) \quad \forall T > 0.$$

Now we can prove the existence of strong solutions to (ACP) with " $u_0 \in D(\varphi) \cap D(\psi)$ ".

Lemma 2.4. *Let $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume that conditions (A1) – (A5) are satisfied. Then for any $u_0 \in D(\varphi) \cap D(\psi)$ there exists a unique strong solution $u(\cdot) \in C([0, \infty); X)$ to (ACP) such that*

- (a) $u(\cdot) \in C^{0,1/2}([0, T]; X) \quad \forall T > 0$, with $\|u(t)\| \leq e^{\gamma t} \|u_0\| \quad \forall t \geq 0$;
- (b) $Su(\cdot), \partial\psi(u(\cdot)), (du/dt)(\cdot) \in L^2(0, T; X) \quad \forall T > 0$;
- (c) $\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are absolutely continuous on $[0, T] \quad \forall T > 0$, with

$$(2.8) \quad 2\lambda \int_0^t \varphi(u(s)) ds + q\kappa \int_0^t \psi(u(s)) ds \leq \frac{1}{2} e^{2\gamma+t} \|u_0\|^2 \quad \forall t \geq 0.$$

Furthermore, let $v(\cdot)$ be a unique strong solution to (ACP) with $v(0) = v_0 \in D(\varphi) \cap D(\psi)$. Then

$$(2.9) \quad \|u(t) - v(t)\| \leq e^{K_1 t + K_2 e^{2\gamma+t} (\|u_0\| \vee \|v_0\|)^2} \|u_0 - v_0\| \quad \forall t \geq 0,$$

where K_1 and K_2 are the same constants as in Theorem 2.1.

Proof. Let $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ be the family as stated above. Let $T > 0$. Then it follows from (2.5) that $\{Su_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in $L^2(0, T; X)$. As noted in the proof of Lemma 2.3, $\{\partial\psi_\varepsilon(u_\varepsilon(\cdot))\}_{\varepsilon>0}$ is bounded in $L^2(0, T; X)$ and so is $\{(du_\varepsilon/dt)(\cdot)\}_{\varepsilon>0}$ in view of the equation in $(ACP)_\varepsilon$. Since S , $\partial\psi$ and d/dt are demiclosed as operators in $L^2(0, T; X)$, we see from Lemma 2.3 that

$$Su_\varepsilon(\cdot) \rightarrow Su(\cdot), \quad \partial\psi_\varepsilon(u_\varepsilon(\cdot)) = \partial\psi(J_\varepsilon u_\varepsilon(\cdot)) \rightarrow \partial\psi(u(\cdot))$$

and $(du_\varepsilon/dt)(\cdot) \rightarrow (du/dt)(\cdot)$ ($n \rightarrow \infty$) weakly in $L^2(0, T; X)$ and $u(\cdot)$ satisfies properties (a) and (b). Therefore we can conclude that $u(\cdot)$ is a strong solution to (ACP). Property (c) is derived from (a) and (b). Letting $\varepsilon \downarrow 0$ in (2.3) and using (2.6), we obtain (2.8).

To prove (2.9) we use the limiting case of condition (A5): $\forall \eta > 0 \exists C_3 = C_3(\eta) > 0$ such that for $u, v \in D(\partial\varphi) \cap D(\partial\psi)$,

$$(2.10) \quad |(\partial\psi(u) - \partial\psi(v), u - v)| \leq \eta\varphi(u - v) + C_3 \left(\frac{\psi(u) + \psi(v)}{2} \right)^\theta \|u - v\|^2;$$

note that for $u \in D(\partial\psi)$, $\partial\psi_\varepsilon(u) \rightarrow \partial\psi(u)$ ($\varepsilon \downarrow 0$) in X . Now let $u(\cdot)$ and $v(\cdot)$ be strong solutions to (ACP) with $u(0) = u_0$ and $v(0) = v_0$, respectively. As in the proof of Lemma 2.3, it follows from (2.10) that

$$\begin{aligned}
 (2.11) \quad & \frac{1}{2} \frac{d}{dt} \|u - v\|^2 \\
 & \leq \gamma \|u - v\|^2 - 2\lambda \varphi(u - v) + \sqrt{\kappa^2 + \beta^2} |(\partial\psi(u) - \partial\psi(v), u - v)| \\
 & \leq \left\{ \gamma + \tilde{C}_3 \left(\frac{\psi(u) + \psi(v)}{2} \right)^\theta \right\} \|u - v\|^2 \\
 & \leq \Psi(u, v) \|u - v\|^2,
 \end{aligned}$$

where $\tilde{C}_3 := C_3 \sqrt{\kappa^2 + \beta^2}$ and $\Psi(u, v)$ is given by

$$\Psi(u, v) := \gamma + \tilde{C}_3 \left\{ (1 - \theta) + \theta \left(\frac{\psi(u) + \psi(v)}{2} \right) \right\} = K_1 + K_2 q \kappa (\psi(u) + \psi(v))$$

(K_1 and K_2 are the same constants as in Theorem 2.1). Here (2.8) implies that

$$\int_0^t \Psi(u(s), v(s)) ds \leq K_1 t + K_2 e^{2\gamma+t} (\|u_0\| \vee \|v_0\|)^2.$$

Therefore we can obtain (2.9) by integration of (2.11). $\square$

To prove Theorem 2.1 we need the following lemma (cf. [7, Lemma 2.4]).

Lemma 2.5. *Let $u(\cdot)$ be a strong solution to (ACP) with $u(0) = u_0 \in D(\varphi) \cap D(\psi)$ as in Lemma 2.4 constructed under conditions (A1)–(A5). Then*

$$(2.12) \quad t\varphi(u(t)) + \frac{\lambda}{2} \int_0^t s \|Su(s)\|^2 ds \leq \frac{1}{4\lambda} e^{K(t, \|u_0\|) + 2\gamma+t} \|u_0\|^2 \quad \forall t \geq 0,$$

where $K(t, \|u_0\|)$ is the same as in Lemma 2.2.

Proof. We use the limiting case of condition (A3): $\forall \eta > 0 \exists C_2 = C_2(\eta) > 0$ such that for $u \in D(S) \cap D(\partial\psi)$,

$$(2.13) \quad |(Su, \partial\psi(u))| \leq \eta \|Su\|^2 + C_2 \psi(u)^\theta \varphi(u),$$

where $\theta \in [0, 1]$ is the same constant as before; note that for $u \in D(\partial\psi)$, $\partial\psi_\varepsilon(u) \rightarrow \partial\psi(u)$ ($\varepsilon \downarrow 0$) in X and $\psi(J_\varepsilon u) \leq \psi_\varepsilon(u) \leq \psi(u)$. As in the proof of [7, Lemma 2.3], we see from (2.13) that

$$(2.14) \quad \frac{d}{ds} \left[\exp \left(- \int_0^s k(r) dr \right) \varphi(u(s)) \right] + \frac{\lambda}{2} \exp \left(- \int_0^s k(r) dr \right) \|Su(s)\|^2 \leq 0,$$

where $k(r) := k_1 + 2k_2 q \kappa \psi(u(r)) \geq 0$, and

$$(2.15) \quad 0 \leq \int_s^t k(r) dr \leq \int_0^t k(r) dr \leq K(t, \|u_0\|) \quad \forall s \in [0, t].$$

Multiplying the both sides of (2.14) by $s \in [0, t]$ and integrating it on $[0, t]$ yield

$$\begin{aligned} t\varphi(u(t)) + \frac{\lambda}{2} \int_0^t s \cdot \exp\left(\int_s^t k(r) dr\right) \|Su(s)\|^2 ds &\leq \int_0^t \exp\left(\int_s^t k(r) dr\right) \varphi(u(s)) ds \\ &\leq \exp\left(\int_0^t k(r) dr\right) \int_0^t \varphi(u(s)) ds. \end{aligned}$$

Therefore (2.12) follows from (2.8) and (2.15). $\square$

Once Lemmas 2.4 and 2.5 are established, we can prove Theorem 2.1 in the same way as in the proof of [6, Theorem 5.2] (see also [7]).

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying Theorem 2.1 to (CGL). Let $X := L^2(\Omega)$ with inner product $(\cdot, \cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$. Let $2 \leq q \leq 2 + 4/N$. Then we define the nonnegative selfadjoint operator S in X and the proper lower semi-continuous convex function ψ on X as follows:

$$\begin{aligned} Su &:= -\Delta u \text{ for } u \in D(S) := H^2(\Omega) \cap H_0^1(\Omega), \\ \psi(u) &:= \begin{cases} \frac{1}{q} \|u\|_{L^q}^q & \text{if } u \in D(\psi) := L^2(\Omega) \cap L^q(\Omega), \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

As is well-known, the subdifferential of ψ is given by

$$\partial\psi(u) = |u|^{q-2}u \text{ for } u \in D(\partial\psi) = L^2(\Omega) \cap L^{2(q-1)}(\Omega).$$

Therefore we can regard (CGL) as one of (ACP)s.

To apply Theorem 2.1 it suffices to show that all the conditions (A1)–(A5) introduced in Section 2 are satisfied. Here we consider only the new type of condition (A5). For the verification of other conditions (A1)–(A4) see [7]. We begin with the strong differentiability of the resolvent with respect to approximating parameter ε .

Lemma 3.1. *Let $f \in D(\partial\psi)$. For $\varepsilon \in [0, \infty)$ and $x \in \Omega$ put*

$$(3.1) \quad u_\varepsilon(x) := \begin{cases} (1 + \varepsilon\partial\psi)^{-1}f(x) & (\varepsilon > 0), \\ f(x) & (\varepsilon = 0). \end{cases}$$

Then $u_\varepsilon \in C^1([0, E]; L^2(\Omega)) \forall E > 0$ (as a function of ε), with

$$(3.2) \quad \frac{\partial u_\varepsilon}{\partial \varepsilon} = \begin{cases} -\frac{1}{1 + \varepsilon(q-1)|u_\varepsilon|^{q-2}} \partial\psi_\varepsilon(f) & (\varepsilon > 0), \\ -\partial\psi(f) & (\varepsilon = 0). \end{cases}$$

Proof. Using the inverse function theorem, we can show that $u_\varepsilon \in C^1([0, E]; L^2(\Omega))$ for every $E > 0$ (for the proof see [8, Proposition 3.4]). Here we derive only (3.2). To this end let $f \in D(\partial\psi)$ and $\varepsilon > 0$. Then it follows from (3.1) that

$$(3.3) \quad u_\varepsilon(x) + \varepsilon|u_\varepsilon(x)|^{q-2}u_\varepsilon(x) = f(x).$$

Writing as

$$u_\varepsilon(x) = v_\varepsilon(x) + iw_\varepsilon(x), \quad f(x) = g(x) + ih(x),$$

we see that (3.3) is equivalent to

$$\begin{cases} v_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{(q-2)/2}v_\varepsilon(x) = g(x), \\ w_\varepsilon(x) + \varepsilon(v_\varepsilon(x)^2 + w_\varepsilon(x)^2)^{(q-2)/2}w_\varepsilon(x) = h(x). \end{cases}$$

Differentiating the both sides with respect to ε yields

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial \varepsilon} + |u_\varepsilon|^{q-2}v_\varepsilon + \varepsilon(q-2)|u_\varepsilon|^{q-4}\left(v_\varepsilon \frac{\partial v_\varepsilon}{\partial \varepsilon} + w_\varepsilon \frac{\partial w_\varepsilon}{\partial \varepsilon}\right)v_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}\frac{\partial v_\varepsilon}{\partial \varepsilon} = 0, \\ \frac{\partial w_\varepsilon}{\partial \varepsilon} + |u_\varepsilon|^{q-2}w_\varepsilon + \varepsilon(q-2)|u_\varepsilon|^{q-4}\left(v_\varepsilon \frac{\partial v_\varepsilon}{\partial \varepsilon} + w_\varepsilon \frac{\partial w_\varepsilon}{\partial \varepsilon}\right)w_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}\frac{\partial w_\varepsilon}{\partial \varepsilon} = 0. \end{cases}$$

Solving this system of equations with respect to $\partial v_\varepsilon/\partial \varepsilon$ and $\partial w_\varepsilon/\partial \varepsilon$, we have

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial \varepsilon} = -\frac{1}{1 + \varepsilon(q-1)|u_\varepsilon|^{q-2}}|u_\varepsilon|^{q-2}v_\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial \varepsilon} = -\frac{1}{1 + \varepsilon(q-1)|u_\varepsilon|^{q-2}}|u_\varepsilon|^{q-2}w_\varepsilon. \end{cases}$$

This implies that

$$\frac{\partial u_\varepsilon}{\partial \varepsilon} = -\frac{1}{1 + \varepsilon(q-1)|u_\varepsilon|^{q-2}}\partial\psi(u_\varepsilon), \quad \varepsilon > 0.$$

Since $\partial\psi(u_\varepsilon) = \partial\psi_\varepsilon(f)$, we obtain (3.2) with $\varepsilon > 0$. In addition, it follows that

$$\|\varepsilon^{-1}(u_\varepsilon - f) + \partial\psi(f)\|_{L^2} = \|\partial\psi_\varepsilon(f) - \partial\psi(f)\|_{L^2} \rightarrow 0 \quad (\varepsilon \downarrow 0).$$

This shows that $(\partial u_\varepsilon/\partial \varepsilon)|_{\varepsilon=0} = -\partial\psi(f)$ and hence (3.2) is true at $\varepsilon = 0$. □

As a consequence of Lemma 3.1 we have

Lemma 3.2. *Let $q \geq 2$. Then for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,*

$$|(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)_{L^2}| \leq (q-1)|\nu - \mu| \left[\frac{2q-3}{2(q-1)} \|\partial\psi(u)\|_{L^2}^2 + \frac{1}{2(q-1)} \|\partial\psi(v)\|_{L^2}^2 \right].$$

Proof. The computation is almost the same as in [8, Lemma 3.7]. Let $u \in D(\partial\psi) = L^2(\Omega) \cap L^{2(q-1)}(\Omega)$. For $\varepsilon \in [0, \infty)$ and $x \in \Omega$ put

$$u_\varepsilon(x) := \begin{cases} (1 + \varepsilon \partial\psi)^{-1} u(x) & (\varepsilon > 0), \\ u(x) & (\varepsilon = 0). \end{cases}$$

Then Lemma 3.1 implies that $u_\varepsilon \in C^1([0, E]; L^2(\Omega))$ for every $E > 0$. Since $\partial\psi_\varepsilon(u) = \varepsilon^{-1}(u - u_\varepsilon)$ for $\varepsilon > 0$, it follows from (3.2) that

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} [\partial\psi_\varepsilon(u)] &= -\frac{1}{\varepsilon^2}(u - u_\varepsilon) - \frac{1}{\varepsilon} \cdot \frac{\partial u_\varepsilon}{\partial\varepsilon} \\ &= -\frac{1}{\varepsilon} \left[\partial\psi_\varepsilon(u) + \frac{\partial u_\varepsilon}{\partial\varepsilon} \right] \\ &= -\frac{(q-1)|u_\varepsilon|^{q-2}}{1 + (q-1)\varepsilon|u_\varepsilon|^{q-2}} \partial\psi_\varepsilon(u) \\ &= -\frac{(q-1)|u_\varepsilon|^{2(q-2)}u_\varepsilon}{1 + (q-1)\varepsilon|u_\varepsilon|^{q-2}}, \quad \varepsilon > 0. \end{aligned}$$

Since $|u_\varepsilon| \leq |u|$, we obtain

$$\left| \frac{\partial}{\partial\varepsilon} [\partial\psi_\varepsilon(u)] \right| \leq (q-1)|u_\varepsilon|^{2q-3} \leq (q-1)|u|^{2q-3}, \quad \varepsilon > 0.$$

Therefore we see that for $\nu, \mu > 0$,

$$|\partial\psi_\nu(u) - \partial\psi_\mu(u)| = \left| \int_\mu^\nu \frac{\partial}{\partial\varepsilon} [\partial\psi_\varepsilon(u)] d\varepsilon \right| \leq (q-1)|\nu - \mu| \cdot |u|^{2q-3},$$

and hence

$$(3.4) \quad |(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)_{L^2}| \leq (q-1)|\nu - \mu| \int_\Omega |u|^{2q-3} |v| dx.$$

It follows from Hölder's inequality and Young's inequality that

$$\int_\Omega |u|^{2q-3} |v| dx \leq \|u\|_{L^{2(q-1)}}^{2q-3} \|v\|_{L^{2(q-1)}} \leq \left(\frac{2q-3}{2(q-1)} \|u\|_{L^{2(q-1)}}^{2(q-1)} + \frac{1}{2(q-1)} \|v\|_{L^{2(q-1)}}^{2(q-1)} \right).$$

Applying this inequality to the right-hand side of (3.4), we can obtain the desired inequality because of $\|u\|_{L^{2(q-1)}}^{2(q-1)} = \|\partial\psi(u)\|_{L^2}^2$. $\square$

Lemma 3.2 shows that condition (A5) is satisfied with

$$\sigma := \frac{2q-3}{2(q-1)}, \quad \tau := \frac{1}{2(q-1)}.$$

Therefore Theorem 2.1 applies to give the assertion of Theorem 1.1.

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