

新 制
强
501
京大附図

A Measure of Integrity for Local Analytic Algebras

泉 脩 藏

学 位 審 査 報 告

氏 名	泉 脩 藏
学 位 の 種 類	理 学 博 士
学 位 記 番 号	論 理 博 第 号
学 位 授 与 の 日 付	昭 和 年 月 日
学 位 授 与 の 要 件	学 位 規 則 第 5 条 第 2 項 該 当
(学 位 論 文 題 目) A measure of integrity for local analytic algebras (局所解析環に対する整域性の尺度)	
論 文 調 査 委 員	主 査 永 田 雅 宜 戸 田 宏, 中 野 茂 男

理 学 研 究 科

(論文内容の要旨)

申請者は、複素数体 $\mathbb{C}$ 上の解析的集合 X の一点 ξ における解析関数の芽の環 $\mathcal{O}$ を考え、 $\mathcal{O}$ の各元に対する次の三つのものを考察している。

(1) algebraic order $\nu_{\xi}(f)$: $\mathcal{O}$ の極大イデアルを $\mathfrak{m}$ で表すとき、

$$f \in \mathfrak{m}^p, f \notin \mathfrak{m}^{p+1} \text{ となる } p \text{ が } \nu_{\xi}(f)$$

$\Rightarrow$

(2) reduced order $\bar{\nu}_{\xi}(f) = \lim_{k \rightarrow \infty} \nu_{\xi}(f^k) / k$

(3) analytic order $\mu_{A, \xi}(f)$ along A [$A \subseteq X$, (A の閉包) $= \xi$]:

$$\mu_{A, \xi}(f) = \sup \{ p \mid \exists \alpha > 0, \exists U = U(\xi) \text{ (近傍)},$$

$$U \cap A \text{ において, つねに } |f(x)| \leq \alpha |x - \xi|^p \}$$

明らかに $\mu_{A, \xi}(f) \geq \bar{\nu}_{\xi}(f) \geq \nu_{\xi}(f)$ であり、また解析集合の射 $\Phi: Y \rightarrow X$ ($\Phi(\eta) = \xi$) について、 $\nu_{\eta}(f \circ \Phi) \geq \nu_{\xi}(f)$ である。

D. Rees によって $\mathcal{O}$ がベキ零元をもたないときには、 f を動かしたとき、 $\bar{\nu}_{\xi}(f) - \nu_{\xi}(f)$ が有界であることが示された。

申請者は $\mathcal{O}$ が整域である場合について、次の三種の不等式を証明するとともに、実解析的である場合へのこの不等式の一般化をも与えている。

(CI₁) 適当な $a_1 \geq 1, b_1 \geq 0$ をえらべば、

$$\nu_{\xi}(fg) \leq a_1(\nu_{\xi}(f) + \nu_{\xi}(g)) + b_1$$

が、すべての $f, g \in \mathcal{O}$ について成り立つ。

(CI₂) 解析空間の射 $\Phi: (Y, \eta) \rightarrow (X, \xi)$ において、

$$\dim X_{\xi} = \inf \{ \dim \Phi(V) \mid V = V(\eta) \} \text{ であれば、適当な } a_2 \geq 1,$$

$$b_2 \geq 0 \text{ により}$$

$$\nu_{\eta}(f \circ \Phi) \leq a_2 \nu_{\xi}(f) + b_2$$

が、すべての $f \in \mathcal{O}$ について成り立つ。

(CI₃) 解析的開集合 A の閉包が ξ を含めば、 A に依存する定数 $a_3 \geq 1,$

$$b_3 \geq 0 \text{ によって、}$$

$$\mu_{A, \xi}(f) \leq a_3 \nu_{\xi}(f) + b_3$$

が、すべての $f \in \mathcal{O}$ について成り立つ。

これらの不等式において、 a_1, a_2, a_3 はそれぞれなるべく小さい方が不等式として強いものといえるから、 $\inf a_1, \inf a_3$ は環 $\mathcal{O}$ の構造の反映として注目されるものであり、また $\inf a_2$ は写像 Φ の性質の反映の一つである。

申請者は、これらの不等式の応用として、

- (1) $\mathbb{C}$ 上の収束べき級数環 $\mathbb{C}\{x_1, \dots, x_m\}$ において、 u が p 乗根 (p は素数) に開けない元であるとき、 $f, g \in \mathbb{C}\{x_1, \dots, x_m\}$ によって、 $\nu_0(f^p u - g^p)$ (ただし 0 は原点) を大きくしようとすれば、必然的に $\nu_0(f), \nu_0(g)$ も大きくなることを示し、
- (2) $\mathbb{C}\{x_1, \dots, x_m\}$ の元 u が、 p 乗根に開ける元にどの程度近いかを示す数値を与え、
- (3) 1963年に Malgrange が発表した予想 [Bull. Soc. Math. France, Tom. 91] を肯定的に解決した。

氏 名	泉 脩 藏
-----	-------

(論文審査の結果の要旨)

申請者の研究は解析的集合の局所的性質を研究するために有効な結果を多く含んでおり，特に主論文において，三種類の不等式を証明するとともに，それらの有効性を示すいくつかの新しい結果および Malgrange 予想の肯定的解決を与えた。

また，扱われた話題に盛り込まれた着想はすぐれており，今後のこの方面の研究に大きく寄与するものと思われる。

以上により，本論文は理学博士の学位を授与するに値すると認める。

なお，主論文及び参考論文に報告されている研究業績を中心とし，これに関連した研究分野について試問した結果，合格と認めた。

A Measure of Integrity for Local Analytic Algebras

By

Shuzo IZUMI*

§0 Introduction

We will complete the study $[I_1]$, $[I_2]$ on orders of elements of a local analytic algebra. Let $(X, \mathcal{O}_X)$ be a complex space and $(\mathcal{O}, m) := (\mathcal{O}_{X, \xi}, m_\xi)$ its local algebra at $\xi \in X$. $\mathcal{O}$ can be expressed as $\mathcal{O} = \mathbb{C}\{x\}/I$ for some ideal I of the algebra $\mathbb{C}\{x\}$ of convergent power series in $x = (x_1, \dots, x_m)$. We define three kinds of orders for $f \in \mathcal{O}$.

algebraic order: $v(f) = v_\xi(f) := \sup\{p: f \in m^p\} = \sup\{\text{the degree of the lowest non-zero homogeneous term of } \hat{f}: \hat{f} \text{ is a representative of } f \text{ in } \mathbb{C}\{x\}\}$

reduced order ([L-T], cf. [S]): $\bar{v}(f) := \lim_{k \rightarrow \infty} v(f^k)/k$

analytic order along $A \subset |X|$: $\mu_{A, \xi}(f) := \sup\{p: \text{there exist } \alpha > 0, \text{ neighbourhood } U \text{ of } \xi \text{ and representative } \tilde{f} \text{ of } f \text{ over } U \text{ such that } |\tilde{f}(x)| \leq \alpha |x - \xi|^p \text{ (} x \in U \cap A \text{)}\}$

We can easily verify the following.

Communicated by H. Hironaka, June 20, 1984 Revised October 9, 1984.

* Faculty of Science and Technology, Kinki-University, Higashi-Osaka 577, Japan.

inequality for the reduced order:

$$(I_0) \quad \bar{v}(f) \geq v(f) \quad (f \in \mathfrak{O})$$

inequality for products:

$$(I_1) \quad v(fg) \geq v(f) + v(g) \quad (f, g \in \mathfrak{O})$$

inequality for pullbacks: If $\Phi: Y \rightarrow X$ is a morphism such that $\Phi(\eta) = \xi$,

$$(I_2) \quad v_\eta(f \circ \Phi) \geq v_\xi(f) \quad (f \in \mathfrak{O})$$

inequality for the analytic order:

$$(I_3) \quad \mu_{A, \xi}(f) \geq v(f) \quad (f \in \mathfrak{O}).$$

We are interested in linear complementary inequalities (LCI) of these (I_i) . By the strong valuation theorem of Rees [Re₃] on Noetherian rings, we see the following:

If X is reduced at ξ , there exists $b_0 = b_0(\mathfrak{O}) \geq 0$ such that

$$(CI_0) \quad \bar{v}(f) \leq v(f) + b_0 \quad (f \in \mathfrak{O}).$$

We prove the following LCI as our main theorem, (3.4),

assuming that X is reduced and irreducible at ξ i.e. $\mathfrak{O}$

is an integral domain.

(1) There exist $a_1 = a_1(\Theta) \geq 1$, $b_1 = b_1(\Theta) \geq 0$ such that

$$(CI_1) \quad v(fg) \leq a_1(v(f) + v(g)) + b_1 \quad (f, g \in \Theta)$$

(2) For any morphism $\Phi: (Y, \eta) \rightarrow (X, \xi)$ with $\dim X_\xi =$

$\text{grnk}_\eta \Phi$ (see below), there exist $a_2 = a_2(\Phi) \geq 1$, $b_2 = b_2(\Phi) \geq 0$

such that

$$(CI_2) \quad v_\eta(f \circ \Phi) \leq a_2 v_\xi(f) + b_2 \quad (f \in \Theta)$$

(3) For any open subanalytic set $A \subset |X|$ adherent to ξ , there

exist $a_3 = a_3(A) \geq 1$, $b_3 = b_3(A) \geq 0$ such that

$$(CI_3) \quad \mu_{A, \xi}(f) \leq a_3 v(f) + b_3 \quad (f \in \Theta).$$

In the above $\text{grnk}_\eta \Phi$ denotes the generic rank defined

for a morphism of germs of analytic spaces over $k = \mathbb{C}$ or $\mathbb{R}$:

$\text{grnk}_\eta \Phi = \varepsilon \cdot \inf\{\text{topological dimension of } \Phi(U) : U \text{ is a}$

neighbourhood of $\eta\}$ ($\varepsilon = 1/2$ if $k = \mathbb{C}$; $\varepsilon = 1$ if $k = \mathbb{R}$)

This has been used to study properties of a ring homomorphism

induced by a morphism between spaces (see [Be])

The constants a_i (or b_i) in the above LCI measure some

intrinsic properties of Θ , Φ and A as follows. If Θ is not

an integral domain, a_1 does not exist ($a_1 = \infty$) So we
 may consider $\inf a_1 = \inf\{a_1: \text{there exists } b_1 \text{ such that}$
 $(CI_1) \text{ holds}\}$ as a measure of integrity of the algebra $\mathcal{O}$
 (or $1/\inf a_1$ as a distance of $\mathcal{O}$ from non-integral
 domains) $\inf a_2$ can be considered as a kind of order of
 the morphism ϕ . Indeed, in case $(X, \xi) = (\mathbb{C}, 0)$, we see
 that $\inf a_2$ coincides with the reduced order, $\bar{v}(\phi)$, of the
 function ϕ . $\inf a_3$ seems to have a relation with the
 size of the germ A_ξ Of course, if $A \subset B$, $\inf a_3(A)$
 $\geq \inf a_3(B)$

We have already proved that (1) and (2) of the main
 theorem are equivalent ($[I_1]$, (1.2)) So we prove (1)
 by induction on $n := \dim X_\xi$ (The proof of (3) is
 postponed till the end of §4) In §1 we recall the
 valuation theorem in a special form we need later It
 implies the case $n = 1$ of (1) immediately The case
 $n = 2$ can not be reduced to the case $n = 1$. We can

manage to prove this case by the use of the intersection matrix of Du Val-Mumford for a normal surface (§2). In §3 we reduce the case $n (\geq 3)$ to the case $n - 1$ using a Bertini type theorem of Flenner, completing the proof of (1). In §4 we show a real analytic version of the main theorem. There we remark that Risler's criterion for real ideal (= real radical) gives an affirmative answer to Malgrange's conjecture ([Ma], p. 124) on the real analytic set associated to a complex analytic set. In §5, as an application of the main theorem, we treat "near solutions" of the algebraic equation $f^p u - g^p = 0$ in $\mathbb{C}[[x]]$ for a given $u \in \mathbb{C}\{x\}$ and a prime number p . This yields an intrinsic measure of nearness of u to p -th powers in $\mathbb{C}\{x\}$. In Appendix we calculate the explicit values of $\inf a_1$ and $\inf b_1$ for some singularities.

The author wishes to thank Mr M. Tomari for many helpful advices and Professor K. Saito for pointing out a mistake in the formulation of (4.5) in the provisional manuscript.

§1 The Valuation Theorem

Here we introduce the most basic result on orders on an analytic algebra, restricting ourselves to the case of the orders with respect to the maximal ideal. Let X be a positive dimensional complex space reduced at ξ and let $\mathcal{M}$ denote the set of all morphisms $\Pi: Y \rightarrow X$ such that Y are normal, Π induce proper morphisms onto neighbourhoods of ξ and the ideal sheaves defining $\Pi^{-1}(\xi)$ (with the canonical structure) are invertible. $\mathcal{M}$ is not empty; the composition of the normalization after the blowing-up with center ξ belongs to $\mathcal{M}$. If $\Pi \in \mathcal{M}$, the exceptional fiber E over ξ (= the reduction of $\Pi^{-1}(\xi)$) has a finite number of irreducible components $\{E_i\}_{i \in A(\Pi)}$. Let I and I_i denote the sheaves of ideals defining E and E_i respectively. By the normality and the Hauptidealsatz (cf [G-R], p.129, p. 223), there exists an complex analytic set $S \subset |Y|$ of codimension at least 2 such that $Y - S$ ($= Y|_{|Y|-S}$) is a smooth manifold and $E - S$ and $E_i - S$ are submanifolds of $Y - S$ of codimension 1. For $f \in \Gamma(U, \mathcal{O}_X)$, we put

$$v_i(f) := \sup\{p: (f \circ \Pi)_y \in I_i^p \text{ for all (some) } y \in |E_i| - S\}$$

If $f \in \mathcal{O} := \mathcal{O}_{X, \xi}$, we define $v_i(f)$ using its representative.

It is obvious that $v_i(fg) = v_i(f) + v_i(g)$ ($f, g \in \mathcal{O}$)

(1.1) Lemma ([L-T], (4.1.8)) If we put $v_i(m) = \inf\{v_i(g): g \in m\}$ ($m := m_\xi$), we have

$$\inf\{v_i(f)/v_i(m): i \in A(\Pi)\} = \sup\{p/q: (f \circ \Pi)^q \in \Gamma(U, I^p)\}.$$

(1.2) Lemma ([L-T], (4.1.6), (5.5), (7.2) cf. (1.12)) Let

X be a positive dimensional complex space reduced at ξ .

Then, for any $f \in \Gamma(X, \mathcal{O}_X)$, $\bar{v}(f)$ is a rational number and the following conditions are equivalent for any $p, q \in \mathbb{N}$.

(r) $\bar{v}(f) \geq p/q$.

(i) There exist $\sigma_i \in \mathcal{O}$ ($1 \leq i \leq k$) such that $v(\sigma_i) \geq ip/q$ and

$$f^k - \sigma_1 f^{k-1} + \sigma_2 f^{k-2} - \cdots \pm \sigma_k = 0 \quad \text{in } \mathcal{O}.$$

(c) Let D denote the unit open disc in the complex plane $\mathbb{C}$.

Then for any morphism $\phi: D \rightarrow X$ with $\phi(0) = \xi$, we have

$$v_0(f \circ \phi) \geq (p/q) \inf\{v_0(g \circ \phi): g \in m\}$$

(m) For any $\Pi: Y \rightarrow X$ in $\mathcal{M}$,

$$\inf\{v_i(f)/v_i(m): i \in A(\Pi)\} \geq p/q.$$

(m') There exists $\Pi: Y \rightarrow X$ in $\mathcal{M}$ such that

$$\inf\{v_i(f)/v_i(m) : i \in A(\Pi)\} \geq p/q.$$

$$(a) \mu_{|X|, \xi}(f) \geq p/q$$

(1.3) Remark. This theorem was proved in an algebraic form by

Rees [Re₁], [Re₂] and Nagata [N₁] independently, developing

Samuel's paper [S]. They treated more general rings. The

conditions (c) and (d) were considered first by Lejeune and

Teissier. They stated the result for general coherent sheaves

on complex spaces.

(1.4) Remark. By (m) and (m') we see that $\bar{v}(f) =$

$$\inf\{v_i(f)/v_i(m) : i \in A(\Pi)\} \text{ for any } \Pi \in \mathcal{M}.$$

§2 LCI for normal surfaces

Here we use the notation in §1.

(2.1) Theorem. If X is a 2-dimensional complex space normal

at ξ , $\Pi: Y \rightarrow X$ a morphism in $\mathcal{M}$ and if Y is nonsingular,

we have the following.

$$(i) \text{ There exist } c_{ij} > 0 \text{ (} c_{ii} = 1 \text{) such that } c_{ij} v_j(f) \geq$$

$$v_i(f) \quad (i, j \in A(\Pi))$$

$$(ii) \text{ If we put } a_1 = (1 + \max_{i,j} c_{ij} v_j(m)/v_i(m))/2, \text{ we have}$$

$$\bar{v}(fg) \leq a_1(\bar{v}(f) + \bar{v}(g)) \text{ and hence } v(fg) \leq a_1(v(f) + v(g))$$

+ $2b_0$ $(f, g \in \mathcal{O})$ by (CI_0)

Proof (i) Let $A(\Pi) = \{E_1, \dots, E_k\}$ and let S_f denote the strict transform of the divisor (f) of $f \in \mathcal{O}$ with respect to Π . Then no E_i appears in S_f and $(f \cdot \Pi) =$

$S_f + \sum m_i E_i$ ($m_i = v_i(f)$) Taking intersection number

with E_j around E (cf [Mu]), we have

$$\sum_i m_i e_{ij} = (f \cdot \Pi) \cdot E_j - S_f \cdot E_j = -S_f \cdot E_j \leq 0$$

$$(j = 1, \dots, k; e_{ij} = E_i \cdot E_j)$$

Hence the vector $(m_1, \dots, m_k)$ is contained in the convex

cone $K \subset \mathbb{R}^k$ defined by

$$(*) \quad \sum_i x_i e_{ij} \leq 0, \quad x_j \geq 0 \quad (j = 1, \dots, k)$$

We have only to prove that $K \setminus \{0\}$ is included in the

strictly positive quadrant. Assume the contrary By

a rearrangement of the coordinates, we may assume that

$$(x_1, \dots, x_p, 0, \dots, 0) \in K \quad (x_1, \dots, x_p > 0, 1 \leq p < k)$$

Since $e_{ij} \geq 0$ ($i \neq j$), (*) implies $e_{ij} = 0$ ($i \leq p,$

$j > p$) This contradicts to the fact that E is connected.

(ii) We may assume that $v_\alpha(f)/v_\alpha(m) = \bar{v}(f) \leq \bar{v}(g)$

$= v_{\beta}(g)/v_{\beta}(m)$ (see (1.4)) Then we have

$$\begin{aligned} \bar{v}(fg) &= \min(v_i(f) + v_i(g))/v_i(m) \leq (v_{\beta}(f) + v_{\beta}(g))/v_{\beta}(m) \\ &\leq c_{\beta\alpha}(v_{\alpha}(m)/v_{\beta}(m))v_{\alpha}(f)/v_{\alpha}(m) + v_{\beta}(g)/v_{\beta}(m) \\ &\leq (2a_1 - 1)\bar{v}(f) + \bar{v}(g) \leq a_1(\bar{v}(f) + \bar{v}(g)). \quad \square \end{aligned}$$

(2.2) Remark. c_{ij} are obviously determined by (e_{ij})

We can show that a_1 is also majorized by a constant

determined by (e_{ij}) Indeed there exists $f \in O_{X,\xi}$ such

that $v_i(f) = v_i(m)$ ($i = 1, \dots, k$) Then $v_j(m) \leq$

$c_{ji}v_i(m)$ and $a_1 \leq (1 + \max_{i,j} c_{ij}c_{ji})/2$.

§3 LCI in General Case

We borrow an important result, a Bertini type theorem of Flenner:

(3.1) Lemma ([F], (4.10)) Let $(\mathcal{O}, \mathfrak{m})$ be a local k -algebra over a field k of characteristic 0 and let $\varphi_1, \dots, \varphi_m \in \mathfrak{m}$. Suppose that the completion $\hat{\mathcal{O}}$ is an integral domain and $\dim \mathcal{O}_{\mathfrak{p}} \geq 3$, $\text{Prof } \mathcal{O}_{\mathfrak{p}} \geq 2$ for any prime ideal $\mathfrak{p}$ that contains $\varphi_1, \dots, \varphi_m$ ($\mathcal{O}_{\mathfrak{p}}$: the localization). Then for any $e = (e_1, \dots, e_m)$ in a nonempty Zariski open subset of k^m , $\sum e_i \varphi_i$ is a prime element of $\mathcal{O}$ (even of $\hat{\mathcal{O}}$).

(3.2) Lemma. Let B be an infinite set of complex hyperplanes (linear subspaces of codimension 1) through 0 in $\mathbb{C}^m$.

Then $v_0(f) = \min_{H \in B} v_0(f|_H)$ ($f \in \mathbb{C}\{x\}$).

Proof. The inequality $v_0(f) \leq v_0(f|_H)$ is a special case of

the inequality for pullbacks (I_2) The case $f = 0$ is

trivial. If $f \neq 0$ and if H is not contained in the

tangent cone of the hypersurface defined by f , then the initial

form (the non-zero homogeneous term of the lowest degree) survives after restriction to H and $v_0(f) = v_0(f|_H)$ Since B is an infinite set, such an H really exists in B . $\square$

(3.3) Lemma ($[I_1]$, Added in proof) Let $\phi: (Y, \eta) \rightarrow (X, \xi)$ be the normalization of a reduced and irreducible germ of a complex space. If (Y, η) satisfies (CI_1) in (3.4), (X, ξ) also satisfies it.

Proof Since ϕ is finite (CI_2) holds ($[I_1]$, (2.3) or (4.1)):

$$v_\eta(f \circ \phi) \leq a v_\xi(f) + b \quad (f \in \mathcal{O}_{X, \xi}) \quad \text{for some } a \geq 1 \text{ and } b \geq 0.$$

Then we have

$$\begin{aligned} v_\xi(fg) &\leq v_\eta((f \circ \phi)(g \circ \phi)) \leq a_1(v_\eta(f \circ \phi) + v_\eta(g \circ \phi)) + b_1 \\ &\leq a_1 a(v_\xi(f) + v_\xi(g)) + 2a_1 b + b_1 \quad (f, g \in \mathcal{O}_{X, \xi}) \quad \square \end{aligned}$$

(3.4) Main Theorem. Let X be a complex space reduced and irreducible at ξ (i.e. $\mathcal{O} := \mathcal{O}_{X, \xi}$ is an integral domain) such that $n := \dim X_\xi \geq 1$. Then it has the following three properties

(1) There exist $a_1 = a_1(\mathcal{O}) \geq 1$, $b_1 = b_1(\mathcal{O}) \geq 0$ such that

$$(CI_1) \quad v(fg) \leq a_1(v(f) + v(g)) + b_1 \quad (f, g \in \mathcal{O})$$

(2) For any morphism $\phi: (Y, \eta) \rightarrow (X, \xi)$ with $\text{grnk}_\eta \phi = \dim X_\xi$ (see §0 for the symbol grnk), there exist $a_1 = a_1(\Theta) \geq 1$, $b_1 = b_1(\Theta) \geq 0$ such that

$$(CI_2) \quad v_\eta(f \circ \phi) \leq a_2 v_\xi(f) + b_2 \quad (f \in \Theta)$$

(3) For any open subanalytic set $A \subset X$ adherent to ξ , there exist $a_3 = a_3(A) \geq 1$, $b_3 = b_3(A) \geq 0$ such that

$$(CI_3) \quad \mu_{A, \xi}(f) \leq a_3 v(f) + b_3 \quad (f \in \Theta)$$

Proof The proof of (3) is postponed to (4.8) Since (1)

and (2) are equivalent ($[I_1]$, (1.2)), we have only to prove

one of them. The case $n = 1$ of (1) follows from (1.4),

since $v_i(fg) = v_i(f) + v_i(g)$ The case $n = 2$ follows

from (2.1) and (3.3) As to the case $n \geq 3$, we prove (1)

by induction on n .

Suppose that the case $n - 1$ ($n \geq 3$) has been proved

already. We may assume that X is normal by (3.3)

We may also assume that $\xi = 0 \in \mathbb{C}^m$ and X is a closed

complex subspace of a product open neighbourhood $U = V \times W$

($V \subset \mathbb{C}^n$, $W \subset \mathbb{C}^{m-n}$) of 0 such that the projection $\Pi: X \rightarrow V$

is finite proper open and locally biholomorphic on $X - \Pi^{-1}(D)$ for some proper analytic subset $D \subset V$ (local parametrization theorem). If we choose a sufficiently small U and such a coordinate space $\mathbb{C}^{m-n}$ that intersects the tangent cone of X_0 only at 0 , then $|\Pi(x)| \leq |x| \leq c|\Pi(x)|$ ($x \in X$) for some $c > 0$. Let P denote the complex projective space of hyperplanes of $\mathbb{C}^n$ through 0 and, if $H \in P$, let $X(H)$ denote the subspace $\Pi^{-1}(H)$ of X with the canonical complex structure. Let us put

$$B := \{H \in P : X(H) \text{ is reduced and irreducible at } 0\}$$

Since $\text{Prof } \mathcal{O} \geq 2$ by normality (cf [G-R], III, 1, Satz 11)

and since the completion $\hat{\mathcal{O}}$ is also an integral domain by

Nagata's theorem (cf [Ma], 1), we may apply (3.1) to $\mathcal{O}$ and

its parameter system induced from the affine coordinates of $\mathbb{C}^n$:

B includes a nonempty Zariski open subset of P . If $H \in B$,

there exist $a(H) \geq 1$, $b(H) \geq 0$ such that

$$a(H)(v_{X(H)}(u) + v_{X(H)}(v)) + b(H) \geq v_{X(H)}(uv) \quad (u, v \in \mathcal{O}_{X(H), 0})$$

by the inductive hypothesis $(v(\), v_H(\))$ and $v_{X(H)}(\)$

denote the algebraic orders respectively on X , H and $X(H)$

at 0) If we put $B_h = \{H: a(H) \leq h, b(H) \leq h\}$, we have

$B = \bigcup_{1 \leq h < \infty} B_h$ Therefore there exists $k \in \mathbb{N}$ such that B_k is an infinite set.

Now suppose that $f, g \in \mathcal{O}$ and $v(fg) = p$. Since

$v_{X(H)}(fg|X(H)) \geq v(fg)$ by the pullback inequality (I_2), we have

$$k(v_{X(H)}(f|X(H)) + v_{X(H)}(g|X(H))) + k \geq p \quad (H \in B_k)$$

Then , we may assume that $kv_{X(H)}(f|X(H)) + k/2 \geq p/2$

$(H \in B')$ for some infinite subset $B' \subset B_k$ without loss of

generality Let s denote the order of the covering

$X \rightarrow \Pi^{-1}(D) \rightarrow V \rightarrow D$ and let $\sigma_i(y)$ ($1 \leq i \leq s$) denote the

i -th elementary symmetric polynomial of values of f on the

fiber $\Pi^{-1}(y) = \{x_1, \dots, x_s\}$ for $y \in V - D$. $\sigma_i(y)$ has a

holomorphic extension over V If $H \in B'$,

$$\mu_{|X(H)|, 0}(f|X(H)) \geq v_{X(H)}(f|X(H)) \geq (p - k)/2k$$

by (I_3) Hence there exist $\epsilon_H > 0$ and $c_H > 0$ such that

$$|f(x)| \leq c_H |x|^{(p-k)/2k} \quad (x \in |X(H)|, |x| < \epsilon_H; H \in B')$$

Then there exists $\epsilon'_H > 0$ such that

$$\frac{|\sigma_i(y)|}{(c|y|)^{i(p-k)/2k}} \leq \sum_{\alpha_1 \leq \dots \leq \alpha_i} \frac{|f(x_{\alpha_1})|}{|x_{\alpha_1}|^{(p-k)/2k}} \cdot \frac{|f(x_{\alpha_i})|}{|x_{\alpha_i}|^{(p-k)/2k}}$$

$$= \begin{pmatrix} s \\ i \end{pmatrix} c_H^i \quad (y \in |H| \setminus D, \quad |y| < \varepsilon'_H ; H \in B').$$

Since $\{H: |H| \subset D\}$ is a finite set, by a choice of B' , we may assume that $D \cap |H|$ is thin in $|H|$ for $H \in B'$. Then,

if $H \in B'$, we have $|\sigma_i(y)| \leq d_{H,i} |y|^{i(p-k)/2k}$ ($y \in |H|$, $|y| < \varepsilon'_H$) for some $d_{H,i} > 0$. This proves that

$$v_H(\sigma_i|_H) = \mu_{|H|,0}(\sigma_i|_H) \geq i(p-k)/2k \quad (H \in B')$$

(The left equality follows from the regularity of H .)

Thus we have $v(\sigma_i \circ \Pi) \geq v_{\mathbb{C}^n}(\sigma_i) \geq i(p-k)/2k$ by (I_2) and

$$(3.2) \quad \text{Since } f^S - (\sigma_1 \circ \Pi) f^{S-1} + \dots \pm \sigma_s \circ \Pi = 0 \text{ on } X_0,$$

$\bar{v}(f) \geq (p-k)/2k$ by (1.2), $(i) \Rightarrow (r)$. Then we obtain

$$(CI_1) \text{ for the case } n, \text{ using } (CI_0): \bar{v}(f) \leq v(f) + b_0 \quad \square$$

(3.5) Corollary. Let X be a reduced complex space and let

$E \subset |X|$ be an exceptional set such that the set of regular

points of X is connected in an arbitrarily small

neighbourhood of E . Then for any $\xi, \xi' \in E$ there exist

$a \geq 1, b \geq 0$ such that $v_\xi(f) \leq a v_{\xi'}(f) + b$ for any

$f \in \Gamma(U, \mathcal{O}_X)$ defined in a neighbourhood U of E .

Proof. By the definition of exceptional set ([G], p.339), there exists a proper surjective morphism $\phi: X \rightarrow Y$ such that $\phi(E) = \{\eta\}$, ϕ is biholomorphic on $|X| - E$ and, if V is a neighbourhood of η , any $f \in \Gamma(\phi^{-1}(V), \mathcal{O}_X)$ can be expressed as $f = g \circ \phi$ for some $g \in \Gamma(V, \mathcal{O}_Y)$. Since U can be expressed as $\phi^{-1}(V)$, we may apply this to given f . Obviously Y_η is reduced and irreducible and $\text{grnk}_x \phi = \dim Y_\eta$ ($x \in E$). Then $v_\xi(f)$ and $v_{\xi'}(f)$ are linearly comparable to $v_\eta(g)$ by (I₂) and (CI₂) of the theorem. Hence $v_\xi(f)$ and $v_{\eta'}(f)$ are also linearly comparable. $\square$

(3.6) Corollary Let A be a ring finitely generated over $\mathbb{C}$ and $\mathfrak{m}$ one of its maximal ideal. Let us put $v_{\mathfrak{m}}(f) = \sup\{p: f \in \mathfrak{m}^p\}$ for $f \in A$. Then the following conditions are equivalent.

(i) There exist $a_1 \geq 1, b_1 \geq 0$ such that $v_{\mathfrak{m}}(fg) \leq$

$$a_1(v_{\mathfrak{m}}(f) + v_{\mathfrak{m}}(g)) + b_1 \quad (f, g \in A)$$

(ii) The completion $(A_{\mathfrak{m}})^{\wedge}$ of the localization $A_{\mathfrak{m}}$ is an integral domain.

Proof. We may assume that $A = \mathbb{C}[x]/I$ ($x = (x_1, \dots, x_m)$)

Since $\mathbb{C}$ is algebraically closed we may assume that

$I \subset m_0 := (x_1, \dots, x_m)\mathbb{C}[x]$, $m = m_0 A$ without loss of

generality. Then $\tilde{A} = \mathbb{C}\{x\}/I\mathbb{C}\{x\}$ is an analytic local

algebra with the maximal ideal $m\tilde{A}$ such that $A_m \subset \tilde{A}$. $(A_m)^\wedge$

is the completion of both $\tilde{A}$ and A_m and hence faithfully

flat over them. Then $\tilde{A}$ is faithfully flat over A_m and

$m^p A_m = m^p \tilde{A} \cap A_m$ On the other hand symbolic powers

$m^{(p)} := m^p A_m \cap A$ coincide with m^p by the maximality of m .

These prove that $v_m(f) = v_{mA_m}(f) = v_{m\tilde{A}}(f)$ ($f \in A$) Then

(i) follows from (ii) by (CI_1) for the integral domain $\tilde{A}$.

Conversely suppose that (ii) is false. Then there

exist $\hat{f}, \hat{g} \in (A_m)^\wedge$ such that $\hat{f}\hat{g} = 0$, $\hat{f} \neq 0$, $\hat{g} \neq 0$. We

can identify $(A_m)^\wedge$ with $\mathbb{C}[[x]]/I\mathbb{C}[[x]]$ Take

representatives $f, g \in \mathbb{C}[[x]]$ of $\hat{f}, \hat{g}$ respectively such that

$v_{m'}(f) = v_{\hat{m}}(\hat{f})$, $v_{m'}(g) = v_{\hat{m}}(\hat{g})$ ($m' = m_0\mathbb{C}[[x]]$, $\hat{m} = m(A_m)^\wedge$)

Let f_p (resp. g_p) be the polynomial which consists of the

homogeneous terms of f (resp. g) of degree less than $p + 1$.

If $p \geq \max\{v_m(\hat{f}), v_m(\hat{g})\}$, then $v_m(f_p \bmod I) = v_m(\hat{f}) < \infty$,

$v_m(g_p \bmod I) = v_m(\hat{g}) < \infty$. On the other hand $v_m(f_p g_p \bmod I)$

$\geq p + 1$. Hence (i) does not hold. □

§4 Real Analytic Algebras

Let $\mathcal{O}$ be a local real analytic algebra. It can be considered as the local ring $\mathcal{O}_{X,\xi}$ of a germ X_ξ of a real analytic space. We define the dimension of X_ξ by the Krull dimension of $\mathcal{O}$: $\dim X_\xi := \dim \mathcal{O}$. In general $\dim X_\xi \geq \dim |X|_\xi$. We call $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$ the complexification of $\mathcal{O}$. It determines a germ $X_\xi^{\mathbb{C}}$ of a complex space called the complexification of X_ξ . Since $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$ is integral over $\mathcal{O}$,

$$\dim X_\xi^{\mathbb{C}} = \dim \mathbb{C} \otimes_{\mathbb{R}} \mathcal{O} = \dim \mathcal{O} = \dim X_\xi$$

by a theorem of Krull-Cohen-Seidenberg.

(4.1) Theorem. Let $\mathcal{O}$ be a local real analytic algebra of positive dimension whose complexification is an integral domain. Then $\mathcal{O}$ satisfies the real versions of (1), (2), (3) of the main theorem (3.4)

Proof (1) Obvious by (3.4), (1)

(2) Let $\phi^{\mathbb{C}}, f^{\mathbb{C}}$ be the complexifications of ϕ and f

respectively. Since $v_{\xi}(f^{\mathbf{C}}) = v_{\xi}(f)$, $v_{\eta}(f^{\mathbf{C}} \circ \Phi^{\mathbf{C}}) = v_{\eta}(f \circ \Phi)$,
 $\text{grnk}_{\eta}^{\Phi^{\mathbf{C}}} = \text{grnk}_{\eta}^{\Phi}$ (cf. [I₁], p.461, P.465 (correction:
 paracompact $\rightarrow$ σ -compact)), our assertion follows from (3.4), (2)

(3) This follows from (3.4), (1) and [I₁], (3.1) □

If S is a germ of a real analytic set at $0 \in \mathbb{R}^n$, $I(S)$ denotes the ideal of elements of $\mathbf{R}\{x\}$ vanishing on S . If J is an ideal of $\mathbf{R}\{x\}$, $V(J)$ denotes the maximal germ of a real analytic set at 0 on which all elements of J vanish.

(4.2) Lemma ([Ri], (4.1)) The following conditions are equivalent for an ideal J of $\mathbf{R}\{x\}$

(C) $f_1^2 + \dots + f_p^2 \in J$ implies $f_1, \dots, f_p \in J$

(Z) $J = I(S)$ for some germ S of a real analytic set.

J is called real if it satisfies the conditions in (4.2)

(4.3) Lemma. Let J be a prime ideal of $\mathbf{R}\{x\}$

(i) J is real if and only if $\dim \mathbf{R}\{x\}/J = \dim V(J)$ ([Ri],

(4.3)).

(ii) If J is real, $\mathbf{C} \otimes_{\mathbf{R}} J$ is prime in $\mathbf{C}\{x\}$ ([G-T], p.371;

[Ri], (6.1))

(4.4) Remark. Let S be an irreducible germ of a real analytic set at $0 \in \mathbf{R}^n$. Then (4.1) is applicable to the integral domain $\mathbf{R}\{x\}/I(S)$ by (4.3), (ii). There exists, however, a non-real ideal J with a prime complexification $\mathbf{C} \otimes_{\mathbf{R}} J$ (e.g. $J = (x^2 + y^2 + z^2)\mathbf{R}\{x, y, z\}$). In such a case $\dim |X|_{\xi} < \dim X_{\xi}$, $\text{grnk}_{\eta} \phi < \dim X_{\xi}$ and hence (2) and (3) of (4.1) is meaningless. Note also that in such a case the complexifications of X_{ξ} and $|X|_{\xi}$ are not compatible.

For $f(z) = \sum \alpha_k z^k \in \mathbf{C}\{z\}$ ($z = (z_1, \dots, z_m)$), we put

$$f^*(z) := \sum \bar{\alpha}_k z^k \in \mathbf{C}\{z\},$$

$$\text{Re } f(x + iy) := (f(x + iy) + f^*(x - iy))/2 \in \mathbf{R}\{x, y\},$$

$$\text{Im } f(x + iy) := (f(x + iy) - f^*(x - iy))/2 \in \mathbf{R}\{x, y\}$$

If I is an ideal of $\mathbf{C}\{z\}$, $I^{\mathbf{r}}$ denote the ideal of $\mathbf{R}\{x, y\}$ generated by $\{\text{Re } f(x + iy) : f \in I\}$ (or, equivalently, by $\{\text{Im } f(x + iy) : f \in I\}$). We put

$$\mathcal{O} := \mathbf{C}\{z\}/I, \quad \mathcal{O}^{\mathbf{r}} := \mathbf{R}\{x, y\}/I^{\mathbf{r}}, \quad \tilde{I} := \mathbf{C} \otimes_{\mathbf{R}} I^{\mathbf{r}},$$

$$\tilde{\mathcal{O}} := \mathbf{C}\{x, y\}/\tilde{I} = \mathbf{C} \otimes_{\mathbf{R}} \mathcal{O}^{\mathbf{r}}$$

Further we put

$$J := \{f(u) : f(z) \in I\}, \quad J^* := \{f^*(v) : f(z) \in I\},$$

$$K := (J \otimes 1)\mathbb{C}\{u, v\} + (1 \otimes J^*)\mathbb{C}\{u, v\}$$

$$\subset \mathbb{C}\{u\} \hat{\otimes} \mathbb{C}\{v\} = \mathbb{C}\{u, v\}$$

$(\otimes, \hat{\otimes}$: analytic tensor product, see[G-R]) Let us define

an isomorphism $\theta: \mathbb{C}\{u, v\} \rightarrow \mathbb{C}\{x, y\}$ by $\theta(f(u, v))$

$$:= f(x + iy, x - iy)$$

(4.5) Lemma (cf. [Ma], §3, which treats analytic sets)

$$\tilde{I} = \theta(K) \quad \text{i.e.} \quad \tilde{\theta} \cong \theta \hat{\otimes} \theta^* \quad (\theta^* = \mathbb{C}\{v\}/J^*)$$

Proof Obvious by the following implication.

$$f \in \theta(K)$$

$$\iff \exists g \in K \quad \text{s.t.} \quad f(x, y) = g(x + iy, x - iy)$$

$$\iff \exists \alpha_\mu, \exists \beta_\mu \in \mathbb{C}\{u, v\}; \exists h_\mu, \exists k_\mu \in J \quad (\mu = 1, \dots, p)$$

$$\begin{aligned} \text{s.t.} \quad f(x, y) = & \sum_{\mu} \alpha_\mu (x + iy, x - iy) h_\mu (x + iy) \\ & + \beta_\mu (x + iy, x - iy) k_\mu^* (x - iy) \end{aligned}$$

$$\iff f \in \tilde{I} \quad \square$$

(4.6) Proposition. If θ is an integral domain in the above,

we have the following

(i) $\tilde{\mathcal{O}}$ is an integral domain.

(ii) $\mathcal{I}^r$ is a real ideal.

(ii) is just the affirmative answer to Malgrange's conjecture [Ma], p.124.

Proof. (i) By (4.5) we have only to prove that $\mathcal{O} \hat{\otimes} \mathcal{O}^*$ is an integral domain. Since $\mathcal{O}^*$ is isomorphic to $\mathcal{O}$ as a ring, it is an integral domain. Then $\mathcal{O} \hat{\otimes} \mathcal{O}^*$ is also so by Nagata's theorem [N_2], (47.5)

(ii) Since $\tilde{\mathcal{O}}$ is an integral domain and since

$$\dim \mathcal{O}^r = \dim \tilde{\mathcal{O}} = \dim \mathcal{O} \hat{\otimes} \mathcal{O}^* = 2 \dim \mathcal{O} = \dim V(\mathcal{I}^r),$$

we have the assertion by (4.3), (i). $\square$

(4.7) Remark. If $\mathcal{O}$ is an integral domain, we may well call

$\mathcal{O}^r$ the real analytic algebra associated to $\mathcal{O}$. But, if

not, $\mathcal{O}^r$ is not fit to be called so. For example, consider

the case $I = I_1 \cap I_2$, $I_i = z_i \mathbb{C}\{z_1, z_2\}$ ($i = 1, 2$)

Since $K = (u_1 u_2, v_1 v_2) \mathbb{C}\{u, v\}$, $V(\tilde{I}) \cong V(K)$ has four

components at $0 \in \mathbb{C}^4$. Therefore $V(\tilde{I})$ is bigger than

the join of $V(\tilde{I}_1)$ and $V(\tilde{I}_2)$ Returning to the algebras of the real parts, we see that $\mathcal{O}^r$ is too big to be called the real analytic space associated to $\mathcal{O}$.

(4.8) Remark. If f_μ ($\mu = 1, \dots, p$) generate I , $\operatorname{Re} f_\mu(x + iy), \operatorname{Im} f_\mu(x + iy)$ generate I^r Then, for any complex space X , we can construct the real analytic space X^r (defined by a coherent sheaf of ideals) and its complexification $\tilde{X}$ in the sense of [H]

Proof of (3.4), (3). Let X_ξ be a reduced and irreducible germ of a complex space. Then X_ξ^r has a reduced and

irreducible complexification $\tilde{X}_\xi$ by (4.6) Hence (4.1)

holds for X_ξ^r If $f \in \mathcal{O}_{X, \xi}$, we have only to apply

(4.1), (3) for X_ξ^r to the real and the imaginary part of

f . (If we consider the proof of (4.1), (3), we see

that (3.4), (3) for n -dimensional X_ξ results from (3.4),

(2) for $2n$ -dimensional $\tilde{X}_\xi$.) □

§5 Nearness to p -th Power in $\mathbb{C}\{x_1, \dots, x_m\}$

In this section we put $A = \mathcal{O}_{m,0} = \mathbb{C}\{x_1, \dots, x_m\}$,

$\hat{A} = \mathbb{C}[[x_1, \dots, x_m]]$, $v(\cdot) = v_0(\cdot)$ Let p be a prime

number and u not a p -th power in A . The functional

equation $f^p u - g^p = 0$ has unique solution $(f, g) = (0, 0)$

If $v(f^p u - g^p)$ is large, (f, g) is a "near solution" We

claim that a near solution is near to the unique solution:

(5.1) Proposition. If $u \in A$ is not a p -th power in A , there

exist $a \geq p$, $b \geq v(u)$, $b' \geq 0$ depending only on n, p, u

such that $av(f) + b \geq v(f^p u - g^p)$, $av(g) + b' \geq v(f^p u - g^p)$

for any $f, g \in \hat{A}$.

(5.2) Remark. By Artin's strong approximation theorem $[A_2]$,

(6.1), we see that, if u is a polynomial, there exists a

function $e: \mathbb{N} \rightarrow \mathbb{N}$ such that $v(f), v(g) \geq e[v(f^p u - g^p)]$

(5.1) asserts that we can choose a linear e .

Proof. Let us put $F = v(f)$, $G = v(g)$, $K = v(f^p u - g^p)$,

$U = v(u)$ and $T = \max\{F, G, K\} + 1$. We have to prove that

$$(*) \quad aF + b \geq K, \quad aG + b' \geq K.$$

Considering the cases $g = 0$ and $f = 0$ we have necessary conditions

$$(i) \quad a \geq p, \quad b \geq v(u), \quad b' \geq 0$$

This assures (*) if $T = \infty$. Thus we have only to treat the case $T < \infty$. If $f - f', g - g' \in m^T$ then $v(f') = F$, $v(g') = G$, $v(f'^p u - g'^p) = K$. Hence we may assume that $f, g \in A$. If the initial form (the homogeneous term of the lowest degree) in u is not a p -th power, $K = \min(pF + U, pG)$ and (*) is obvious ($a = p$). So we may assume that $U = pV$ ($V \in \mathbf{N}$)

Let us introduce a new variable z and consider

$$\mathfrak{O}_{m+1,0} := \mathbf{C}\{x_1, \dots, x_m, z\},$$

$$\mathfrak{O} := \mathfrak{O}_{m+1,0} / (u - (-z)^p) \mathfrak{O}_{m+1,0}$$

Let v and $v_{\mathfrak{O}}$ denote the algebraic order of $\mathfrak{O}_{m+1,0}$ and $\mathfrak{O}$ with respect to the respective maximal ideals (these are extensions of the original v) $h \in \mathfrak{O}$ has a unique

representative of the form $\sum_{i=0}^{p-1} h_i z^i \in \mathfrak{O}_{m+1,0}$ ($h_i \in A$)

$v_{\mathfrak{O}}(h) = \min (v(h_i) + i)$ by the assumption $U = pV \geq p$.

If we put

$$\begin{aligned} h := & (fg^{p-2}z^{p-1} - g^{p-1}z^{p-2}) + (f^{p-1}uz^{p-3} - f^{p-2}guz^{p-4} \\ & + \dots + f^3(-g)^{p-4}uz + f^2(-g)^{p-3}u) \\ & \qquad \qquad \qquad (\text{if } p = 2, h := fz - g), \end{aligned}$$

we have

$$(fz + g)h \equiv (f^p u - g^p)z^{p-2} \pmod{u} - (-z)^p \quad (f, g \in A)$$

Then, applying (3.4), (1) to the integral domain $\mathfrak{O}$, we have

$$\begin{aligned} \text{(ii)} \quad a_1(v(fz + g) + v(h)) + b_1 & \geq v((f^p u - g^p)z^{p-2}) = K + p - 2 \\ & \qquad \qquad \qquad (f, g \in A) \end{aligned}$$

If $pG \neq pF + U$, we can directly prove (*) using (i) So

we have only to treat the case $pG = pF + U$. Then, since

$$v(g^{p-1}z^{p-2}) - v(fg^{p-2}z^{p-1}) = V - 1,$$

$$v(f^{p-1}uz^{p-3}) - v(fg^{p-2}z^{p-1}) = 2(V - 1),$$

$$v(f^{p-2}guz^{p-4}) - v(fg^{p-2}z^{p-1}) = 3(V - 1),$$

.

$$v(f^2g^{p-3}u) - v(fg^{p-2}z^{p-1}) = (p - 1)(V - 1),$$

we have

$$\begin{aligned} \text{(iii)} \quad v(h) &= v(fg^{p-2}z^{p-1}) = (p-1)F + (p-2)V + p-1 \\ &= (p-1)G - V + p-1. \end{aligned}$$

Since $G - (F + 1) = V - 1 \geq 0$,

$$\text{(iv)} \quad v(fz + g) = F + 1 = G - V + 1.$$

Combining (ii), (iii) and (iv), we have

$$a_1(pF + (p-2)V + p) + b_1 - p + 2 \geq K,$$

$$a_1(pG - 2V + p) + b_1 - p + 2 \geq K.$$

Hence (*) holds by a suitable choice of a and b ($a = a_1p$) $\square$

(5.3) Definition. $\theta(p, u) := \inf a/p \quad (a: \text{in (5.1)})$

This can be regarded as a measure of nearness to p-th power In view of the proof above, we have an estimate

$$1 \leq \theta(p, u) \leq a_1$$

(5.4) Example. If u is not a p-th power, $\theta(p, u) = 1$.

This means that u is very far from p-th powers.

(5.5) Example. Suppose that $u := v^p - w$ is not a p-th power

in A and $k := v(w) \geq pv(v) + 1$. Using Tailor expansion

$\sqrt[p]{1+t+t^2+\dots} = 1 + c_1 t + c_2 t^2 + \dots$, we have

$$(1 + c_1 t + c_2 t^2 + \dots + c_n t^n)^p (1 - t) = 1$$

$$= d_{n+1} t^{n+1} + d_{n+2} t^{n+2} + \dots + d_{np+1} t^{np+1}$$

for suitable $d_{n+1}, \dots, d_{np+1} \in \mathbb{C}$. Hence

$$(v^{np} + c_1 v^{(n-1)p} w + c_2 v^{(n-2)p} w^2 + \dots + c_n w^n)^p (v^p - w)$$

$$= v^{(np+1)p}$$

$$= d_{n+1} v^{(np-n)p} w^{n+1} + d_{n+2} v^{(np-n-1)p} w^{n+2} + \dots + d_{np+1} w^{np+1}$$

If we put $f_n = v^{np} + c_1 v^{(n-1)p} w + \dots + c_n w^n$, $g_n = v^{np+1}$,

we have

$$v(f_n^p u - g_n^p) = v(v^{(np-n)p} w^{n+1}) = n(p-1)ph + (n+1)k$$

$$(h := v(v))$$

Hence $\theta(p, u) \geq (p^2 h - ph + k)/p^2 h = 1 + (k - hp)/hp^2 (> 1)$

(5.6) Example. If $u = x_1^p - x_2^k$ ($k > p$, $k \not\equiv 0 \pmod{p}$), a_1

for θ in the proof of (5.1) is estimated as $a_1 \leq k/p$

by the proof of $[I_1]$, (1.5) Hence we have

$$1 - 1/p + k/p^2 \leq \theta(p, u) \leq k/p.$$

Appendix: $\inf a_1$ and $\inf b_1$ for Some Elementary Singularities

Let $\underline{a}_1$ denote the infimum of a_1 in (3.4) for a given local complex analytic integral domain $\mathcal{O} = \mathcal{O}_{X,\xi}$. In case $\underline{a}_1$ is attained, $\underline{b}_1$ denotes the infimum of b_1 that satisfies (3.4), (1) with $a_1 = \underline{a}_1$. The author does not know whether $\underline{a}_1$ is always attained and whether $\underline{a}_1$ is a rational number. Here we exhibit the explicit values of $\underline{a}_1$ and $\underline{b}_1$ for some elementary singularities. $\underline{a}_1$ in (i), (iii), (iv) and $\underline{b}_1$ in (i) were first taught by Tomari.

(i) $v(fg) = v(f) + v(g)$ ($f, g \in \mathcal{O}$) (i.e. $\underline{a}_1$ and $\underline{b}_1$ are attained and $\underline{a}_1 = 1, \underline{b}_1 = 0$) iff $\text{Gr}_{\mathfrak{m}}^{\mathcal{O}} := \sum_{p=0}^{\infty} \mathfrak{m}^p / \mathfrak{m}^{p+1}$ is an integral domain. The proof is standard.

(ii) If $\dim \mathcal{O} = 1$, the exceptional fiber of the normalization of the blowing-up of X with center $\mathfrak{m}_{\xi}$ is irreducible.

Then $\underline{a}_1$ is attained and $\underline{a}_1 = 1$ by (1.4)

(iii) Let $(X, 0)$ be a hypersurface singularity in $\mathbb{C}^3$ defined by $z^2 - \varphi(x, y) = 0$ ($\varphi(0, 0) = 0$). Let v and $v_{\mathcal{O}}$ denote the algebraic orders on $\mathbb{C}\{x, y, z\}$ and on $\mathcal{O} := \mathbb{C}\{x, y, z\}/(z^2 - \varphi)\mathbb{C}\{x, y, z\}$ respectively. Further we assume that $r := v(\varphi)$ is an odd number greater than 2. Here we claim about $\mathcal{O}$ that $\underline{a}_1 = 1$ and $\underline{b}_1 = r - 2$ and that they are attained. Obviously every $\bar{f} \in \mathcal{O}$ has a unique representative of the form $f = f_1(x, y)z + f_2(x, y) \in \mathbb{C}\{x, y, z\}$ such that $v_{\mathcal{O}}(\bar{f}) = v(f) = \min(v(f_1) + 1, v(f_2))$. Suppose that

$$(*) \quad v_{\mathcal{O}}(\bar{f}\bar{g}) > v_{\mathcal{O}}(\bar{f}) + v_{\mathcal{O}}(\bar{g}) + r - 2$$

for some $\bar{f}, \bar{g} \in \mathcal{O}$. This implies that

$$v(f_1g_2 + f_2g_1) > v(f) + v(g) + r - 3,$$

$$v(f_1g_1\varphi + f_2g_2) > v(f) + v(g) + r - 2$$

By a standard calculation we have

$$v(f_1^2\varphi - f_2^2) > 2v(f) + r - 2.$$

On the other hand, since r is odd,

$$v(f_1^2\varphi - f_2^2) = \min(2v(f_1) + r, 2v(f_2)) \leq 2v(f) + r - 2,$$

a contradiction. Thus (*) never holds and we may put

$$a_1 = 1, b_1 = r - 2 \quad \text{Since}$$

$$v_{\mathcal{O}}(z^2) = r = (v_{\mathcal{O}}(z) + v_{\mathcal{O}}(z)) + r - 2,$$

b_1 can never be smaller than $r - 2$ Hence we have

proved the assertion.

In the case $\mathcal{O}$ is an isolated singularity of X (i.e. X is normal at $\mathcal{O}$), we can prove that the exceptional fiber of the normalization of the blowing-up of X with center m_0 is irreducible, which again implies that $\underline{a}_1$ is attained and $\underline{a}_1 = 1$ This case includes the rational singularities of types D_k ($k \geq 4$), E_6 , E_7 , E_8 (cf. [Br])

(iv) The remaining two dimensional rational singularity of multiplicity 2 is A_k ($k \geq 1$) defined by the equation

$$x^2 + y^2 + z^{k+1} = 0 \quad \text{in } \mathbb{C}^3 \quad \text{By the proof of [I}_1\text{], (1.5),}$$

we see that (3.4), (1) holds for $a_1 = (k + 1)/2$, $b_1 = 0$

On the other hand

$$\begin{aligned} a_1 &\geq \{v((x + iy)^p(x - iy)^p) - b_1\}/\{v((x + iy)^p) + v((x - iy)^p)\} \\ &= \{(k + 1)p - b_1\}/2p \quad (p \in \mathbb{N}) \end{aligned}$$

Hence $a_1 \geq (k + 1)/2$. Thus we have proved $a_1 = (k + 1)/2$,

$b_1 = 0$ and they are attained.

References

- [A] Artin, M.: Algebraic approximation of structures over complete local rings Publ. Math. IHES 36, 23-58 (1969)
- [Be] Becker, J : On the composition of power series. In: Commutative algebra (analytic methods) (LN in pure & ap. math. 68), (159-172) New York: Marcel Dekker 1982.
- [Br] Brieskorn, E.: Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. Math. Ann. 166, 76-102 (1966)
- [F] Flenner, H.: Die Sätze von Bertini für lokale Ringe. Math. Ann. 229, 97-111 (1977)
- [G] Grauert, H.: Über Modifikationen und exceptionelle analytische Mengen. Math. Ann. 146, 331-368 (1962)
- [G-R] Grauert, H., Remmert, R.: Analytische Stellenalgebren (GMW 176) Berlin-Heidelberg-New York: Springer 1971.
- [G-T] Galbiati, M., Tognoli, A.: Alcune proprietà delle varietà algebriche reali. Ann. Scuola Norm. Sup. Pisa 27, 359-404 (1973)

- [H] Hironaka, H.: Subanalytic sets.: In: Number theory, algebraic geometry and commutative algebra, in honor of Y Akizuki, (453-493) Tokyo: Kinokuniya 1973.
- [I₁] Izumi, S.: Linear complementary inequalities for orders of germs of analytic functions. Invent. Math. 65, 459-471 (1982)
- [I₂] Izumi, S.: Inequalities for orders on a rational singularity of a surface. J Math. Kyoto Univ 24, 239-241 (1984)
- [L-T] Lejeune-Jalabert, M., Teissier, B.: Clôture intégrale des idéaux et équisingularité. École Polytechnique 1974.
- [Ma] Malgrange, B.: Sur les fonctions différentiables et les ensembles analytiques. Bull. Soc. Math. France 91, 113-127 (1963)
- [Mu] Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Pub. Math. IHES 11, 229-246 (1961)

- [N₁] Nagata, M.: Note on a paper of Samuel concerning asymptotic properties of ideals. Mem. Coll. Sci Univ Kyoto, Ser. A, 30, 165-175 (1957)
- [N₂] Nagata, M.: Local rings (Interscience tracts in Pure & applied Math. 13) New York-London: Interscience Publishers 1962.
- [Re₁] Rees, D.: Valuations associated with a local ring (I). Proc. London Math. Soc 5. 107-128 (1955)
- [Re₂] Rees, D.: Valuations associated with ideals (II) J London Math. Soc. 31, 221-228 (1956)
- [Re₃] Rees, D.: Valuations associated with a local ring (II) J London Math. Soc. 31, 228-235 (1956)
- [Ri] Risler, J-J.: Le théorème des zéros en géométries algébrique et analytique réelles. Bull. Soc. Math. France 104, 113-127 (1976)
- [S] Samuel, P.: Some asymptotic properties of powers of ideals, Ann. of Math. 56, 11-21 (1952)