数理解析研究所講究録
第 1660 巻 2009 年 128-138

Complexified Penner's coordinates and its applications

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1 Penner's λ-lengths

1.1 A coordinate-system for Teichmüller space

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk, a model of hyperbolic plane and

$$SU(1,1)=\left\{\left(\begin{array}{cc} a & b\\ \bar{b} & \bar{a}\end{array}\right):a,b\in\mathbb{C},|a|^{2}-|b|^{2}=1\right\}.$$

Then $PSU(1,1)$ is the group of orientation preserving hyperbolic motions of $\mathbb{D}$.

Let $G=G_{g,n}$ be the punctured surface group of type (g,n), where $2g-2+n>0$:

$$G=\langle a_{1},b_{1},\ldots,a_{g},b_{g},d_{1},\ldots,d_{n}:(\prod_{k=1}^{g}a_{k}b_{k}a_{k}^{-1}b_{k}^{-1})d_{1}\cdots d_{n}=1\rangle.$$

A point of the *Teichmüller space* $\mathcal{T}=\mathcal{T}_{g,n}$ is a class of faithful Fuchsian representations of G into $PSU(1,1)$ which have finite covolume. We denote points in $\mathcal{T}$ by *marked* groups Γ_{m}, where Γ is a Fuchsian group and $m:G\rightarrow\Gamma$ is an isomorphism.

Elements $D_{1},\ldots,D_{n}$ in $\Gamma_{m}\in\mathcal{T}$ corresponding to $d_{1},\ldots,d_{n}$ are *parabolic.* Choose a horocycle H_{k} invariant under D_{k} such that action of D_{k} on H_{k} is the translation of length one. Then the identification of Γ_{m} with $(\Gamma_{m},H_{1},\ldots,H_{n})$ gives the following statement.

$\mathcal{T}_{g,n}$ is naturally embedded in the decorated Teichmüller space $\tilde{\mathcal{T}}_{g,n}$.

Therefore, by restricting them to this embedded subspace, Penner's λ-length coordinates for $\tilde{\mathcal{T}}_{g,n}$ give also global coordinates for the Teichmüller space $\mathcal{T}_{g,n}$.

*A joint work with M. Näätänen. The author is grateful to Professor Robert Penner for helpful discussions. He thanks Professor Michihiko Fujii for organizing a series of workshops on hyperbolic geometry and its related topics.

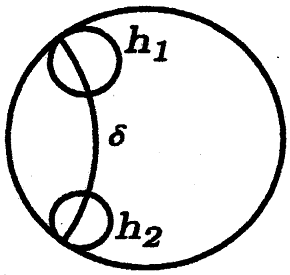
1.2 Distance between horocycles

Let p be a point of the unit circle. A *horocycle* h at p is a Euclidean circle in $\mathbb{D}$ tangent at p to the unit circle. The point p is called the *base point* of h.

Let h_1 and h_2 be horocycles based at different points p_1 and p_2 and γ the hyperbolic line between p_1 and p_2. Define

$$\lambda = e^{\delta/2}, \tag{1}$$

where δ is the *signed* length of the portion of the geodesic γ intercepted between the two horocycles h_1 and h_2, $\delta > 0$ if h_1 and h_2 are disjoint and $\delta < 0$ otherwise. In this way we can assign a positive number λ to the pair (h_1, h_2).



1.3 λ-length of an ideal arc

Let S be the oriented closed surface of genus g, $P = \{p_1, ..., p_n\}$ a set of n points. An *ideal arc* c of (S, P) is a path joining two points p_i and p_j in $S - P$. The ideal arc c is *simple* if $c \cap (S - P)$ is a simple arc.

Let $\Gamma_m \in \mathcal{T}_{g,n}$, then there exists an orientation preserving homeomorphism

$$f : S - P \to \mathbb{D}/\Gamma$$

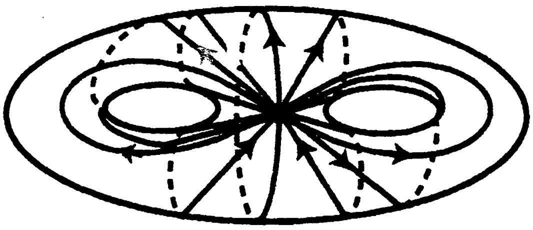
inducing m. Let γ be the geodesic representative in the homotopy class of $f(c)$ for the Poincaré metric of the punctured surface $\mathbb{D}/\Gamma$. By the identification of Γ_m with $(\Gamma_m, H_1, ..., H_n)$, the horocycles at the endpoints of γ defines the λ-length $\lambda(c, \Gamma_m)$.

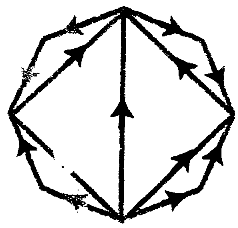
Let $\Delta = \{c_1, c_2, ..., c_q\}$, $q = 6g - 6 + 3n$, be an ideal triangulation of (S, P). Then

Theorem 1 (Penner [1])

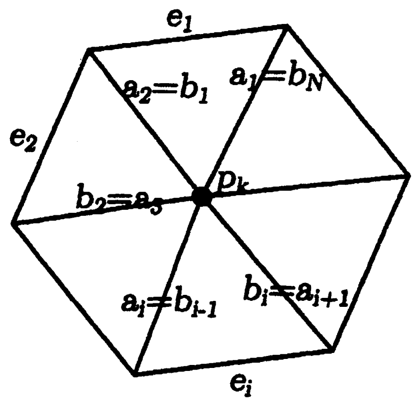
$$\lambda_\Delta = \prod_{i=1}^{q} \lambda(c_i) : \mathcal{T}_{g,n} \to (\mathbb{R}_+)^q$$

is an embedding.





The image of λ_Δ is a real algebraic variety determined by n polynomials. A component of $S - \cup_{j=1}^{q} c_j$ is called a *triangle* in Δ. The image of λ_Δ is a real algebraic variety determined by zero loci of n algebraic equations $D_1, \ldots, D_n$, where D_k is easily obtained by triangles abutting on the kth puncture p_k.



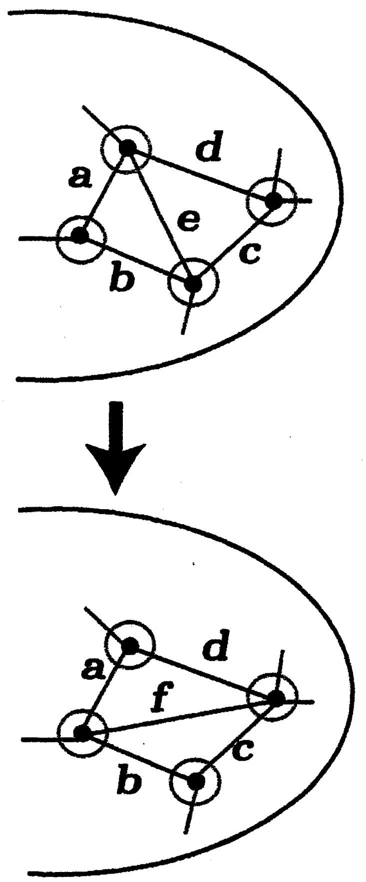
$$D_k(\lambda_1, \ldots, \lambda_q) = \sum_{i=1}^{N} \frac{\lambda(e_i)}{\lambda(a_i)\lambda(b_i)} - 1. \tag{2}$$

1.4 The Ptolemy identity

Let $\Delta = \{c_1, c_2, \ldots, c_q\}$ be an ideal triangulation of (S, P). Let $e \in \Delta$ and T_1 and T_2 be triangles being on the different sides of e. It is possible that $T_1 = T_2$. Lift $T_1 \cup e \cup T_2$ to a quadrangle $Q = \tilde{T}_1 \cup \tilde{e} \cup \tilde{T}_2$ in $\mathbb{D}$. Then $\tilde{e}$ is a diagonal of Q. Let $\tilde{f}$ be the other diagonal and project $\tilde{f}$ to an ideal arc f in $T_1 \cup e \cup T_2$. Then

$$\Delta' = (\Delta - \{e\}) \cup \{f\}$$

is another ideal triangulation of (S, P). We say that Δ' arises from Δ by the *elementary move* on e



Let $(\tilde{a}, \tilde{b}, \tilde{e})$ be the sides of $\tilde{T}_1$ and $(\tilde{c}, \tilde{d}, \tilde{e})$ be the sides of $\tilde{T}_2$. Suppose that $\tilde{a}$ and $\tilde{c}$ are opposite sides of Q. Let a, b, c, $d \in \Delta \cap \Delta'$ be the projections of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$. The following theorems are proved in Penner's paper :

Theorem 2 (the Ptolemy identity, Penner [1])
The λ-lengths function satisfy the identity

$$\lambda(a)\lambda(c) + \lambda(b)\lambda(d) = \lambda(e)\lambda(f) \tag{3}$$

This theorem describes the coordinate-change between $\lambda_\Delta(\mathcal{T})$ and $\lambda_{\Delta'}(\mathcal{T})$:

$$\begin{aligned} &\lambda_{\Delta'} \circ \lambda_\Delta^{-1}(\cdots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \lambda(e), \cdots) \\ &\quad = (\cdots, \lambda(a), \lambda(b), \lambda(c), \lambda(d), \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(e)}, \cdots) \end{aligned} \tag{4}$$

Theorem 3 (Penner [1]) *For arbitrary ideal triangulations Δ and Δ' of (S, P), there exists a finite sequence of ideal triangulations*

$$\Delta = \Delta_0, \Delta_1, \cdots, \Delta_m = \Delta',$$

where each Δ_i arises from Δ_{i-1} by an elementary move.

Using this theorem it can be shown that coordinate change between λ-length coordinates associated with two ideal triangulations is a bi-rational map:

Theorem 4 *If Δ and Δ' are ideal triangulations of F, then the coordinate change*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\lambda_\Delta} & \lambda_\Delta(\mathcal{T}) \subset (\mathbb{R}_+)^q \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \lambda_{\Delta'} \circ \lambda_\Delta^{-1}} \\ \mathcal{T} & \xrightarrow{\lambda_{\Delta'}} & \lambda_{\Delta'}(\mathcal{T}) \subset (\mathbb{R}_+)^q \end{array}$$

extends to a rational transformation of $\mathbb{R}^q$

Let $\mathcal{MC} = \mathcal{MC}_{g,n}$ denote the *mapping class group* of (S, P). Each $\varphi \in \mathcal{MC}$ acts on the Teichmüller space $\mathcal{T}$. The theorem above yields

Theorem 5 *The correspondence*

$$\phi \mapsto \phi_* = \lambda_{\varphi^{-1}(\Delta)} \circ \lambda_\Delta^{-1}$$

gives an isomorphism of $\mathcal{MC}$ to a group of rational transformations.

2 $SL(2,\mathbb{C})$-representation space of a punctured surface group

Let $\mathcal{R} = \mathcal{R}_{g,n}$ be the space of classes of faithful representations $[m]$ of the punctured surface group G into $SL(2,\mathbb{C})$ such that $m(d_i)$ is parabolic with tr $m(d_i) = -2$ for $i = 1, 2, ..., n$. The Teichmüller space $\mathcal{T}_{g,n}$ is a subspace of $\mathcal{R}_{g,n}$.

Our purpose is to give a coordinate-system for $\mathcal{R}_{g,n}$ whose restriction to $\mathcal{T}_{g,n}$ coincides with Penner's λ-lengths coordinate-system.

2.1 Parabolic elements of $SL(2,\mathbb{C})$

Define

$$\mathcal{P} = \{P \in SL(2,\mathbb{C}) : P \text{ is parabolic with } \mathrm{tr}P = -2\}.$$

If P_1 and $P_2 \in P$ do not commute, then the square root of $-P_1P_2$ in $SL(2,\mathbb{C})$

$$Q = \pm\frac{1}{\sqrt{2 - \mathrm{tr}P_1P_2}}(I - P_1P_2), \tag{5}$$

is unique up to sign and satisfies

$$P_2 = Q^{-1}P_1Q. \tag{6}$$

For the rest of this paper, the diagram

$$P_1 \xrightarrow{Q} P_2$$

will mean that $Q^2 = -P_1P_2$.

Cycles of parabolic elements

Let P_1, ..., P_n, $P_{n+1} = P_1 \in \mathcal{P}$. Suppose that no consecutive elements P_i and P_{i+1} commute. Let Q_i be a square root of $-P_iP_{i+1}$, $(i = 1, 2, .., n)$. Then, since $P_{i+1} = Q_i^{-1}P_iQ_i$, $Q_1Q_2 \cdots Q_n$ commutes with P_1,

$$\mathrm{tr}Q_1Q_2 \cdots Q_n = +2 \text{ or } -2. \qquad (7)$$

Definition
$(Q_1, Q_2, ..., Q_n)$ is a $(+)$-*system* or a $(-)$-*system* according to if $\mathrm{tr}Q_1Q_2 \cdots Q_n = +2$ or -2.

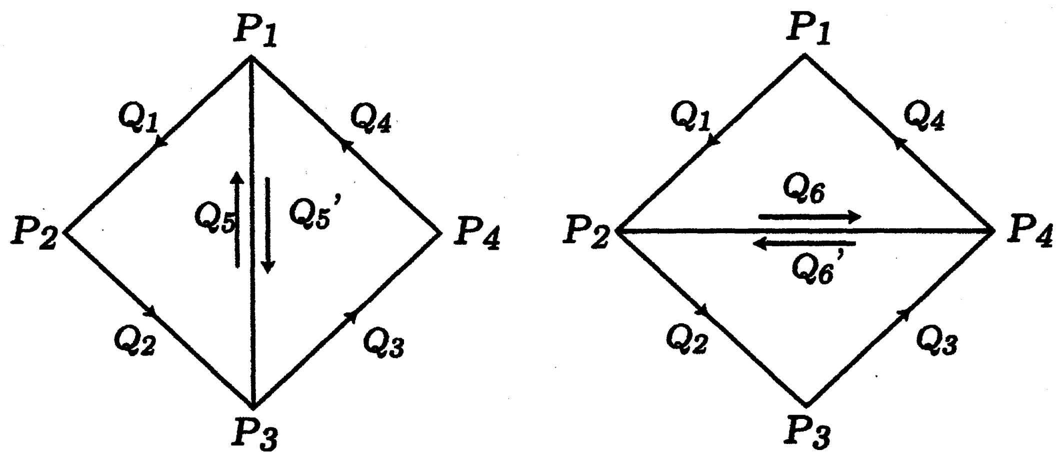
2.2 A trace identity of Ptolemy type

Let P_1, P_2, P_3 and P_4. Suppose that P_i and P_j do not commute unless $i = j$. Choose Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_5', $Q_6' \in SL(2, \mathbb{C})$ so that

$$\begin{array}{lll} Q_1^2 = -P_1P_2, & Q_2^2 = -P_2P_3, & Q_3 = -P_3P_4, \\ Q_4^2 = -P_4P_1, & Q_5^2 = -P_3P_1, & Q_6 = -P_2P_4, \\ (Q_5')^2 = -P_1P_3, & (Q_6')^2 = -P_4P_2, & \end{array}$$

where

$$Q_5' = P_1Q_5P_1^{-1}, \qquad Q_6' = P_4Q_6P_4^{-1}.$$



Theorem 6 *If* (Q_1, Q_2, Q_5), (Q_5', Q_3, Q_4) *and* (Q_1, Q_6, Q_4) *are* $(-)$-*systems, then*

$$\mathrm{tr}Q_5\mathrm{tr}Q_6 = \mathrm{tr}Q_1\mathrm{tr}Q_3 + \mathrm{tr}Q_2\mathrm{tr}Q_4 \qquad (8)$$

3 Complexified λ-length

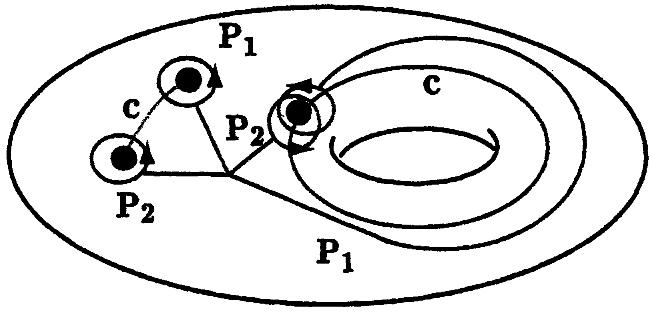
3.1 Definition of λ-length

A point of $\mathcal{R}$ is represented by a marked group Γ_m. Let $\mathcal{P}_+(\Gamma)$ be the set of parabolic elements in $[m(d_1)] \cup \cdots \cup [m(d_n)]$, where $[m(d_i)]$ is the conjugacy class of $m(d_i)$.

Let c be an ideal arc in (S, P). Then for each $\Gamma_m \in \mathcal{R}$, c defines two parabolic elements P_1, P_2 of $\mathcal{P}_+(\Gamma)$, see the following figure. We define the *λ-length* of c with respect to Γ_m by

$$\lambda(c, \Gamma_m) = \mathrm{tr}Q, \tag{9}$$

where Q is a square root of $-P_1P_2$. The λ-length is defined up to sign.



3.2 λ-length coordinates for $\mathcal{R}_{g,n}$

Let $\Delta = (c_1, c_2, ..., c_q)$ be an ideal triangulation of (S, P). Let T be a triangle in Δ. T inherites the orientation of the surface S. Label the sides of T by a, b, c in order. Then those sides determine matrices Q_a, Q_b, Q_c whose traces give λ-lengths of a, b and c for Γ_m.

Lemma 1 *It is possible to choose branches of λ-length functions $\lambda(c_1)$, $\lambda(c_2)$, ..., $\lambda(c_q)$ so that (Q_a, Q_b, Q_c) is a $(-)$-system for each triangle T in Δ.*

With the choice of branches of λ-lengths as depicted in the lemma, we obtain

Theorem 7 *For each ideal triangulation Δ,*

$$\lambda_\Delta = \prod_{i=1}^{q} \lambda(c_i) : \mathcal{R}_{g,n} \to (\mathbb{C}^*)^q$$

is an embedding. The image is contained in an algebraic variety.

3.3 Rational representation of the mapping class group

As in the case of $\mathcal{T}$, the Ptolemy identity (8) yields

Theorem 8 *The mapping class group $\mathcal{MC}$ acts on $\mathcal{R}$ as a group of rational transformations.*

4 Invariant holomorphic two-form

Let $T_1,\dots, T_p$, $p = 4g-2$, be triangles in an ideal triangulation of a once-punctured surface. Let the sequence of sides a_i, b_i, c_i of T_i agree with the positive orientation of T_i, then the 2-form

$$\sum_{i=1}^{p}(d\log\lambda(a_i)\wedge d\lambda(b_i)+d\log\lambda(b_i)\wedge d\log\lambda(c_i)+d\log\lambda(c_i)\wedge d\log\lambda(a_i)) \quad (10)$$

is invariant under the mapping class group $\mathcal{MC}$. The proof is similar to the one of the corresponding result in [2].

5 A characterization of the rational map induced by a mapping class

5.1 Example: Once punctured torus

The Teichmüller space $\mathcal{T}_{1,1}$ of once punctured tori is represented as the subspace of $(\mathbb{R}_+)^3$ defined by

$$x^2+y^2+z^2=xyz, \quad (11)$$

where x, y, z are λ-length functions related to an (essentially unique) triangulation of the once punctured torus (or x, y, z are trace functions tr_A, tr_B, tr_{AB}, with $\{A, B\}$ the canonical generator-system of $G_{1,1}$.)

The mapping class group $\mathcal{MC}_{1,1}$ has generators

$$\sigma(x,y,z)=(x,z,\frac{x^2+z^2}{y}) \quad \text{and} \quad \tau(x,y,z)=(\frac{x^2+y^2}{z},y,x),$$

with relations

$$(\tau\circ\sigma)^3=1, \quad (\sigma\circ\tau\circ\sigma)^2=1.$$

Since $\mathcal{MC}_{1,1}$ acts on $\mathcal{T}_{1,1}$, the group of rational transformations generated by σ and τ preserves the equation (11) and $(x, y, z) = (3, 3, 3)$ gives integer solutions of (11).

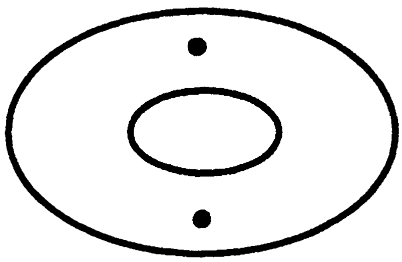
Theorem 9 (Markoff) *All positive integer solutions of (11) are in the oribit of $(3,3,3)$ under the action of $\mathcal{MC}_{1,1}$.*

The viewpoint of understanding the Markoff transformations as mapping classes actiong on $\mathcal{T}_{1,1}$ is given in Penner's paper [1].

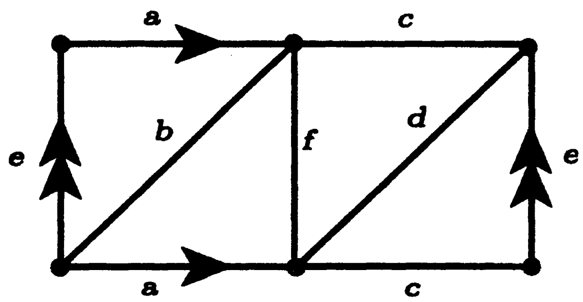
With λ-length coordinates, the Teichmüller space $\mathcal{T}_{g,n}$ is determined by n algebraic equations and the group of rational transformations induced by the mapping class group $\mathcal{MC}_{g,n}$ keep this space. So we can pursue analogies of the above result.

5.2 Example: twice punctured torus

Let Δ be the ideal triangulation of the twice punctured torus as depicted in the following figure.



twice punctured torus



Consider the λ-lengths

$$\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e$$

associated with Δ. Then it holds that $\lambda_e = \lambda_f$. The Teichmüller space $\mathcal{T}_{1,2}$ (or the space $\mathcal{R}_{1,2}$) is represented by the λ-lengths as the space

$$\frac{\lambda_e}{\lambda_a\lambda_b} + \frac{\lambda_a}{\lambda_b\lambda_e} + \frac{\lambda_b}{\lambda_a\lambda_e} + \frac{\lambda_c}{\lambda_d\lambda_e} + \frac{\lambda_d}{\lambda_c\lambda_e} + \frac{\lambda_e}{\lambda_c\lambda_d} = 1$$

or

$$\lambda_c\lambda_d(\lambda_a^2 + \lambda_b^2 + \lambda_e^2) + \lambda_a\lambda_b(\lambda_c^2 + \lambda_d^2 + \lambda_e^2) = \lambda_a\lambda_b\lambda_c\lambda_d\lambda_e. \tag{12}$$

The mapping class group $\mathcal{MC}_{1,2}$ (as a group of rational transformations) has generators

$$\begin{aligned}
\omega_{1*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_d, \lambda_b, \lambda_c, \frac{\lambda_e^2 + \lambda_d^2}{\lambda_a}, \lambda_e) \\
\omega_{2*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_d, \lambda_a, \lambda_b, \lambda_c, \frac{\lambda_a\lambda_c + \lambda_b\lambda_c}{\lambda_e}) \\
\omega_{3*}(\lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e) &= (\lambda_a, \frac{\lambda_b^2 + \lambda_e^2}{\lambda_c}, \lambda_b, \lambda_d, \lambda_e),
\end{aligned}$$

with relations

$$\omega_{2*}^2\omega_{1*}\omega_{2*}^2 = \omega_{3*} \qquad \omega_{1*}\omega_{3*} = \omega_{3*}\omega_{1*}$$

$$(\omega_{1*}\omega_{2*})^3 = 1, \qquad (\omega_{3*}\omega_{2*})^3 = 1$$

The point $p = (6, 6, 6, 6, 6)$ gives integer solutions of (12). An analogous result to the Markoff equation holds:

Theorem 10 *The orbit* $\{\varphi_*(6, 6, 6, 6, 6) : \varphi \in \mathcal{MC}_{1,2}\}$, *gives integer solutions of (12), but not all of its integer solutions.*

5.3 Diophantine equations

We consider a once punctured surface.

Lemma 2 *Let $(\lambda_1, \lambda_2, ..., \lambda_q)$ be the λ-length coordinate-system for $\mathcal{R}_{g,1}$ associated to an ideal triangulation $(c_1, c_2, ..., c_q)$, where $q = 6g - 3$. Then the λ-length of a simple ideal arc c is expressed by a rational function of the form*

$$\frac{P(\lambda_1, \lambda_2, ..., \lambda_q)}{\lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_q^{m_q}}, \tag{13}$$

where $P(\lambda_1, \lambda_2, ..., \lambda_q)$ is a homogeneous polynomial of degree

$$d = 1 + m_1 + m_2 + \cdots + m_q,$$

with positive integer coefficients and m_i is the geometric intersection number of c and c_i in $S - P$ for $i = 1, 2, ..., q$

For $\varphi \in \mathcal{MC}_{g,1}$ let φ_* denote the rational transformation induced by φ. Then entries of $\varphi_*(\lambda_1, \lambda_2, ..., \lambda_q)$ are of the form as in (13). This fact leads us to the following observation.

Let

$$D(\lambda_1, ..., \lambda_q) = 0 \tag{14}$$

be the algebraic equation which determines $\mathcal{T}_{g,1}$ in the λ-length coordinates. Then the rational transformation φ_* induced by $\varphi \in \mathcal{MC}_{g,1}$ preserves $D(\lambda_1, ..., \lambda_q)$. Moreover, if

$$(\lambda, \lambda, ..., \lambda)$$

gives integer solutions of (14), then so does $\varphi_*(\lambda, \lambda, ..., \lambda)$.

We remark that it is not true in general that all integer solutions are in the orbit of $(\lambda, \lambda, ..., \lambda)$ under $\mathcal{MC}$.

6 3-manifolds which fiber over the circle

Let $\varphi \in \mathcal{MC}_{g,n}$. Let M_φ be a manifold which fibers over the circle and whose monodoromy is φ. If φ_* denotes the action of φ on the fundamental group $G = G_{g,n}$ of the surface S of type (g, n), then the fundamental group of M_φ has the presentation

$$\tilde{G} = \langle G, t : \varphi_*(g) = tgt^{-1} \text{ for all } g \in G \rangle \tag{15}$$

If $m : \tilde{G} \to SL(2, \mathbb{C})$ is a faithful representation of $\tilde{G}$, then for all $g \in G$

$$(\varphi_* \circ m)(g) = m(t)m(g)m(t)^{-1}.$$

Hence the class $[m]$ is a fixed point of φ_* for its action on $\mathcal{R}_{g,n}$.

The λ-length coordinates of $\mathcal{R}_{g,n}$ represent φ_* as a rational function. Hence the fixed point $[m]$ corresponds to a solution of the algebraic equation

$$\varphi_*(\lambda_1, ..., \lambda_q) = (\lambda_1, ..., \lambda_q). \tag{16}$$

If φ is reducible, then one of the solutions of (16) gives a faithful and *discrete* representation m of G. We can find the Möbius transformation $m(t)$ easily, because $m(t)$ sends the fixed point of $m(g)$ to that of $m(\varphi_*(g))$ for each parabolic element $g \in G$. In this way *hyperbolization* of M_φ can be done. However, to carry this hyperbolization program into effect, we need efficient discreteness criteria.

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