

$L^q - L^r$ estimate of the Stokes semigroup in a perturbed half-space

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1 Introduction

Let Ω be a perturbed half-space with smooth boundary $\partial\Omega$ in $\mathbb{R}_+^n$ ($n \geq 3$); to be precise, we call an open set Ω the exterior domain of $\mathbb{R}_+^n$ if there is a positive number R such that $\Omega \cap \{x \in \mathbb{R}_+^n; |x| > R\} = \{x \in \mathbb{R}_+^n; |x| > R\}$. In $\Omega \times (0, \infty)$, we consider the nonstationary Stokes initial boundary value problem concerning the velocity field $u(x, t)$ and the scalar pressure $p(x, t)$:

$$\begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = a(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $u_t = \partial u / \partial t$, Δ is the Laplacian in $\mathbb{R}_+^n$, $\nabla = (\partial_1, \dots, \partial_n)$ with $\partial_j = \partial / \partial x_j$ is gradient, and $\nabla \cdot u = \operatorname{div} u = \sum_{j=1}^n \partial_j u_j$ is the divergence of u .

To discuss our results more precisely, at first we outline at this point our notation used throughout the paper. To denote the special sets, we use the following symbols:

$$B_R = \{x \in \mathbb{R}_+^n; |x| < R\}, \quad \Omega_R = \Omega \cap B_R.$$

We will use the standard notations $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)}$ (or $\|\cdot\|_q$ if the underling domain is known from the context). We put

$$\begin{aligned} \mathbb{L}^p(\Omega) &= \{u = (u_1, \dots, u_n); u_j \in L^p(\Omega), j = 1, \dots, n\}, \\ \mathbb{L}_R^p(\Omega) &= \{u \in \mathbb{L}^p(\Omega); u(x) = 0 \text{ for } |x| > R\}, \\ \mathbb{J}^p(\Omega) &= \text{the completion in } \mathbb{L}^p(\Omega) \text{ of the set } \{u \in C_0^\infty(\Omega); \nabla \cdot u = 0 \text{ in } \Omega\}, \\ \mathbb{J}_R^p(\Omega) &= \{u \in \mathbb{J}^p(\Omega); u(x) = 0 \text{ for } |x| > R\}, \\ \mathbb{G}^p(\Omega) &= \{\nabla p \in \mathbb{L}^p(\Omega); p \in L_{loc}^p(\Omega)\}. \end{aligned}$$

For Banach spaces X and Y , $B(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y . We write $B(X) = B(X, X)$.

For the exterior domain Ω , R. Farwig and H. Sohr [7] proved that the Banach space $\mathbb{L}^p(\Omega)$ ($1 < p < \infty$) admits the Helmholtz decomposition: $\mathbb{L}^p(\Omega) = \mathbb{J}^p(\Omega) \oplus \mathbb{G}^p(\Omega)$, where

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$\oplus$ denotes the direct sum. Let $\mathbb{P}$ be a continuous projection from $\mathbb{L}^p(\Omega)$ to $\mathbb{J}^p(\Omega)$. The Stokes operator A is defined by $A = -\mathbb{P}\Delta$ with domain $D(A) = \{u \in \mathbb{J}^p(\Omega) \cap \mathbb{W}^{2,p}(\Omega); u|_{\partial\Omega} = 0\}$. It is proved by R. Farwig and H. Sohr that $-A$ generates an analytic semigroup e^{-tA} in $\mathbb{J}^p(\Omega)$.

Now we state our main results.

Theorem 1.1 (Local energy decay) *Let $1 < p < \infty$ and let m be a nonnegative integer. R is any positive number such that $\Omega \setminus B_R = \mathbb{R}_+^n \setminus B_R$. Then, for any $t \geq 1$, $a \in \mathbb{J}_R^p(\Omega)$, there exists a positive constant $C_{p,m}$ such that*

$$\|\partial_t^m e^{-tA} a\|_{\mathbb{L}^p(\Omega_R)} \leq C_{p,m} t^{-\frac{n+1}{2}-m} \|a\|_{\mathbb{L}^p(\Omega)}. \quad (1.2)$$

Theorem 1.2 ($L^q - L^r$ estimates) *Let $n \geq 3$.*

1. *For all $t > 0$, $a \in \mathbb{J}^q(\Omega)$ and $1 \leq q \leq r \leq \infty$ ($q \neq \infty$, $r \neq 1$), there holds the estimate:*

$$\|e^{-tA} a\|_{\mathbb{L}^r(\Omega)} \leq C_{q,r} t^{-(n/q-n/r)/2} \|a\|_{\mathbb{L}^q(\Omega)}. \quad (1.3)$$

2. *For all $t > 0$, $a \in \mathbb{J}^q(\Omega)$ and $1 < q \leq r \leq n$, there holds the estimate:*

$$\|\nabla e^{-tA} a\|_{\mathbb{L}^r(\Omega)} \leq C_{q,r} t^{-(n/q-n/r)/2-\frac{1}{2}} \|a\|_{\mathbb{L}^q(\Omega)}. \quad (1.4)$$

2 The representation formula of the solution to the Stokes resolvent problem in $\mathbb{R}_+^n$

In this section, we shall give the solution formula of the Stokes resolvent problem:

$$\begin{cases} (\lambda - \Delta)u + \nabla p = f, & \nabla \cdot u = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & & \text{on } x_n = 0. \end{cases} \quad (2.1)$$

Let $P_\lambda f$ and $\pi_\lambda f$ be defined by $P_\lambda f = u$ and $\pi_\lambda f = p$ which satisfy (2.1). By Farwig-Sohr[7], we can construct $P_\lambda f \in W^{2,p}(\mathbb{R}_+^n)$ and $\pi_\lambda f \in \widehat{W}^{1,p}(\mathbb{R}_+^n)$ by using partial Fourier transform, which satisfies the estimate:

$$\|P_\lambda f\|_{W^{2,p}(\mathbb{R}_+^n)} + \|\nabla \pi_\lambda f\|_{L^p(\mathbb{R}_+^n)} \leq C_{\varepsilon,\lambda} \|f\|_{L^p(\mathbb{R}_+^n)},$$

provided that $\lambda \in \Sigma_\varepsilon$ and $|\lambda| \geq \lambda_0$. We shall investigate the property of $P_\lambda f$ and $\pi_\lambda f$ near $\lambda = 0$ when f has compact support.

In the course of our argument below, we shall use the following proposition which is proved by using the residue theorem.

Proposition 2.1 *For nonnegative integer k , the following equalities are valid:*

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\xi_n} \xi_n^k}{\lambda + |\xi|^2} d\xi_n &= \begin{cases} \frac{i^k (\lambda + |\xi'|^2)^{\frac{k-1}{2}}}{2} e^{-\sqrt{\lambda+|\xi'|^2}|z|} & z > 0, \\ \frac{(-i)^k (\lambda + |\xi'|^2)^{\frac{k-1}{2}}}{2} e^{-\sqrt{\lambda+|\xi'|^2}|z|} & z < 0, \end{cases} \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\xi_n} \xi_n^k}{|\xi|^2 (\lambda + |\xi|^2)} d\xi_n &= \begin{cases} \frac{i^k}{2\lambda} \left[|\xi'|^{k-1} e^{-|\xi'| |z|} - (\lambda + |\xi'|^2)^{\frac{k-1}{2}} e^{-\sqrt{\lambda+|\xi'|^2}|z|} \right] & z > 0, \\ \frac{(-i)^k}{2\lambda} \left[|\xi'|^{k-1} e^{-|\xi'| |z|} - (\lambda + |\xi'|^2)^{\frac{k-1}{2}} e^{-\sqrt{\lambda+|\xi'|^2}|z|} \right] & z < 0. \end{cases} \end{aligned}$$

In order to obtain representation formula of solution $(P_\lambda f, \pi_\lambda f)$, first we extend a given external force $f = (f_1, \dots, f_n)$ to $F = (f_1^e, \dots, f_{n-1}^e, f_n^o)$, where $f^e(x)$ and $f^o(x)$ denote the even extension and the odd extension respectively defined by

$$f^e(x) = \begin{cases} f(x', x_n) & x_n > 0, \\ f(x', -x_n) & x_n < 0, \end{cases} \quad f^o(x) = \begin{cases} f(x', x_n) & x_n > 0, \\ -f(x', -x_n) & x_n < 0. \end{cases}$$

Let (U, Ψ) be the solution in a whole space corresponding to this F , namely (U, Ψ) is the solution of the equation:

$$(\lambda - \Delta)U + \nabla \Psi = F, \quad \nabla \cdot U = 0 \quad \text{in } \mathbb{R}^n. \quad (2.2)$$

By the Fourier transform, (2.2) is reduced to the following equations:

$$(\lambda + |\xi|^2)\widehat{U}(\xi) + i\xi\widehat{\Psi}(\xi) = \widehat{F}(\xi) \quad \text{for } \mathbb{R}^n. \quad (2.3)$$

Since $\Delta \Psi = \nabla \cdot F$ follows from (2.2), by (2.3) we have

$$\Psi(x) = \mathcal{F}_\xi^{-1} \left[-\frac{i\xi \cdot \widehat{F}(\xi)}{|\xi|^2} \right] (x), \quad (2.4)$$

$$U(x) = \mathcal{F}_\xi^{-1} \left[\frac{1}{\lambda + |\xi|^2} \left(\widehat{F}(\xi) - \frac{\xi\xi \cdot \widehat{F}(\xi)}{|\xi|^2} \right) \right] (x), \quad (2.5)$$

where $\widehat{f}(\xi)$ and $\mathcal{F}_\xi^{-1}[f(\xi)](x)$ denote the Fourier transform of $f(x)$ and the Fourier inverse transform of $f(\xi)$ respectively defined by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}_\xi^{-1}[f(\xi)](x) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

For the argument below, we calculate $\widetilde{U}_j(\xi', 0)$, where $\widetilde{f}(\xi', x_n)$ denotes the partial Fourier transform of $f(x)$ with respect to $x' = (x_1, \dots, x_{n-1})$ defined by

$$\widetilde{f}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix' \cdot \xi'} f(x', x_n) dx'.$$

Noticing the formula:

$$\widetilde{U}_j(\xi', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix_n \xi_n} \widehat{U}_j(\xi) d\xi_n \Big|_{x_n=0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{U}_j(\xi', \xi_n) d\xi_n,$$

and applying Proposition 2.1 to (2.5), we obtain

$$\begin{aligned} \widetilde{U}_j(\xi', 0) &= \int_0^{+\infty} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda+|\xi'|^2}} \widetilde{f}_j(\xi', x_n) + \sum_{k=1}^{n-1} \xi_j \xi_k \widetilde{f}_k(\xi', x_n) \frac{1}{\lambda} \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|x_n}}{|\xi'|} \right) \\ &\quad - i\xi_j \widetilde{f}_n(\xi', x_n) \frac{1}{\lambda} \left(e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-x_n|\xi'|} \right) dx_n \quad \text{for } j = 1, \dots, n-1, \end{aligned} \quad (2.6)$$

$$\widetilde{U}_n(\xi', 0) = 0. \quad (2.7)$$

Secondary, we shall construct a Riemann function (v, θ) . To this end, put $u = U + v$ and $\pi = \Phi + \theta$, then (v, θ) enjoys the equation:

$$\begin{cases} (\lambda - \Delta)v + \nabla \theta = 0, & \nabla \cdot v = 0 & \text{in } \mathbb{R}_+^n, \\ v_j(x', 0) = -U_j(x', 0) & \text{for } j = 1, \dots, n-1, \\ v_n(x', 0) = 0. \end{cases} \quad (2.8)$$

A solution (v, θ) to (2.8) is given by the following formula:

$$\begin{aligned} \tilde{v}_j(\xi', x_n) &= -\tilde{U}_j(\xi', 0)e^{-\sqrt{\lambda+|\xi'|^2}x_n} \\ &\quad - \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \frac{\xi_j}{|\xi'|} \xi' \cdot \tilde{U}'(\xi', 0), \end{aligned} \quad (2.9)$$

$$\tilde{v}_n(\xi', x_n) = -\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} i\xi' \cdot \tilde{U}'(\xi', 0), \quad (2.10)$$

$$\tilde{\theta}(\xi', x_n) = \frac{\sqrt{\lambda+|\xi'|^2} + |\xi'|}{|\xi'|} e^{-|\xi'|x_n} i\xi' \cdot \tilde{U}'(\xi', 0), \quad (2.11)$$

where $\xi' \cdot \tilde{U}'(\xi', 0) = \sum_{j=1}^{n-1} \xi_j \tilde{U}_j'(\xi', 0)$. In fact, since $\Delta\theta = 0$ by (2.8), applying the Laplacian reduces (2.8) to the following equation:

$$\begin{cases} (\lambda - \Delta)v_n = 0, \\ v_n(x', 0) = 0, \\ \partial_n v_n(x', 0) = \nabla' \cdot U'(x', 0), \end{cases} \quad (2.12)$$

where $U'(x) = (U_1, \dots, U_{n-1})(x)$ and $\nabla' = (\partial_1, \dots, \partial_{n-1})$. Application of the partial Fourier transform converts (2.12) into

$$\begin{cases} (\partial_n^2 - \lambda - |\xi'|^2)(\partial_n^2 - |\xi'|^2)\tilde{v}_n(\xi', x_n) = 0, \\ \tilde{v}_n(\xi', 0) = 0, \\ \partial_n \tilde{v}_n(\xi', 0) = i\xi' \cdot \tilde{U}'(\xi', 0). \end{cases} \quad (2.13)$$

Also application of the partial Fourier transform converts $(\lambda - \Delta)v_n + \partial_n \theta = 0$ in (2.8) and $\Delta\theta = 0$ into

$$\begin{cases} (|\xi'|^2 - \partial_n^2)\tilde{\theta}(\xi', x_n) = 0, \\ \partial_n \tilde{\theta}(\xi', 0) = -(\lambda + |\xi'|^2 - \partial_n^2)\tilde{v}_n(\xi', x_n)|_{x_n=0}. \end{cases} \quad (2.14)$$

Solving (2.13)-(2.14), we obtain (2.10)-(2.11).

Next we shall show (2.9). Application of (2.8) implies that $\tilde{v}_j(\xi', x_n)$ satisfies the equation:

$$\begin{cases} -(\partial_n^2 - \lambda - |\xi'|^2)\tilde{v}_j(\xi', x_n) = i\xi_j \tilde{\theta}(\xi', x_n), \\ \tilde{v}_j(\xi', 0) = -\tilde{U}_j(\xi', 0), \end{cases} \quad (2.15)$$

for $j = 1, 2, \dots, n-1$. Solving (2.15), we obtain (2.9). Summing up, we have obtained the

solution formulas of (u, p) :

$$\begin{aligned} \tilde{u}_j(\xi', x_n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix_n \xi_n} \left\{ \frac{1}{\lambda + |\xi|^2} \left[\widehat{f}_j^e(\xi) - \sum_{k=1}^{n-1} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k^e(\xi) - \frac{\xi_j \xi_n}{|\xi|^2} \widehat{f}_n^o(\xi) \right] \right\} d\xi_n \\ &\quad - e^{-\sqrt{\lambda + |\xi'|^2} x_n} \widetilde{U}_j(\xi', 0) - \sum_{k=1}^{n-1} \frac{e^{-\sqrt{\lambda + |\xi'|^2} x_n} - e^{-|\xi'| x_n}}{\sqrt{\lambda + |\xi'|^2} - |\xi'|} \frac{\xi_j \xi_k}{|\xi'|} \widetilde{U}_k(\xi', 0), \\ &\quad \text{for } j = 1, 2, \dots, n-1 \end{aligned} \quad (2.16)$$

$$\begin{aligned} \widetilde{u}_n(\xi', x_n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix_n \xi_n} \left\{ \frac{1}{\lambda + |\xi|^2} \left[\widehat{f}_n^o(\xi) - \sum_{k=1}^{n-1} \frac{\xi_n \xi_k}{|\xi|^2} \widehat{f}_k^e(\xi) - \frac{\xi_n^2}{|\xi|^2} \widehat{f}_n^o(\xi) \right] \right\} d\xi_n \\ &\quad - i \sum_{k=1}^{n-1} \frac{e^{-\sqrt{\lambda + |\xi'|^2} x_n} - e^{-|\xi'| x_n}}{\sqrt{\lambda + |\xi'|^2} - |\xi'|} \xi_k \widetilde{U}_k(\xi', 0), \end{aligned} \quad (2.17)$$

$$\begin{aligned} \tilde{p}(\xi', x_n) &= \frac{-i}{2\pi} \int_{-\infty}^{\infty} e^{ix_n \xi_n} \left\{ \sum_{k=1}^{n-1} \frac{\xi_k}{|\xi|^2} \widehat{f}_k^e(\xi) + \frac{\xi_n}{|\xi|^2} \widehat{f}_n^o(\xi) \right\} d\xi_n \\ &\quad + i \frac{\sqrt{\lambda + |\xi'|^2} + |\xi'|}{|\xi'|} e^{-|\xi'| x_n} \sum_{k=1}^{n-1} \xi_k \widetilde{U}_k(\xi', 0). \end{aligned} \quad (2.18)$$

When we consider the expansion formula of (u, p) when $|\lambda| \leq \frac{1}{2}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, we devide the solution formula of (2.16)-(2.18) into the parts where $|\xi'| \leq 2$ and where $|\xi'| \geq 1$ by using the cut-off function. And to analyse the part where $|\xi'| \leq 2$, we use the following more detailed formulas obatined by applying Proposition 2.1 to (2.16) - (2.18);

$$\begin{aligned} \tilde{u}_j(\xi', x_n) &= \frac{1}{2} \int_0^{\infty} \left(\frac{e^{-\sqrt{\lambda + |\xi'|^2} |x_n - y_n|}}{\sqrt{\lambda + |\xi'|^2}} - \frac{e^{-\sqrt{\lambda + |\xi'|^2} (x_n + y_n)}}{\sqrt{\lambda + |\xi'|^2}} \right) \tilde{f}_j(\xi', y_n) dy_n \\ &\quad + \sum_{k=1}^{n-1} \frac{\xi_j \xi_k}{2\lambda} \left\{ \int_0^{\infty} \left(\frac{e^{-\sqrt{\lambda + |\xi'|^2} |x_n - y_n|}}{\sqrt{\lambda + |\xi'|^2}} - \frac{e^{-|\xi'| |x_n - y_n|}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \right. \\ &\quad \left. + \int_0^{\infty} \left(\frac{e^{-\sqrt{\lambda + |\xi'|^2} (x_n + y_n)}}{\sqrt{\lambda + |\xi'|^2}} - \frac{e^{-|\xi'| (x_n + y_n)}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \right. \\ &\quad \left. - 2e^{-\sqrt{\lambda + |\xi'|^2} x_n} \int_0^{\infty} \left(\frac{e^{-\sqrt{\lambda + |\xi'|^2} y_n}}{\sqrt{\lambda + |\xi'|^2}} - \frac{e^{-|\xi'| y_n}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \right\} \\ &\quad + \frac{i\xi_j}{2\lambda} \left\{ \int_0^{x_n} \left(e^{-\sqrt{\lambda + |\xi'|^2} (x_n - y_n)} - e^{-|\xi'| (x_n - y_n)} \right) \widetilde{f}_n(\xi', y_n) dy_n \right. \\ &\quad \left. - \int_{x_n}^{\infty} \left(e^{-\sqrt{\lambda + |\xi'|^2} (y_n - x_n)} - e^{-|\xi'| (y_n - x_n)} \right) \widetilde{f}_n(\xi', y_n) dy_n \right. \\ &\quad \left. - \int_0^{\infty} \left(e^{-\sqrt{\lambda + |\xi'|^2} (x_n + y_n)} - e^{-|\xi'| (x_n + y_n)} \right) \widetilde{f}_n(\xi', y_n) dy_n \right. \\ &\quad \left. - 2e^{-\sqrt{\lambda + |\xi'|^2} x_n} \int_0^{\infty} \left(e^{-\sqrt{\lambda + |\xi'|^2} y_n} - e^{-|\xi'| y_n} \right) \widetilde{f}_n(\xi', y_n) dy_n \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^{n-1} \frac{\xi_j \xi_k}{|\xi'|} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \tilde{f}_k(\xi', y_n) dy_n \\
& - \sum_{k=1}^{n-1} \frac{\xi_j \xi_k |\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \\
& + \frac{i \xi_j |\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty (e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}) \tilde{f}_n(\xi', y_n) dy_n, \quad (2.19)
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_n(\xi', x_n) &= \frac{1}{2} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)}}{\sqrt{\lambda+|\xi'|^2}} \right) \tilde{f}_n(\xi', y_n) dy_n \\
& + \sum_{k=1}^{n-1} \frac{i \xi_k}{2\lambda} \left\{ \int_0^{x_n} (e^{-\sqrt{\lambda+|\xi'|^2}(x_n-y_n)} - e^{-|\xi'|(x_n-y_n)}) \tilde{f}_k(\xi', y_n) dy_n \right. \\
& \quad - \int_{x_n}^\infty (e^{-\sqrt{\lambda+|\xi'|^2}(y_n-x_n)} - e^{-|\xi'|(y_n-x_n)}) \tilde{f}_k(\xi', y_n) dy_n \\
& \quad \left. + \int_0^\infty (e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)}) \tilde{f}_k(\xi', y_n) dy_n \right\} \\
& - \frac{1}{2\lambda} \left\{ \int_0^\infty (\sqrt{\lambda+|\xi'|^2} e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|} - |\xi'| e^{-|\xi'|(x_n-y_n)}) \tilde{f}_n(\xi', y_n) dy_n \right. \\
& \quad \left. - \int_0^\infty (\sqrt{\lambda+|\xi'|^2} e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)} - |\xi'| e^{-|\xi'|(x_n+y_n)}) \tilde{f}_n(\xi', y_n) dy_n \right\} \\
& - \sum_{k=1}^{n-1} i \xi_k \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \tilde{f}_k(\xi', y_n) dy_n \\
& - \sum_{k=1}^{n-1} \frac{i |\xi'|^2 \xi_k}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \\
& - \frac{|\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty (e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}) \tilde{f}_n(\xi', y_n) dy_n, \quad (2.20)
\end{aligned}$$

$$\begin{aligned}
\tilde{p}(\xi', x_n) &= -i \sum_{k=1}^{n-1} \frac{\xi_k}{2|\xi'|} \left\{ \int_0^\infty (e^{-|\xi'|(x_n-y_n)} + e^{-|\xi'|(x_n+y_n)}) \tilde{f}_k(\xi', y_n) dy_n \right\} \\
& - \frac{i}{2} \left\{ \int_0^{x_n} e^{-|\xi'|(x_n-y_n)} \tilde{f}_n(\xi', y_n) dy_n - \int_{x_n}^\infty e^{-|\xi'|(y_n-x_n)} \tilde{f}_n(\xi', y_n) dy_n \right. \\
& \quad \left. - \int_0^\infty e^{-|\xi'|(x_n+y_n)} \tilde{f}_n(\xi', y_n) dy_n \right\} \\
& + \sum_{k=1}^{n-1} \frac{i \xi_k (\sqrt{\lambda+|\xi'|^2} + |\xi'|)}{|\xi'|} e^{-|\xi'|x_n} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \tilde{f}_k(\xi', y_n) dy_n \\
& + \sum_{k=1}^{n-1} \frac{i \xi_k |\xi'| (\sqrt{\lambda+|\xi'|^2} + |\xi'|)}{\lambda} e^{-|\xi'|x_n} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n \\
& + \frac{(\sqrt{\lambda+|\xi'|^2} + |\xi'|) |\xi'|}{\lambda} e^{-|\xi'|x_n} \int_0^\infty (e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}) \tilde{f}_n(\xi', y_n) dy_n. \quad (2.21)
\end{aligned}$$

3 An expansion of Solution Operator of (2.1)

The aim of this section is to get an expansion of solution operator of (2.1). To this end, we choose $\varphi_0(r) \in C^\infty(\mathbb{R}^{n-1})$ so that

$$\varphi_0(r) = 1 \quad \text{for } r \leq 1 \quad \text{and} \quad \varphi_0(r) = 0 \quad \text{for } r \geq 2. \quad (3.1)$$

Put $\varphi_\infty(r) = 1 - \varphi_0(r)$ and set

$$R_j^\infty(\lambda)f = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(|\xi'|)\tilde{u}_j(\xi', x_n)](x'), \quad j = 1, 2, \dots, n \quad (3.2)$$

$$\Pi^\infty(\lambda)f = \mathcal{F}_{\xi'}^{-1}[\varphi_\infty(|\xi'|)\tilde{p}(\xi', x_n)](x'),$$

$$R_j^0(\lambda)f = \mathcal{F}_{\xi'}^{-1}[\varphi_0(|\xi'|)\tilde{u}_j(\xi', x_n)](x'), \quad j = 1, 2, \dots, n \quad (3.3)$$

$$\Pi^0(\lambda)f = \mathcal{F}_{\xi'}^{-1}[\varphi_0(|\xi'|)\tilde{p}(\xi', x_n)](x'),$$

where $\mathcal{F}_{\xi'}^{-1}[f(\xi', x_n)](x)$ denotes the partial Fourier inverse transform with respect to ξ' defined by

$$\mathcal{F}_{\xi'}^{-1}[f(\xi', x_n)](x) = \left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} f(\xi', x_n) d\xi'.$$

In particular, setting

$$R(\lambda)f = {}^t(R_1^\infty(\lambda)f, \dots, R_n^\infty(\lambda)f) + {}^t(R_1^0(\lambda)f, \dots, R_n^0(\lambda)f) \quad (3.4)$$

$$\Pi(\lambda)f = \Pi^\infty(\lambda)f + \Pi^0(\lambda)f, \quad (3.5)$$

$(u, p) = (R(\lambda)f, \Pi(\lambda)f)$ is the solution operator which gives us the solution of (2.1). We get the following theorem for resolvent expansion around the origin.

Theorem 3.1 *Let $n \geq 2$ and $\Omega = \mathbb{R}_+^n$. We put $U_{1/2} = \{\lambda \in \mathbb{C}; |\lambda| < 1/2\}$. Then $(R(\lambda), \Pi(\lambda))$ has the following expansion of with respect $\lambda \in U_{1/2} \setminus (-\infty, 0]$:*

$$(R(\lambda), \Pi(\lambda)) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}} \log \lambda + G_3(\lambda) & n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}} \log \lambda + G_3(\lambda) & n \text{ is odd,} \end{cases} \quad (3.6)$$

where $G_1(\lambda), G_2(\lambda)$ and $G_3(\lambda)$ are $B(L_R^p(\mathbb{R}_+^n), W^{2,p}(B_R^+) \cap W^{1,p}(B_R^+))$ -valued holomorphic function in $U_{1/2}$.

In order to prove Theorem 3.1, we introduce the following proposition.

Proposition 3.2 *Let $\varphi_0(r)$ be the same as (3.1). Then, the following assertion are valid. For any non-negative integers a and b , we have the formulas:*

$$\int_0^\infty \varphi_0(r) r^a \left(\sqrt{\lambda + r^2}\right)^{b-1} dr = H_{a,b}(\lambda) + B_{a,b}^1 \lambda^{\frac{a+b}{2}} + B_{a,b}^2 \lambda^{\frac{a+b}{2}} \log \lambda, \quad (3.7)$$

$$\int_0^\infty \varphi_0(r) \frac{r^a (\sqrt{\lambda + r^2})^{b-1}}{\sqrt{\lambda + r^2} + r} dr = \widetilde{H}_{a,b}(\lambda) + \widetilde{B}_{a,b}^1 \lambda^{\frac{a+b}{2}} + \widetilde{B}_{a,b}^2 \lambda^{\frac{a+b}{2}} \log \lambda, \quad (3.8)$$

where $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{1}{2}$. Here $H_{a,b}(\lambda)$ and $\widetilde{H}_{a,b}(\lambda)$ are the holomorphic functions in $\lambda \in U_{1/2} = \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{1}{2}\}$; $B_{a,b}^1, B_{a,b}^2, \widetilde{B}_{a,b}^1$ and $\widetilde{B}_{a,b}^2$ are the real numbers. Moreover, they have the following properties:

$$\begin{aligned} B_{a,b}^2 &= 0 & \text{when } a+b \text{ is an odd number;} \\ \widetilde{B}_{a,b}^2 &= 0 & \text{when } a+b \text{ is an even number;} \end{aligned}$$

$$|B_{a,b}^1| \leq 1, \quad |\widetilde{B_{a,b}^1}| \leq 1, \quad |B_{a,b}^2| \leq \frac{1}{2}, \quad |\widetilde{B_{a,b}^2}| \leq \frac{1}{2},$$

and there exist constants C and L independence of a and b such that

$$|H_{a,b}(\lambda)| \leq CL^{a+b}, \quad |\widetilde{H_{a,b}}(\lambda)| \leq CL^{a+b} \quad \lambda \in U_{\frac{1}{2}}.$$

[Proof of Proposition 3.2] By substituting $t = r + \sqrt{\lambda + r^2}$, we see easily the formula (3.7) and (3.8). In fact,

$$\begin{aligned} H_{a,b}(\lambda) &= \int_1^\infty \varphi_0(r) r^a (\sqrt{\lambda + r^2})^{b-1} dr \\ &+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m \neq \frac{a+b}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{\lambda^{\ell+m} (1 + \sqrt{1 + \lambda})^{a+b-2(\ell+m)}}{a+b-2(\ell+m)} \\ &+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m = \frac{a+b}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \lambda^{\frac{a+b}{2}} \log(1 + \sqrt{1 + \lambda}), \\ B_{a,b}^1 &= - \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m \neq \frac{a+b}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{1}{a+b-2(\ell+m)}, \\ B_{a,b}^2 &= - \left(\frac{1}{2}\right)^{a+b+1} \sum_{\ell=0, \ell+m = \frac{a+b}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m}, \\ \widetilde{H_{a,b}}(\lambda) &= \int_1^\infty \varphi_0(r) \frac{r^a (\sqrt{\lambda + r^2})^{b-1}}{\sqrt{\lambda + r^2} + r} dr \\ &+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m \neq \frac{a+b-1}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{\lambda^{\ell+m} (1 + \sqrt{1 + \lambda})^{a+b-1-2(\ell+m)}}{a+b-1-2(\ell+m)} \\ &+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m = \frac{a+b-1}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \lambda^{\frac{a+b-1}{2}} \log(1 + \sqrt{1 + \lambda}), \\ \widetilde{B_{a,b}^1} &= - \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0, \ell+m \neq \frac{a+b-1}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{1}{a+b-1-2(\ell+m)}, \\ \widetilde{B_{a,b}^2} &= - \left(\frac{1}{2}\right)^{a+b+1} \sum_{\ell=0, \ell+m = \frac{a+b-1}{2}}^a \sum_{m=0}^b (-1)^\ell \binom{a}{\ell} \binom{b}{m}. \end{aligned}$$

[Proof of Theorem 3.1] The assertion of Theorem 3.1 for $|\xi'| \geq 1$ follows from Fourier multiplier theorem, but to get the expansion for $|\xi'| \leq 1$, we need the different method from the $|\xi'| \geq 1$ case. We explain how to expand the $v_n(x)$ only because we can expand all the other term by the same method. To develop the expansion of $v_n(x)$, we separate $v_n(x)$

into three parts :

$$\begin{aligned}
v_n(x) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \varphi_0(|\xi'|) e^{ix' \cdot \xi'} \\
&\quad \left\{ -\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \sum_{j=1}^{n-1} \frac{i\xi_j}{\sqrt{\lambda+|\xi'|^2}} \int_0^\infty e^{-\sqrt{\lambda+|\xi'|^2}y_n} \tilde{f}_j(\xi', y_n) dy_n \right. \\
&\quad + \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \sum_{j=1}^{n-1} \frac{|\xi'|^2}{\lambda} i\xi_j \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \tilde{f}_j(\xi', y_n) dy_n \\
&\quad \left. - \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \frac{|\xi'|^2}{\lambda} \int_0^\infty (e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}) \tilde{f}_n(\xi', y_n) dy_n \right\} d\xi' \\
&= : I + II + III
\end{aligned}$$

where $\varphi_0(r)$ is the same as in (3.1). At first, we consider the first term I :

$$I = -\left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} \varphi_0(|\xi'|) e^{ix' \cdot \xi'} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \sum_{j=1}^{n-1} \frac{i\xi_j e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \int_0^\infty \tilde{f}_j(\xi', y_n) dy_n.$$

By using Maclaurin expansion of e^x , we can expand the first factor and the second factor as follows:

$$\begin{aligned}
\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} &= -x_n \int_0^1 e^{-|\xi'|(1-\theta)x_n} e^{-\theta\sqrt{\lambda+|\xi'|^2}x_n} d\theta \\
&= \sum_{p,t=0}^\infty \frac{(-x_n)^{p+t+1}}{(p+t+1)!} |\xi'|^t (\sqrt{\lambda+|\xi'|^2})^p. \quad (3.9)
\end{aligned}$$

$$\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} = \sum_{q=0}^\infty \frac{(-y_n)^q}{q!} (\sqrt{\lambda+|\xi'|^2})^{q-1}. \quad (3.10)$$

Moreover, we rewrite $e^{ix' \cdot \xi'} \tilde{f}_j(\xi', x_n)$ as follows:

$$e^{ix' \cdot \xi'} \tilde{f}_j(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{i(x'-y') \cdot \xi'} f_j(y', y_n) dy' = \sum_{s=0}^\infty \int_{\mathbb{R}^{n-1}} \frac{(i(x'-y') \cdot \xi')^s}{s!} f_j(y) dy'. \quad (3.11)$$

By substituting (3.9)-(3.11) into I , we can rewrite I as follows:

$$\begin{aligned}
I &= -\left(\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} f_j(y) dy \\
&\quad \times \sum_{p,q,s,t} \frac{(-x_n)^{p+t+1} (-y_n)^q}{(p+t+1)! q!} \int_{\mathbb{R}^{n-1}} \varphi_0(|\xi'|) i\xi_j |\xi'|^t (\sqrt{\lambda+|\xi'|^2})^{p+q-1} (i(x'-y') \cdot \xi')^s d\xi' \\
&= -\left(\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} f_j(y) dy \\
&\quad \times \sum_{p,q,s,t} \frac{(-x_n)^{p+t+1} (-y_n)^q}{(p+t+1)! q!} \int_{S^{n-1}} i\omega_j \frac{(i(x'-y') \cdot \omega')^s}{s!} d\sigma \int_0^\infty \varphi_0(r) r^{s+t+n-1} (\sqrt{\lambda+r^2})^{p+q-1} dr
\end{aligned}$$

Applying Proposition 3.2, when n is odd we have

$$I = G^I(\lambda)f + \lambda^{\frac{n-1}{2}} G^{II}(\lambda)f + \lambda^{\frac{n}{2}} G^{III}(\lambda)f + \lambda^{\frac{n-1}{2}} \log \lambda G^{IV}(\lambda)f,$$

where we have put

$$\begin{aligned} G_{p,q,s,t}(\lambda)f &= -\left(\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{n-1} \int_{\mathbb{R}_+^n} f_j(y) dy \int_{S^{n-1}} i\omega_j (-i(x' - y') \cdot \omega')^s d\sigma \frac{(-x_n)^{p+t+1} (-y_n)^q}{s!(p+t+1)!q!}, \\ G^I(\lambda)f &= \sum_{p,q,s,t} (G_{p,q,s,t}(\lambda)f) h_{s+t+n-1,p+q}(\lambda), \\ G^{II}(\lambda)f &= \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m} (G_{p,q,s,t}(\lambda)f) B_{s+t+n-1,p+q}^1 \lambda^m, \\ G^{III}(\lambda)f &= \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m+1} (G_{p,q,s,t}(\lambda)f) B_{s+t+n-1,p+q}^1 \lambda^m, \\ G^{IV}(\lambda)f &= \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m} (G_{p,q,s,t}(\lambda)f) B_{s+t+n-1,p+q}^2 \lambda^m, \end{aligned}$$

and when n is even we have

$$I = G^I(\lambda)f + \lambda^{\frac{n-1}{2}} G^{II}(\lambda)f + \lambda^{\frac{n}{2}} G^{III}(\lambda)f + \lambda^{\frac{n}{2}} \log \lambda G^{IV}(\lambda)f,$$

where $G_{p,q,s,t}(\lambda)f, G^I(\lambda)f, G^{II}(\lambda)f$ and $G^{III}(\lambda)f$ are the same as in the odd dimension case, but

$$G^{IV}(\lambda)f = \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m+1} (G_{p,q,s,t}(\lambda)f) B_{s+t+n-1,p+q}^2 \lambda^m.$$

When $|x| < R$ and $\text{supp } f \subset \{y \in \mathbb{R}_+^n; |y| < R\}$, we can show that the right hand sides of $G^i(\lambda)f$ ($i = I, II, III, IV$) are absolutely convergent when $\lambda \in U_{\frac{1}{2}}$. In fact, since

$$\sup_{|x| \leq R} |G_{p,q,s,t}(\lambda)f| \leq C_R \|f\|_{L^p(\mathbb{R}_+^n)} \frac{R^{p+q+t+1} (2R)^s}{s!(p+t+1)!q!},$$

by Proposition 3.2, we have

$$\begin{aligned} \sup_{|x| \leq R} |G^I(\lambda)f| &\leq \sum_{p,q,s,t} \frac{R^{p+q+t+1} (2R)^s L^{s+t+n-1+p+q}}{s!(p+t+1)!q!} \times CC_R \|f\|_{L^p(\mathbb{R}_+^n)} \\ &\leq CC_R \|f\|_{L^p(\mathbb{R}_+^n)} L^{n-1} e^{2LR} e^{LR} \sum_{p,t} \frac{(LR)^{p+t}}{(p+t+1)!} \\ &\leq CC_R \|f\|_{L^p(\mathbb{R}_+^n)} L^{n-1} e^{2LR} e^{LR} \sum_{p,t} \frac{(LR)^{p+t}}{p!t!} \\ &\leq CC_R \|f\|_{L^p(\mathbb{R}_+^n)} L^{n-1} e^{5LR} e^{LR}, \end{aligned}$$

where we have used the fact that $(p+t+1)! \geq p!t!$. In the same manner, by Proposition 3.2 we have

$$\begin{aligned}
|G^{II}(\lambda)f| &\leq C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m} \frac{1}{s!(p+t+1)!q!} \\
&= C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \sum_{p+t \leq 2m} \frac{1}{(p+t+1)!} \sum_{s=0}^{2m-p-t} \frac{1}{s!(2m-p-t-s)!} \\
&= C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \sum_{p+t \leq 2m} \frac{2^{2m-p-t}}{(p+t+1)!(2m-p-t)!} \\
&\leq C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \sum_{\ell=0}^{2m} \frac{2^{2m}}{\ell!(2m-\ell)!} \leq C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \frac{2^{2m}}{(2m)!} = C_R e^4 \|f\|_{L^p(\mathbb{R}_+^n)},
\end{aligned}$$

and also we have

$$\begin{aligned}
|G^{III}(\lambda)f| &\leq C_R \|f\|_{L^p(\mathbb{R}_+^n)} \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m+1} \frac{1}{s!(p+t+1)!q!} \leq C_R e^4 \|f\|_{L^p(\mathbb{R}_+^n)}, \\
|G^{IV}(\lambda)f| &\leq C_R \frac{e^4}{2} \|f\|_{L^p(\mathbb{R}_+^n)}.
\end{aligned}$$

Therefore, $G^i(\lambda)f$ ($i = I, II, III, IV$) are the holomorphic with respect to $\lambda \in U_{\frac{1}{2}}$. In the same way, we obtain the similar expansion formula of II and III . This completes the proof of the proposition. $\blacksquare$

4 Continuity property of $(R(\lambda), \Pi(\lambda))$ near $\lambda = 0$

In this section we shall prove the following theorem:

Theorem 4.1 *Let $1 < p < \infty$ and $f = (f_1, \dots, f_n) \in L^p(\mathbb{R}_+^n)$ with $\text{supp } f \subset B_R$. Let P_λ and π_λ be the operators defined in section 2. If we put $u = P_0 f$ and $\pi = \pi_0 f$, then $(u, \pi) \in W_{loc}^{2,p}(\mathbb{R}_+^n)^n \times W_{loc}^{1,p}(\mathbb{R}_+^n)$, and (u, π) satisfies the equation:*

$$-\Delta u + \nabla \pi = f, \quad \nabla \cdot u = 0, \quad \text{in } \mathbb{R}_+^n, \quad u(x', 0) = 0,$$

Moreover (u, π) satisfies the estimates:

$$\begin{aligned}
&\|u\|_{W^{2,p}(B_L^+)} + \|\pi\|_{W^{1,p}(B_L^+)} \leq C_{R,L} \|f\|_{L^p(\mathbb{R}_+^n)}, \quad \text{for } L > 0 \\
&\sup_{|x| \geq 1, x \in \mathbb{R}_+^n} [|x|^{n-2}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|] \leq C_R \|f\|_{L^p(\mathbb{R}_+^n)}.
\end{aligned}$$

and the formula:

$$\lim_{\lambda \rightarrow 0} \|P_\lambda f - P_0 f\|_{W^{2,p}(B_L^+)} + \|\pi_\lambda f - \pi_0 f\|_{W^{1,p}(B_L^+)} = 0 \quad \text{for any } L > 0,$$

when $n \geq 3$.

In order to show Theorem 4.1, we consider the limit of $P_\lambda f$ and $\pi_\lambda f$ as $\lambda \rightarrow 0$. To this end, we only consider $\tilde{v}_j(\xi', x_n)$ ($j = 1, \dots, n$) defined in (2.19) and (2.20). The

estimates of $U(x)$ follow from the Fourier multiplier theorem, so it suffices to investigate the Riemann function $v_j(x)$ ($j = 1, \dots, n$). Let $v_j^0(x)$, $\theta^0(x)$ ($j = 1, \dots, n$) be limits to which $v_j(x)$, $\theta(x)$ ($j = 1, \dots, n$) converge formally as $\lambda \rightarrow 0$. Below, we consider only the term $I_\lambda(x)$ defined by

$$\tilde{I}_\lambda(\xi', x_n) = \sum_{k=1}^{n-1} \frac{i|\xi'|^2 \xi_k}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2} x_n} - e^{-|\xi'| x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2} y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'| y_n}}{|\xi'|} \right) \tilde{f}_k(\xi', y_n) dy_n,$$

which appears as the 7th. term of $\tilde{u}_n(\xi', x_n)$ (cf. (2.20)). Put $I^0(x) = \lim_{\lambda \rightarrow 0} I_\lambda(x)$. Noticing that as $\lambda \rightarrow 0$,

$$\begin{aligned} \frac{e^{-\sqrt{\lambda+|\xi'|^2} x_n} - e^{-|\xi'| x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} &= \int_0^1 e^{[|\xi'| + \theta(\sqrt{\lambda+|\xi'|^2} - |\xi'|)] x_n} d\theta x_n \rightarrow x_n e^{-|\xi'| x_n} \\ \frac{1}{\lambda} \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2} y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'| y_n}}{|\xi'|} \right) &= \frac{1}{\lambda} \left\{ \frac{e^{-\sqrt{\lambda+|\xi'|^2} y_n} - e^{-|\xi'| y_n}}{\sqrt{\lambda+|\xi'|^2}} + \left(\frac{1}{\sqrt{\lambda+|\xi'|^2}} - \frac{1}{|\xi'|} \right) e^{-|\xi'| y_n} \right\} \\ &= \frac{\sqrt{\lambda+|\xi'|^2} - |\xi'|}{\lambda \sqrt{\lambda+|\xi'|^2}} \left(y_n \int_0^1 e^{-(\theta \sqrt{\lambda+|\xi'|^2} + (1-\theta)|\xi'|) y_n} d\theta - \frac{e^{-|\xi'| y_n}}{|\xi'|} \right) \\ &\rightarrow \frac{e^{-|\xi'| y_n}}{2|\xi'|^3} (|\xi'| y_n - 1). \end{aligned}$$

$I^0(x)$ has the representation formula:

$$\begin{aligned} I^0(x) &= \frac{1}{2} \left(\frac{1}{2\pi} \right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} x_n e^{-|\xi'| x_n} \\ &\quad \left(\sum_{k=1}^{n-1} \frac{i \xi_k}{|\xi'|} \int_0^\infty -e^{-|\xi'| y_n} \tilde{f}_k(\xi', y_n) dy_n + \sum_{k=1}^{n-1} i \xi_k \int_0^\infty e^{-|\xi'| y_n} y_n \tilde{f}_k(\xi', y_n) dy_n \right) d\xi'. \end{aligned}$$

In order to obtain the decay with respect to $|x|$, we introduce the following lemma.

Lemma 4.2 Let B be a Banach space and $|\cdot|_B$ its corresponding norm. Let α be a number $> -n$ and set $\alpha = N + \sigma - n$ where $N \geq 0$ is an integer and $0 < \sigma \leq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n \setminus \{0\}; B)$ such that

$$\partial_\xi^\gamma f(\xi) \in L^1(\mathbb{R}^n; B), \quad (4.1)$$

$$|\partial_\xi^\gamma f(\xi)|_B \leq C_\gamma |\xi|^{\alpha-|\gamma|} \quad \forall \xi \neq 0 \quad \forall \gamma, \quad (4.2)$$

$$g(x) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(\xi) d\xi.$$

Then, we have

$$|g(x)|_B \leq C_{n,\alpha} \left(\max_{|\gamma| \leq N+2} C_\gamma \right) |x|^{-(n+\alpha)} \quad \forall x \neq 0. \quad (4.3)$$

[proof of Lemma 4.2] See Shibata and Shimizu [13] ■

At first, we consider the following function $g(x)$ corresponding to $v^0(x)$:

$$g(x) = \left(\frac{1}{2\pi} \right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \frac{\xi_j}{|\xi'|} x_n e^{-|\xi'| x_n} \int_0^\infty y_n |\xi'| e^{-|\xi'| y_n} \tilde{f}_j(\xi', y_n) dy_n d\xi'.$$

There holds the estimate:

$$|\partial_{\xi'}^{\alpha'} e^{-|\xi'|x_n}| \leq C_{\alpha'} |\xi'|^{-|\alpha'|} e^{-\frac{1}{2}|\xi'|x_n}.$$

In fact, we have

$$\begin{aligned} |\partial_{\xi'}^{\alpha'} e^{-|\xi'|x_n}| &= \sum_{\sigma=1}^{|\alpha'|} x_n^{\sigma} e^{-|\xi'|x_n} \sum_{|\alpha'_1|+|\alpha'_2|+\dots+|\alpha'_\sigma|=|\alpha'|, |\alpha_j| \geq 1} \left(\partial_{\xi'_1}^{\alpha'_1} |\xi'| \right) \dots \left(\partial_{\xi'_\sigma}^{\alpha'_\sigma} |\xi'| \right) \\ &\leq C \sum_{\sigma=1}^{|\alpha'|} |\xi'|^{\sigma-|\alpha'|} x_n^{\sigma} e^{-|\xi'|x_n} < C |\xi'|^{-|\alpha'|} e^{-\frac{1}{2}|\xi'|x_n}. \end{aligned}$$

Therefore

$$\left| \partial_{\xi'}^{\alpha'} \xi_j x_n y_n e^{-|\xi'|(x_n+y_n)} \right| \leq |x_n y_n| |\xi'|^{1-|\alpha'|} e^{-\frac{1}{2}|\xi'|(x_n+y_n)} \leq |\xi'|^{-1-|\alpha'|} e^{-\frac{1}{4}|\xi'|(x_n+y_n)},$$

And

$$\left| \partial_{\xi'}^{\alpha'} [\xi_j (x_n y_n) e^{-|\xi'|(x_n+y_n)} \tilde{f}_j(\xi', y_n)] \right| \leq C_{\alpha'} |\xi'|^{-1-|\alpha'|} \sum_{|\gamma'| \leq |\alpha'|} |\partial_{\xi'}^{\gamma'} \tilde{f}_j(\xi', y_n)|.$$

Applying Lemma 4.2 with $N = n - 3$, $\sigma = 1$,

$$|g(x)| \leq C |x'|^{-(n-2)} \int_{\mathbb{R}_+^n} \int_0^\infty \sum_{|\gamma'| \leq n-1} |f_j(x', y_n) (x')^{\gamma'}| dx' dy_n \leq C |x'|^{-(n-2)} \int_{\mathbb{R}_+^n} |f(x)| dx.$$

On the other hand, taking $(y_n |\xi'|) e^{-|\xi'|y_n} \leq 1$ into consideration, there holds the estimate:

$$|g(x)| \leq \left(\frac{1}{2\pi} \right)^{n-1} \int_{\mathbb{R}^{n-1}} x_n e^{-|\xi'|x_n} d\xi' \int_{\mathbb{R}_+^n} |f(x)| dx = C_n x_n^{-(n-2)} \int_{\mathbb{R}_+^n} |f(x)| dx.$$

Hence

$$|g(x)| \leq \begin{cases} C_n |x'|^{-(n-2)} \int_{\mathbb{R}_+^n} |f(x)| dx \\ C_n x_n^{-(n-2)} \int_{\mathbb{R}_+^n} |f(x)| dx \end{cases} \leq C_n |x|^{-(n-2)} \int_{\mathbb{R}_+^n} |f(x)| dx.$$

Similarly applying Lemma 4.2 with $N = n - 2$, $\sigma = 1$, we obtain

$$|\nabla g(x)| \leq C_n |x|^{-(n-1)} \int_{\mathbb{R}_+^n} |f(x)| dx.$$

Next we consider

$$g_0(x) = \left(\frac{1}{2\pi} \right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \phi_0(\xi') \frac{\xi_j}{|\xi'|} x_n e^{-|\xi'|x_n} \int_0^\infty y_n |\xi'| e^{-|\xi'|y_n} \widehat{f}_j(\xi', y_n) dy_n d\xi'.$$

then the following estimate holds:

$$|\partial_x^\alpha g_0(x)| \leq C_\alpha \int_{|\xi'| \leq 1} \frac{d\xi'}{|\xi'|} \int_{\mathbb{R}_+^n} |f_j(y)| dy \leq C_\alpha \int_{\mathbb{R}_+^n} |f_j(y)| dy.$$

We choose $\phi_\infty(\xi') = 1 - \phi_0(\xi') \in C^\infty(\mathbb{R}^n)$ so that $\phi_\infty(\xi') = 1$ for $|\xi'| \geq 2$ and $\phi_\infty(\xi') = 0$ for $|\xi'| \leq 1$. Set

$$U_\infty(x) = \mathcal{F}_\xi^{-1} \left[\frac{\phi_\infty(\xi)}{|\xi|^2} \left(\widehat{F}(\xi) - \frac{\xi}{|\xi|^2} (\xi \cdot \widehat{F}(\xi)) \right) \right],$$

And then for $1 < p < \infty$

$$U_\infty(x) \in W^{2,p}(\mathbb{R}^n) \quad ; \|U_\infty\|_{W^{2,p}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}_+^n)}. \quad (4.4)$$

In fact, noticing that

$$\partial_{\xi_n}^{\alpha_n} |\xi|^{-2} = \sum_{\sigma=1}^{\alpha_n} C_\alpha |\xi|^{-2(\sigma+1)} \sum_{a_1+\dots+a_\sigma=\alpha_n} \partial_{\xi_n}^{a_1} \xi_n^2 \dots \partial_{\xi_n}^{a_\sigma} \xi_n^2 = \sum_{\sigma \geq n/2}^{\alpha_n} C_\alpha |\xi|^{-2(\sigma+1)} \sum_{p=2\sigma-\alpha_n} \xi_n^p,$$

and

$$|\partial_{\xi'}^{\alpha'} |\xi|^{-2(\sigma+1)}| \leq \sum_{\ell=1}^{|\alpha'|} |\xi|^{-2(\sigma+\ell+1)} |\xi'|^{2\ell-|\alpha'|} \leq C_\alpha |\xi|^{-2(\sigma+1)} |\xi'|^{-|\alpha'|},$$

we have

$$|\partial_{\xi'}^{\alpha'} \partial_{\xi_n}^{\alpha_n} [\phi_\infty(\xi') |\xi|^{-2}]| \leq \begin{cases} \sum_{\sigma \geq \alpha_n/2}^{\alpha_n} C |\xi|^{-2(\sigma+1)} \xi_n^{2\sigma-\alpha_n} |\xi'|^{-|\alpha'|} & |\xi'| \geq 2 \\ 0 & |\xi'| \leq 1 \end{cases}$$

thus there holds the estimate for any β with $|\beta| \leq 2$ and α ,

$$|\xi^\alpha \partial_{\xi'}^{\alpha'} \partial_{\xi_n}^{\alpha_n} [\phi_\infty(\xi') |\xi|^{-2} \xi^\beta]| \leq C_\alpha.$$

Hence by Fourier multiplier theory, we have

$$\|U_\infty\|_{W^{2,p}(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

Furthermore, we choose $\psi_\infty(\xi') \in C^\infty(\mathbb{R}^n)$ so that $\psi = 1$ for $|\xi'| \geq 1$ and $\psi = 0$ for $|\xi'| \leq \frac{1}{2}$. Setting $\widetilde{v_{n,\infty}}(\xi', x_n) = \psi_\infty(\xi') x_n e^{-|\xi'| x_n} i \xi' \cdot \widetilde{U_\infty}(\xi', 0)$, we consider the following estimates for $|\beta'| \leq 2$:

$$(\xi')^{\beta'} \widetilde{v_{n,\infty}}(\xi', x_n) = x_n |\xi'| e^{-\frac{1}{2} |\xi'| x_n} \frac{\psi_\infty(\xi')}{|\xi'|} (\xi')^\beta i \xi' \cdot \widetilde{U_\infty}(\xi', 0) e^{-\frac{1}{2} |\xi'| x_n}.$$

In the case where $|\beta'| \leq 1$, we set $\mathcal{F}_{\xi'}^{-1} [i \xi' \cdot \widetilde{U_\infty}(\xi', 0) e^{-|\xi'| x_n}] (x') = W_\infty(x)$. We have

$$\left| \partial_{\xi'}^{\alpha'} [x_n |\xi'| e^{-\frac{1}{2} |\xi'| x_n} \frac{\psi_0(\xi')}{|\xi'|} (\xi')^{\beta'}] \right| \leq C_\alpha |\xi'|^{-|\alpha'|},$$

so that by Fourier multiplier theory, we obtain the following estimate:

$$\|\partial_{x'}^{\beta'} v_{n,\infty}\|_{L^p(\mathbb{R}_+^n)} \leq C \|W_\infty\|_{L^p(\mathbb{R}_+^n)} \leq C \|\nabla U_\infty\|_{L^p(\mathbb{R}_+^n)}$$

In the case where $|\beta'| = 2$, we set $\beta' = \beta'_1 + \beta'_2$ with $|\beta'_1| = |\beta'_2| = 1$ then we have

$$\|\partial_{x'}^{\beta'} W_\infty\|_{L^p(\mathbb{R}_+^n)} = \left\| \mathcal{F}_{\xi'}^{-1} [(\xi')^{\beta'_1} i \xi' \cdot \widetilde{U_\infty}(\xi', 0) e^{-|\xi'| x_n}] (x') \right\| \leq C \|\nabla^2 U_\infty\|_{L^p(\mathbb{R}_+^n)}.$$

Thus we obtain the following estimate:

$$\|\partial_x^{\beta_1} v_{n,\infty}\|_{L^p(\mathbb{R}_+^n)} \leq C \|\partial_x^{\beta'} W_\infty\|_{L^p(\mathbb{R}_+^n)} \leq C \|\nabla^2 U_\infty\|_{L^p(\mathbb{R}_+^n)}.$$

Moreover noticing that

$$\partial_n \widetilde{v_{n,\infty}}(\xi', x_n) = (1 - x_n |\xi'|) e^{-\frac{1}{2}|\xi'|x_n} e^{-\frac{1}{2}|\xi'|x_n} i \xi' \cdot \widetilde{U_\infty}(\xi', 0),$$

we have the following estimates :

$$\|\partial_x^{\beta'} \partial_n \widetilde{v_{n,\infty}}\|_{L^p(\mathbb{R}_+^n)} \leq C \|\partial_x^{\beta'} W_\infty\|_{L^p(\mathbb{R}_+^n)} \leq C \|\nabla^2 U_\infty\|_{L^p(\mathbb{R}_+^n)},$$

and

$$\partial_n^2 \widetilde{v_{n,\infty}}(\xi', x_n) = \sum_{j=1}^{n-1} (2 - x_n |\xi'|) \frac{\xi_j}{|\xi'|} e^{-\frac{1}{2}|\xi'|x_n} e^{-\frac{1}{2}|\xi'|x_n} \xi_j i \xi' \cdot \widetilde{U_\infty}(\xi', 0).$$

Therefore we obtain

$$\|\partial_n^2 \widetilde{v_{n,\infty}}(\xi', x_n)\|_{L^p(\mathbb{R}_+^n)} \leq C \|\nabla^2 U_\infty\|_{L^p(\mathbb{R}_+^n)}.$$

We have

$$\|v_{n,\infty}\|_{W^{2,p}(\mathbb{R}_+^n)} \leq C \|U_\infty\|_{W^{2,p}(\mathbb{R}_+^n)} \leq C \|f\|_{L^p(\mathbb{R}_+^n)}.$$

The other term of I_n enjoys the same estimate. As a result we see $I_n \in W_{loc}^{2,p}(\mathbb{R}_+^n)$ and

$$\begin{aligned} \|I_n\|_{W^{2,p}(|x| \leq R)} &\leq C_R \|f\|_{L^p(\mathbb{R}_+^n)}, \\ |I_n(x)| &\leq C |x|^{-(n-2)} \|f\|_{L^p(\mathbb{R}_+^n)}, \quad |\nabla I_n(x)| \leq C |x|^{-(n-1)} \|f\|_{L^p(\mathbb{R}_+^n)}. \end{aligned}$$

In the same way, we can obtain the decay of velocity term and pressure term:

$$\begin{aligned} \|v_n\|_{W^{2,p}(|x| \leq R)} &\leq C_R \|f\|_{L^p(\mathbb{R}_+^n)}, \\ |v_n(x)| &\leq C |x|^{-(n-2)} \|f\|_{L^p(\mathbb{R}_+^n)}, \quad |\nabla v_n(x)| \leq C |x|^{-(n-1)} \|f\|_{L^p(\mathbb{R}_+^n)}. \\ \theta_n &\in W^{1,p}(\mathbb{R}_+^n) \quad \text{and} \quad \|\theta_n\|_{W^{1,p}(|x| < R, x \in \mathbb{R}_+^n)} \leq C_R \|f\|_{L^p(\mathbb{R}_+^n)}. \end{aligned}$$

Lemma 4.3 Let $n \geq 3$ and $1 < p < \infty$. Let $\Omega = \mathbb{R}_+^n$ or a perturbed half-space. Let $u \in W_{loc}^{2,p}(\Omega)$ and $\pi \in W_{loc}^{1,p}(\Omega)$ enjoy

$$-\Delta u + \nabla \pi = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (4.5)$$

Moreover

$$\sup_{x \in B_{R+3}^+} [|x|^{n-2} |u(x)| + |x|^{n-1} |\nabla u(x)| + |x|^{n-1} |\pi(x)|] < \infty.$$

Then $u = 0$ and $\pi = 0$.

[proof of Lemma 4.3] Taking local regularity into account, we may assume that $u \in W_{loc}^{2,q}(\Omega)$, $\pi \in W_{loc}^{1,p}(\Omega)$ for $1 < q < \infty$. In particular we may assume that $u \in W_{loc}^{2,2}(\Omega)$, $\pi \in W_{loc}^{1,2}(\Omega)$. Now we choose $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| < 1$ and $\psi(x) = 0$ for $|x| > 2$. Set $\psi_L(x) = \psi(x/L)$. Since $u(x), \pi(x)$ satisfy (4.5), we have

$$\begin{aligned} 0 &= (-\Delta u + \nabla \pi, \psi_L u) \\ &= (\nabla u, (\nabla \psi_L) u) + (\nabla u, (\psi_L) \nabla u) - (\pi, (\nabla \psi_L) u) - (\pi, \psi_L (\nabla \cdot u)) \\ &= (\nabla u, (\nabla \psi_L) u) + (\nabla u, (\psi_L) \nabla u) - (\pi, (\nabla \psi_L) u). \end{aligned}$$

Then the first and the third terms of right hand side tend to 0 as $L \rightarrow \infty$, and therefore we have

$$0 = \|\nabla u\|_{L^2}^2,$$

which implies that $\nabla u = 0$. Therefore $u = \text{constant}$. Since $u(x)|_{\partial\Omega} = 0$, we obtain $u = 0$. By the equation $\nabla\pi = 0$, which implies $\pi = \text{constant}$. Since $\pi(x) = O(|x|^{-(n-1)})$ as $|x| \rightarrow \infty$, $\pi = 0$. This completes the proof. ■

5 Analysis in Ω by constructing a parametrix

First of all, to construct a parametrix in Ω , we introduce the Bogovskii lemma [3] which plays an important role in this paper. To introduce the Bogovskii lemma for any domain D in $\mathbb{R}^n$, we define the spaces $W_0^{N,p}(D)$ and $\dot{W}^{N,p}(D_R)$ as follows.

$$\begin{aligned} W_0^{N,p}(D) &= \{f \in W^{N,p}(D); \partial_x^\alpha f|_{\partial D} = 0 \text{ for } |\alpha| \leq N-1\} \quad N \geq 1, \\ \dot{W}^{N,p}(D) &= \{f \in W_0^{N,p}(D); \int_{D_R} f dx = 0\} \quad N \geq 1, \\ \dot{W}^{0,p}(D) &= \{f \in L^p(D); \int_{D_R} f dx = 0\}. \end{aligned}$$

Proposition 5.1 (Bogovskii lemma) *Let $1 < p < \infty$. For $D_R = \{x \in \mathbb{R}_+^n; R < |x| < R+1\}$ and any integer $N \geq 0$, there is a linear operator $\mathbb{B}_R$ from $\dot{W}_0^{N,p}(D_R)$ into $W_0^{N+1,p}(D_R)$ such that*

$$\nabla \cdot \mathbb{B}_R f = f, \quad \|\mathbb{B}_R f\|_{W^{N+1,p}(D_R)} \leq C_{N,p,R} \|f\|_{W^{N,p}(D_R)},$$

for any $f \in \dot{W}_0^{N,p}(D_R)$.

Next we shall introduce the notation in order to constructing a parametrix. Fix $R > 0$ such that $B_+^R \cap \Omega = B_+^R = \{x \in \mathbb{R}_+^n; |x| > R\}$. Let E_R be a bounded domain with smooth boundary ∂E_R such that $E_R \cap B_{R+4} = \Omega \cap B_{R+4}$. And let D_R be a bounded domain such that $D_R = \{x \in \mathbb{R}_+^n; R+1 \leq |x| \leq R+2\}$. In particular $D_R \subset \bar{\Omega} \cap B_{R+3} \subset E_R$.

Let $P_\lambda f$ and $\pi_\lambda f$ be operators given in section 2. Recall that w and θ satisfy the equation:

$$(\lambda - \Delta)w + \nabla \theta = f_0, \quad \nabla \cdot w = 0 \quad \text{in } \mathbb{R}_+^n, \quad w|_{x_n=0} = 0,$$

and the estimate:

$$\|P_\lambda f\|_{W^{2,p}(D_R)} + \|\pi f\|_{W^{1,p}(D_R)} \leq C_R \|f\|_{L^p(\Omega)}.$$

Given $f \in \mathbb{L}_{R+3}^p$, we set $Af = w$, $\Phi f = \pi$ where w and π are the solution to the equation;

$$-\Delta w + \nabla \pi = f, \quad \nabla \cdot w = 0 \quad \text{in } E_R, \quad w|_{\partial E_R} = 0. \quad (5.1)$$

We know the unique existence of $Af \in W^{2,p}(E_R)$ and $\Phi f \in W^{1,p}(E_R)$ satisfying the estimate:

$$\|Af\|_{W^{2,p}(E_R)} + \|\nabla \Phi f\|_{L^p(E_R)} \leq C_{R,p} \|f\|_{L^p(E_R)}.$$

(cf. Farwig and Sohr [7]). By addition some constant to Φf , we may assume that

$$\int_{D_R} (\Phi f - \pi f) dx = 0.$$

Given function f defined on Ω , let $f_0(x)$ be defined by the formula; $f_0(x) = f(x)$ for $|x| > R$ and $f_0(x) = 0$ for $|x| \leq R$. We choose $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ so that $\phi(x) = 1$ for $|x| \leq R+1$ and $\phi(x) = 0$ for $|x| \geq R+2$. We set

$$R_\lambda f = (1 - \phi)P_\lambda f + \phi A f + \mathbb{B}[(\nabla \phi) \cdot (P_\lambda - A_0 f)].$$

Then $R_\lambda f|_{\partial\Omega} = 0$. Since $\text{supp}(\nabla \phi) \subset D_R$ and

$$\int_{\Omega} (\nabla \phi) \cdot (P_\lambda f - A f) dx = 0,$$

we see $(\nabla \phi) \cdot (P_\lambda f - A f) \in \dot{W}^{1,p}(D_R)$ and $\mathbb{B}[(\nabla \phi) \cdot (P_\lambda f - A f)] \in W^{2,p}(D_R)$. Thus $R_\lambda f \in W_{loc}^{2,p}(\Omega)$. Now set $\Pi_\lambda f = (1 - \phi)\pi f + \phi \Phi_0 f$, then

$$(\lambda - \Delta)R_\lambda f + \nabla \Pi_\lambda f = f + S_\lambda f, \quad \nabla \cdot R_\lambda f = 0 \quad \text{in } \Omega, \quad R_\lambda f|_{\partial\Omega} = 0,$$

where

$$\begin{aligned} S_\lambda f = & 2(\nabla \phi) : (\nabla P_\lambda f) + (\Delta \phi)P_\lambda f + (\lambda - \Delta)\mathbb{B}[(\nabla \phi) \cdot (P_\lambda f - A f)] \\ & + \lambda \phi A f - 2(\nabla \phi) : (\nabla A f) - (\Delta \phi)(A f) - (\nabla \phi)\pi f + (\nabla \phi)\Phi f. \end{aligned}$$

Then we know $S_\lambda : \mathbb{L}_{R+3}^p(\Omega) \rightarrow \mathbb{L}_{R+3}^p(\Omega)$ is compact operator (see [8]). What is more we have the two following lemmas.

Lemma 5.2 For $1 < p < \infty$, the following relation holds

$$\lim_{\lambda \rightarrow 0, \lambda \in \Sigma_*} \|S_\lambda - S_0\|_{\mathcal{L}(\mathbb{L}_{p,R+3}(\Omega))} = 0.$$

[proof of lemma 5.2] Lemma 5.2 follows immediately from lemma 4.1 . ■

Lemma 5.3 It holds that

$$(1 + S_0)^{-1} \in \mathcal{L}(\mathbb{L}_{p,R+3}(\Omega)).$$

[proof of Lemma 5.3] Since

$$S_0 f = 2(\nabla \phi) : (\nabla P_0 f - A f) + (\Delta \phi)(P_0 f - A f) - \Delta \mathbb{B}[(\nabla \phi) \cdot (P_0 f - A f)]$$

is the compact operator in $\mathbb{L}_{R+3}^p(\Omega)$, we will show that $1 + S_0$ is injective in $\mathbb{L}_{R+3}^p$. Let $f \in \mathbb{L}_{R+3}^p(\Omega)$ satisfy $(1 + S_0)f = 0$. Set $u = R_0 f$ and $\pi = \Pi f$, and by uniqueness we see $u(x) = 0$, $\pi(x) = 0$. Therefore

$$\begin{cases} (1 - \phi)P_0 f + \phi A f + \mathbb{B}[(\nabla \phi) \cdot (P_0 - A_0 f)] = 0 & \text{in } \Omega, \\ (1 - \phi)\pi f + \phi \Phi f = 0 & \text{in } \Omega. \end{cases} \quad (5.2)$$

Since $\phi(x) = 0$ for $|x| \geq R+2$, $P_0 f(x) = 0$, $\pi f(x) = 0$. And since $\phi(x) = 1$ for $|x| \leq R+1$, we have $A f(x) = 0$, $\Phi f(x) = 0$. Put $\widetilde{E}_R = \{x \in E_R; |x| \geq R\} \cup \{x \in \mathbb{R}_+^n; |x| < R\}$. If we put

$$w(x) = \begin{cases} A f(x) & |x| \geq R, x \in E_R, \\ 0 & |x| < R, \end{cases} \quad p(x) = \begin{cases} \Phi f(x) & |x| \geq R, x \in E_R, \\ 0 & |x| < R, \end{cases}$$

then $w \in \mathbb{W}^{2,p}(\widetilde{E}_R)$ and $p \in W^{1,p}(\widetilde{E}_R)$ and (w, p) satisfies the equation;

$$-\Delta w + \nabla p = f_0, \quad \nabla \cdot w = 0 \quad \text{in } \widetilde{E}_R, \quad w|_{\partial \widetilde{E}_R} = 0.$$

On the other hand, we have

$$\begin{cases} -\Delta P_0 f + \nabla \pi f = f_0, \quad \nabla \cdot (P_0 f) = 0 & \text{in } \widetilde{E}_R, \\ P_0 f|_{\partial \widetilde{E}_R} = 0. \end{cases}$$

By uniqueness, we obtain $P_0 f = w$ in $\widetilde{E}_R$. Notice that $\nabla(\pi f - p) = 0$ and

$$\int_{D_R} (\pi f - p) dx = \int_{D_R} (\pi f - \Phi_0 f) dx = 0,$$

we see $\pi f - p = \text{constant} = c$ in $\widetilde{E}_R$. Since

$$0 = \int_{D_R} (\pi f - p) dx = \int_{D_R} c dx = c|D_R|,$$

we obtain $c = 0$. Therefore $\pi f = p$ in $\widetilde{E}_R$. As a result, we have

$$P_0 f = w = A_0 f, \quad \pi f = p = \Phi_0 f \quad \text{in } \widetilde{E}_R.$$

In particular it holds $(\nabla \phi) \cdot [P_0 f - A_0 f] = 0$ in Ω . By (5.2), we have

$$\begin{aligned} 0 &= P_0 f + \phi(A_0 f - P_0 f) = P_0 f & \text{for } |x| \geq R+1, \quad x \in \Omega, \\ 0 &= \pi f + \phi(\Phi_0 f - \pi f) = \pi f & \text{for } |x| \geq R+1, \quad x \in \Omega. \end{aligned}$$

Since $A_0 f = 0$ and $\Phi_0 f(x) = 0$ for $x \in \Omega$, $|x| \leq R+1$,

$$\begin{aligned} 0 &= -\Delta A_0 f(x) + \nabla \Phi_0 f(x) = f & \text{for } |x| \leq R+1, \quad x \in \Omega, \\ 0 &= -\Delta P_0 f + \nabla \pi f = f & \text{for } |x| \geq R+1, \quad x \in \Omega. \end{aligned}$$

Consequently we obtain $f = 0$. ■

We get the following lemma from lemma 5.2 and 5.3.

Lemma 5.4 *There exists $\lambda_0 > 0$ such that for $\lambda \in \Sigma_\epsilon \cup \{0\}$, $|\lambda| \leq \lambda_0$, the following relations holds:*

$$(1 + S_\lambda)^{-1} \in \mathcal{L}(\mathbb{L}_{R+3}^p(\Omega)), \quad \|(1 + S_\lambda)^{-1}\|_{\mathcal{L}(\mathbb{L}_{R+3}^p(\Omega))} \leq C$$

By lemma 5.4, we can denote the solution (u, π) as follows:

$$\begin{aligned} u(x) &= R_\lambda (1 + S_\lambda)^{-1} f \\ &= (1 - \phi) P_\lambda (1 + S_\lambda)^{-1} f + \phi A_0 (1 + S_\lambda)^{-1} f + \mathbb{B}[(\nabla \phi) \cdot (P_\lambda (1 + S_\lambda)^{-1} f - A_\lambda (1 + S_\lambda)^{-1} f)], \\ \pi(x) &= \Pi_\lambda (1 + S_\lambda)^{-1} f = (1 - \phi) \pi (1 + S_\lambda)^{-1} f + \phi \Phi_0 (1 + S_\lambda)^{-1} f, \end{aligned}$$

where

$$\begin{aligned} (1 + S_\lambda)^{-1} f &= [1 + S_0 + (S_\lambda - S_0)]^{-1} f \\ &= (1 + S_0)^{-1} [1 + (1 + S_0)^{-1} (S_\lambda - S_0)]^{-1} f \\ &= (1 + S_0)^{-1} \sum_{j=0}^{\infty} [(1 + S_0)^{-1} (S_\lambda - S_0)]^j f, \\ S_\lambda - S_0 &= 2(\nabla \phi) : \nabla (P_\lambda - P_0) + (\Delta \phi)(P_\lambda - P_0) + \lambda \phi A_0 f \\ &\quad + \lambda \mathbb{B}[(\nabla \phi) \cdot (P_\lambda - P_0)] - \Delta \mathbb{B}[(\nabla \phi)(P_\lambda - P_0)]. \end{aligned}$$

6 Proofs of Theorem 1.1 and Theorem 1.2

We study coerciveness estimates for A^m and $(A - \lambda)^{-1}$ when $\lambda \in \Sigma = \{\lambda \in \mathbb{C} \setminus \{0\}; |\arg \lambda| < \delta\}$ with $|\lambda| \geq 1$.

Proposition 6.1 *Let $1 < q < \infty$ and let A be the Stokes operator in $J_q(\Omega)$.*

(i) *Assume that $u \in \mathfrak{D}_q(A)$ and $Au \in W_q^m(\Omega)$ for a nonnegative integer m . Then $u \in W_q^{m+2}(\Omega)$ and for some constant $C_m > 0$,*

$$\|u\|_{W_q^{m+2}(\Omega)} \leq C_m (\|Au\|_{W_q^m(\Omega)} + \|u\|_{L^q(\Omega)}). \quad (6.1)$$

(ii) *$u \in \mathfrak{D}_q(A^m)$, $m \geq 0$, then $u \in W_q^{2m}(\Omega)$ and*

$$\|u\|_{W_q^{2m}(\Omega)} \leq C_m (\|A^m u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}). \quad (6.2)$$

[proof of Proposition 6.1] The proof of (6.2) is carried out by applying (6.1) and the following estimate: for any $\varepsilon > 0$ and integer $\ell \geq 1$,

$$\|A^{\ell-1} u\|_{L^q(\Omega)} \leq \varepsilon \|A^\ell u\|_{L^q(\Omega)} + C_\varepsilon \|u\|_{L^q(\Omega)}, \quad u \in \mathfrak{D}_q(A^\ell)$$

with some $C_\varepsilon > 0$. ■

Proposition 6.2 *Let $1 < q < \infty$ and let A be the Stokes operator in $J_q(\Omega)$.*

(i) *For a nonnegative integer m ,*

$$\|A^m u\|_{L^q(\Omega)} \leq C_m \|u\|_{W_q^{2m}(\Omega)}, \quad u \in \mathfrak{D}_q(A^m). \quad (6.3)$$

(ii) *Let $0 < \delta < \pi$ and let m be a nonnegative integer. If $f \in \mathfrak{D}_q(A^m)$, then*

$$\|(A + \lambda)^{-1} f\|_{W_q^{2m+2}(\Omega)} \leq C_m \|f\|_{W_q^{2m}(\Omega)}, \quad (6.4)$$

for any $\lambda \in \Sigma(\delta)$.

[proof of Proposition 6.2]

(i) Since $Au = -P\Delta u$ for $u \in \mathfrak{D}_q(A)$ and P is bounded in $W_q^\ell(\Omega)$ for any nonnegative integer ℓ , we have

$$\|A^m u\|_{L^q(\Omega)} = \|P\Delta(A^{m-1}u)\|_{L^q(\Omega)} \leq C \|A^{m-1}u\|_{W_q^{2, q}(\Omega)} \leq C \|A^{m-2}u\|_{W_q^{4, q}(\Omega)}.$$

Repeating this manipulation leads us to (6.3).

(ii) The estimate (6.4) is an immediate consequence of Proposition 6.1, (6.2) and (6.3). In fact,

$$\begin{aligned} \|(A + \lambda)^{-1} f\|_{W_q^{2m+2}(\Omega)} &\leq C_m (\|A^{m+1}(A + \lambda)^{-1} f\|_{L^q(\Omega)} + \|(A + \lambda)^{-1} f\|_{L^q(\Omega)}) \\ &\leq C_m (\|A^m f\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}) \leq C_m \|f\|_{W_q^{2m}(\Omega)} \end{aligned}$$

We shall prove Theorem 1.1 and 1.2 from what we showed in section 5 in the same way as in Iwashita [9]. First, we shall show Theorem 1.1. For this purpose, it suffices to prove the following theorem: ■

Theorem 6.3 Let $n \geq 3$ and $1 < q < \infty$. Then there exists a positive constant $C = C(q)$ such that the inequality

$$\|e^{-tA}f\|_{L^q(\Omega)} \leq C(1+t)^{-\frac{n+1}{2}}\|f\|_{L^q(\Omega)} \quad t \geq 0 \quad (6.5)$$

is valid for any $f \in \mathbb{J}^q(\Omega)$.

[proof of Theorem 6.3] Since the semigroup e^{-tA} is bounded in $J_q(\Omega)$, it suffices to show (6.5) for large $t > 0$. Then let $\frac{\pi}{2} < \delta_0 < \delta < 2\pi$ and $0 < \varepsilon < \varepsilon_0$. Let Γ be a contour as follows: $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{\lambda \in \mathbb{C}; 0 < |\lambda| < \varepsilon, \arg \lambda = \pm\delta\}, \quad \Gamma_2 = \{\lambda \in \mathbb{C}; |\lambda| > \varepsilon, \arg \lambda = \pm\delta\}.$$

The semigroup is described as follows :

$$e^{-tA} = \frac{-1}{2\pi i} \int_{\Gamma_1} e^{-t\lambda} R(\lambda) d\lambda + \frac{-1}{2\pi i} \int_{\Gamma_2} e^{-t\lambda} (A + \lambda)^{-1} d\lambda. \quad (6.6)$$

The second term of the right hand side of (6.6) is estimated as

$$\left\| \frac{-1}{2\pi i} \int_{\Gamma_2} e^{-t\lambda} (A + \lambda)^{-1} d\lambda \right\|_{\mathcal{L}(J_q(\Omega))} \leq C \int_{\varepsilon}^{\infty} e^{-tr} dr \leq C e^{-c\varepsilon t}. \quad (6.7)$$

for some $c, C > 0$. In order to estimate the first term of the right hand side of (6.6), we need the following lemma that is a direct consequence of the formula of the gamma function $\Gamma(\sigma)$.

Lemma 6.4 (i) For $\sigma > 0$ and $t > 0$, it holds that

$$\frac{-1}{2\pi i} \int_{\Gamma} e^{-tz} z^{\sigma-1} dz = -\frac{\sin \sigma \pi}{\pi} \Gamma(\sigma) e^{i\pi\sigma} t^{-\sigma}.$$

(ii) For a nonnegative integer j and any $t > 0$,

$$\frac{-1}{2\pi i} \int_{\Gamma} e^{-tz} z^j \log z dz = -\frac{d}{dz} \left[\frac{\sin \sigma \pi}{\pi} \Gamma(\sigma) e^{i\pi\sigma} \right] \Big|_{\sigma=j+1} t^{-j-1}.$$

Since $R(\lambda)$ is described as

$$R(\lambda) = \begin{cases} G_1(\lambda) \lambda^{\frac{n-1}{2}} + G_2(\lambda) \lambda^{\frac{n}{2}} \log \lambda + G_3(\lambda) & n \text{ is even,} \\ G_1(\lambda) \lambda^{\frac{n}{2}} + G_2(\lambda) \lambda^{\frac{n-1}{2}} \log \lambda + G_3(\lambda) & n \text{ is odd,} \end{cases} \quad (6.8)$$

we can apply Lemma 6.4 to obtain

$$\left\| \frac{-1}{2\pi i} \int_{\gamma_1} e^{-t\lambda} G_1(\lambda) \lambda^{\frac{n-1}{2}} f d\lambda \right\|_{L^q(\Omega)} \leq C t^{-\frac{n}{2}-\frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad (6.9)$$

$$\left\| \frac{-1}{2\pi i} \int_{\gamma_1} e^{-t\lambda} G_2(\lambda) \lambda^{\frac{n}{2}} (\log \lambda) f d\lambda \right\|_{L^q(\Omega)} \leq C t^{-\frac{n}{2}-1} \|f\|_{L^q(\Omega)}. \quad (6.10)$$

Finally the operator $G_3(\lambda)$ is bounded so that we have

$$\left\| \frac{-1}{2\pi i} \int_{\gamma_1} e^{-t\lambda} G_3(\lambda) f d\lambda \right\|_{L^q(\Omega)} \leq C e^{-c\varepsilon t} \|f\|_{L^q(\Omega)}. \quad (6.11)$$

Combining (6.7) and (6.9)-(6.11) completes the proof of Theorem 6.3. $\blacksquare$

We can show Theorem 1.2 from Theorem 1.1 and cut-off technique in the same way as in Iwashita [9]. In order to show Theorem 1.2, we introduce the Ukai operator $E(t)f(x)$ (cf. Ukai [14]) which solves the equation:

$$\begin{cases} u_t - \Delta u + \nabla p = 0, & \nabla \cdot u = 0 & \text{in } (0, \infty) \times \mathbb{R}_+^n, \\ u|_{x_n=0} = 0, & u|_{t=0} = f, \end{cases}$$

with $u = E(t)f(x)$ and some pressure term $p(t, x)$. We know the following fact (cf. Ukai [14], Borchers and Miyakawa [2]).

Lemma 6.5 *Let $1 \leq q \leq r \leq \infty$ and put $\sigma = (\frac{n}{q} - \frac{n}{r})/2$. Then, we have*

$$\begin{aligned} \|E(t)f\|_{L^r(\mathbb{R}_+^n)} &\leq C_\alpha t^{-\sigma} \|f\|_{L^q(\mathbb{R}_+^n)}, \\ \|\nabla E(t)f\|_r &\leq C_\alpha t^{-\sigma-\frac{1}{2}} \|f\|_{L^q(\mathbb{R}_+^n)}, \end{aligned} \quad (6.12)$$

for $t > 0$.

The next lemma is concerned with the estimate of derivatives of $e^{-tA}f$.

Lemma 6.6 *Let $n \geq 3, 1 < q < \infty, d > R_0$, and let m be a nonnegative integer.*

(i) *There exists a constant $C = C(d, m) > 0$ such that*

$$\|e^{-tA}f\|_{\Omega_{d,q,2m}} \leq C(1+t)^{-\frac{n}{2}-\frac{1}{2}} \|f\|_{\Omega_{d,q,2m}}, \quad (6.13)$$

for any $f \in \mathcal{D}_q(A^m)$ with $f = 0$ for $|x| > d$.

(ii) *If $f \in \mathcal{D}_q(A^{m+1})$ and $f = 0$ for $|x| > d$, then*

$$\|\partial_t e^{-tA}f\|_{\Omega_{d,q,2m}} \leq C(1+t)^{-\frac{n}{2}-\frac{3}{2}} \|f\|_{\Omega_{d,q,2m+2}}, \quad (6.14)$$

where the constant $C > 0$ is independent of f .

[proof of Lemma 6.6] By proposition 6.1 and proposition 6.2, it suffices to verify the assertions for t large. The proof of (i) can be carried out similarly as in proof of theorem 6.3 with the aid of (6.4) and expansion of $R(\lambda)$.

(ii) For $t \gg 1$ and the contour γ introduced in proof of theorem 6.3, the identity

$$\partial_t e^{-tA}f = \frac{-1}{2\pi i} \int_{\gamma_1} e^{-t\lambda} (-\lambda) R(\lambda) f d\lambda + \frac{-1}{2\pi i} \int_{\gamma_2} e^{-t\lambda} (-\lambda) (A - \lambda)^{-1} f d\lambda \quad (6.15)$$

is valid so that (6.14) is obtained by using the expansion and (6.4) in (6.15). $\blacksquare$

Put $u_0 = e^{-A}f$ for $f \in J_q(\Omega)$. Since $u_0 \in \mathcal{D}_q(A^N)$ for any integer $N \geq 0$, it follows from proposition 6.1 that $u_0 \in W_q^{2N}(\Omega)$ and

$$\|u_0\|_{\Omega_{q,2N}} \leq C_N \|f\|_{\Omega_{q,2}}. \quad (6.16)$$

Put $u(t) = e^{-tA}u_0$ and $u(t)$ is satisfied with the following:

$$\begin{cases} \partial_t u(t) - \Delta u(t) + \nabla p(t) = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot u(t) = 0 & \text{in } (0, \infty) \times \Omega, \\ u(t)|_{\partial\Omega} = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

In the same manner as in Iwashita [9], by Lemma 6.5 and Lemma 6.6, we have the two lemmas.

Lemma 6.7 Let $u(t)$ be as above and $d \geq R_0 + 5$. For a nonnegative integer m , the inequalities

$$\|u(t)\|_{\Omega_{d,q,2m}} \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + 2m + 2}}, \quad (6.17)$$

$$\|\partial_t u(t)\|_{\Omega_{d,q,2m}} \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + 2m + 4}}, \quad (6.18)$$

are valid for any $t \geq 0$, where the constant C depends only on d, m , and q .

Lemma 6.8 Let $p(t)$ be a certain pressure associated with $u(t)$. Then,

$$\|p(t)\|_{\Omega_{d,q,2m}} \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + 2m + 4}}. \quad (6.19)$$

We choose $\psi \in C^\infty(\mathbb{R}_+^n)$ so that $\psi(x) = 1$ for $|x| \geq d$ and $= 0$ for $|x| \leq d-1$. By proposition 5.1, Lemma 6.7, we can find $v_3(t)$ to satisfy the following relation:

$$\nabla \cdot v_3(t) = \nabla \cdot [\psi u(t)], \quad \text{supp } v_3(t) \subset \{d-1 \leq |x| \leq d\} \cap \mathbb{R}_+^n.$$

Then $v_3(t)$ satisfies the two following estimates:

$$\begin{aligned} \|v_3(t)\|_{q,m} &\leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + m + 2}}, \\ \|\partial_t v_3(t)\|_{q,m} &\leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + m + 4}}, \end{aligned} \quad (6.20)$$

In fact

$$\begin{aligned} \|v_3(t)\|_{q,m} &\leq C \|\nabla \cdot [\psi u(t)]\|_{q,m-1} = C (\|\nabla \psi \cdot u(t)\|_{q,m-1} + \|\psi \nabla \cdot u(t)\|_{q,m-1}) \\ &\leq C \|u(t)\|_{\Omega_{d,q,m-1}} \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + m + 2}}. \end{aligned}$$

Set $v_4(t) = \psi u(t) - v_3(t)$.

Lemma 6.9 Let $q < r < \infty$ and $v_4(t)$ be as above. Then

$$\|v_4(t)\|_r \leq C(1+t)^{-(\frac{n}{q} - \frac{n}{r})/2} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + [2\sigma] + 7}}, \quad (6.21)$$

[proof of Lemma 6.9] 1st step. We set

$$h(t) = -\{2(\nabla \psi) \cdot \nabla + (\Delta \psi)\} u(t) - (\partial_t - \Delta) v_3(t) + p(t) \nabla \psi,$$

and then see $\text{supp } h(t) \subset \{d-1 \leq |x| \leq d\} \cap \mathbb{R}_+^n$. Moreover, $h(t)$ satisfies the estimate:

$$\|h(t)\|_{q,m} \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega_{q, [\frac{n}{q}] + m + 4}}. \quad (6.22)$$

Here we set $v_4 = \psi u_0 - v_3(0)$ and see $\nabla \cdot v_4 = 0$ in $\mathbb{R}_+^n$ and

$$\|v_4\|_{q,m} \leq C \|u_0\|_{\Omega_{q,m}}. \quad (6.23)$$

$v_4(t)$ satisfies the following problem:

$$\begin{cases} \partial_t v_4(t) - \Delta v_4(t) + \nabla(\psi p(t)) = h(t) & \text{in } (0, \infty) \times \mathbb{R}_+^n, \\ \nabla \cdot v_4(t) = 0 & \text{in } (0, \infty) \times \mathbb{R}_+^n, \\ v_4(0) = v_4 & \text{in } \mathbb{R}_+^n. \end{cases} \quad (6.24)$$

and is hence described as

$$v_4(t) = E(t)v_4 + v_5(t), \quad v_5(t) = \int_0^t E(t-s)P_0h(s)ds.$$

By (6.12), (6.23), we see

$$\|E(t)v_4\|_r \leq C(1+t)^{-\sigma}\|v_4\|_{q,[2\sigma]+1} \leq C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[2\sigma]+1}.$$

Next we will estimate $v_5(t)$ in the second step.

2nd step: Case 1. We consider the case of $\frac{n}{2q} > 1$. We can estimate $v_5(t)$ as follows ;

$$\begin{aligned} \|v_5(t)\|_r &\leq C \int_0^t (1+(t-s))^{-\sigma} \|P_0h(s)\|_{q,[2\sigma]+1} ds \\ &\leq C \int_0^t (1+(t-s))^{-\sigma} \|h(s)\|_{q,[2\sigma]+1} ds \\ &= C \int_0^t (1+(t-s))^{-\sigma} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+5}, \end{aligned}$$

where

$$\begin{aligned} \int_{\frac{1}{2}}^t (1+(t-s))^{-\sigma} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-\sigma}, \\ \int_0^{\frac{1}{2}} (1+(t-s))^{-\sigma} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-\sigma} \int_0^{\frac{1}{2}} (1+s)^{-\frac{n}{2q}} ds \leq C(1+t)^{-\sigma}. \end{aligned}$$

We see

$$\|v_5(t)\|_r \leq C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+5}.$$

Case 2: We consider the case of $\frac{n}{2q} \leq 1$. Taking $\sigma = (\frac{n}{q} - \frac{n}{r})/2 < 1$ and $n(1 - \frac{1}{r}) > 1$ into account, we see $r > \frac{n}{n-2}$. When $n \geq 4$, $r > \frac{n}{n-2}$ follows from $r > q > \frac{n}{2}$. Then We can choose $\rho > 1$ so that $(\frac{n}{\rho} - \frac{n}{r})/2 = 1 + \kappa$ ($0 < \kappa < \frac{1}{2}$). By (6.22), we can estimate $v_5(t)$ as follows:

$$\begin{aligned} \|v_5(t)\|_r &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} \|P_0h(s)\|_{\rho,3} ds \\ &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} \|h(s)\|_{\rho,3} ds \\ &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \\ &\leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7}. \end{aligned}$$

Case 3 : We will show $\frac{3}{2} \leq q < r \leq 3$ for $n = 3$. We choose $\rho > 1$ so that $\frac{3}{2\rho} > 1$ and $3(\frac{1}{\rho} - \frac{1}{r})/2 < 1$. We have

$$\begin{aligned} \|v_5(t)\|_r &\leq C \int_0^t (1+(t-s))^{-3(\frac{1}{\rho} - \frac{1}{r})/2} \|P_0h(s)\|_{\rho,2} ds \\ &\leq C \int_0^t (1+(t-s))^{-3(\frac{1}{\rho} - \frac{1}{r})/2} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega,q,[\frac{n}{q}]+6}. \end{aligned}$$

When $q > \frac{3}{2}$, taking account of $-3(\frac{1}{\rho} - \frac{1}{r})/2 - \frac{3}{2q} + 1 < -\sigma$ we see

$$\|v_5(t)\|_r \leq C(1+t)^{-3(\frac{1}{\rho} - \frac{1}{r})/2 - \frac{3}{2q} + 1} \|u_0\|_{\Omega, q, [\frac{n}{q}] + 6},$$

and

$$\|v_5(t)\|_r \leq C(1+t)^{-\sigma} \|u_0\|_{\Omega, q, [\frac{n}{q}] + 6}.$$

For $q = \frac{3}{2}$, noticing that $\rho < \frac{3}{2} = q$ we have

$$\|v_5(t)\|_r \leq C(1+t)^{-3(\frac{1}{\rho} - \frac{1}{r})/2} \log(1+t) \|u_0\|_{\Omega, q, [\frac{n}{q}] + 6} \leq C(1+t)^{-\sigma} \|u_0\|_{\Omega, q, [\frac{n}{q}] + 6},$$

since $-\frac{3}{2\rho} + \frac{3}{2r} + \varepsilon < -\sigma$. Summing up, we obtain the assertion of lemma. $\blacksquare$

[proof of Theorem 1.2 (1)] At first we shall show (1.3) for $t \geq 1$. Since (6.16), (6.17) and Sobolev's embedding theorem, we have

$$\begin{aligned} \|u(t)\|_{\Omega, d, r} &\leq C \|u(t)\|_{\Omega, d, q, [2\sigma] + 1} \\ &\leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega, q, [\frac{n}{q}] + [2\sigma] + 3} \leq C(1+t)^{-\frac{n}{2q}} \|f\|_{\Omega, q}, \end{aligned} \quad (6.25)$$

for $d \geq R_0 + 5$. On the other hand, by (6.16), (6.20) and (6.21), we see

$$\begin{aligned} \|u(t)\|_{\{x \geq d\}, r} &\leq \|v_4(t)\|_r + \|v_3(t)\|_r \\ &\leq C(1+t)^{-\sigma} \|u_0\|_{\Omega, q, [\frac{n}{q}] + [2\sigma] + 7} + C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega, q, [\frac{n}{q}] + [2\sigma] + 3} \\ &\leq C(1+t)^{-\sigma} \|f\|_{\Omega, q}. \end{aligned} \quad (6.26)$$

Since $u(t) = e^{-tA} u_0 = e^{-(1+t)A} f$, we obtain (1.3) for $t \geq 1$.

Secondly we will show (1.3) for $0 < t < 1$. Set $N = [2\sigma]$. When N is even, by proposition 6.1 we see

$$\|e^{-tA} f\|_{\Omega, q, N} \leq C \left(\|A^{\frac{N}{2}} e^{-tA} f\|_{\Omega, q} + \|e^{-tA} f\|_{\Omega, q} \right) \leq C t^{-\frac{N}{2}} \|f\|_{\Omega, q}.$$

Similary

$$\|e^{-tA} f\|_{\Omega, q, N+2} \leq C t^{-\frac{N+2}{2}} \|f\|_{\Omega, q}.$$

By using Sobolev's embedding theorem and an interpolation method, we have

$$\begin{aligned} \|e^{-tA} f\|_{\Omega, r} &\leq C \|e^{-tA} f\|_{\Omega, q, 2\sigma} \\ &\leq C \left(t^{-\frac{N}{2}} \right)^{-\sigma + \frac{N}{2} + 1} \left(t^{-\frac{N+2}{2}} \right)^{1 + \sigma - \frac{N}{2} - 1} \|f\|_{\Omega, q} \\ &= C t^{-\sigma} \|f\|_{\Omega, q}. \end{aligned}$$

When N is odd, we consider $N - 1$ instead of N in the same way. $\blacksquare$

[proof of Theorem 1.2 (2)] We shall prove (1.4) for large t only. Since $v_4(t)$ satisfy (6.24), $\partial v_4(t)$ is described as the integral form :

$$\partial v_4(t) = \partial E(t) v_4 + v_6, \quad v_6(t) = \int_0^t \partial E(t-s) P_0 h(s) ds.$$

We can estimate $v_4(t)$ as follows:

$$\|\partial v_4(t)\|_r \leq C(1+t)^{-\sigma - \frac{1}{2}} \|u_0\|_{\Omega, q, [\frac{n}{q}] + [2\sigma] + 7}. \quad (6.27)$$

In fact, since by (6.23) and (6.23) we have

$$\|\partial E(t)v_4\|_r \leq C(1+t)^{-\sigma-\frac{1}{2}}\|v_4\|_{q,[2\sigma]+2} \leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{q,[2\sigma]+2},$$

it is sufficient that we consider the estimate of $v_6(t)$.

Case 1: We consider the estimate of $v_6(t)$ for $\sigma > \frac{1}{2}$. We have

$$\begin{aligned} \|v_6(t)\|_r &\leq C \int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}} \|P_0 h(s)\|_{q,[2\sigma]+2} ds \\ &\leq C \int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6}. \end{aligned}$$

Since $q \leq r \leq n$, it follows that $\frac{n}{2r} \geq \frac{1}{2}$, that is, $\sigma + \frac{1}{2} \leq \frac{n}{2q}$, we can see

$$\begin{aligned} \int_{\frac{1}{2}}^t (1+(t-s))^{-\sigma-\frac{1}{2}} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-\sigma-\frac{1}{2}}, \\ \int_0^{\frac{1}{2}} (1+(t-s))^{-\sigma-\frac{1}{2}} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-\sigma-\frac{1}{2}-\frac{n}{2q}+1} \leq C(1+t)^{-\sigma-\frac{1}{2}}. \end{aligned}$$

Therefore we can show

$$\|v_6(t)\|_r \leq C(1+t)^{-\sigma-\frac{1}{2}} \|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6}.$$

Case 2: We consider the case that $\sigma \leq \frac{1}{2}$, $n(1 - \frac{1}{r}) > 1$ ie. $r > \frac{n}{n-1}$. We can find $\rho > 1$ so that $\rho < q$ and $\frac{n}{\rho} - \frac{n}{r} = 1 + 2\kappa$ for $0 < \kappa < \frac{1}{2}$. We have

$$\begin{aligned} \|v_6(t)\|_r &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} \|P_0 h(s)\|_{\rho,3} ds \\ &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} \|h(s)\|_{\rho,3} ds \\ &\leq C \int_0^t (1+(t-s))^{-(1+\kappa)} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega,q,[\frac{n}{q}]+7}. \end{aligned}$$

If $\frac{n}{2q} \geq 1$, we have

$$\begin{aligned} \int_{\frac{1}{2}}^t (1+(t-s))^{-(1+\kappa)} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-1}, \\ \int_0^{\frac{1}{2}} (1+(t-s))^{-(1+\kappa)} (1+s)^{-\frac{n}{2q}} ds &\leq C(1+t)^{-(1+\kappa)-\frac{n}{2q}+1} \leq C(1+t)^{-1}. \end{aligned}$$

Therefore we can obtain the estimate of $v_6(t)$:

$$\|v_6(t)\|_r \leq C(1+t)^{-1} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma-\frac{1}{2}} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.$$

If $\frac{n}{2q} < 1$, then $\sigma + \frac{1}{2} \leq \frac{n}{2q}$. So we have

$$\|v_6(t)\|_r \leq C(1+t)^{-\frac{n}{2q}} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma-\frac{1}{2}} \|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.$$

Case 3 : We consider the case that $\sigma \leq \frac{1}{2}$ and $r \leq \frac{n}{n-1}$. Then $\frac{n}{2q} \geq \frac{n-1}{2}$. Provided that $\frac{n}{2q} > 1$, which is always valid when $n \geq 4$, we have

$$\begin{aligned} \|v_6(t)\|_r &\leq C \int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}} \|P_0 h(s)\|_{q, [2\sigma]+2} ds \\ &\leq C \int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}} (1+s)^{-\frac{n}{2q}} ds \|u_0\|_{\Omega, q, [\frac{n}{q}]+[2\sigma]+6} \\ &\leq C(1+t)^{-\sigma-\frac{1}{2}} \|u_0\|_{\Omega, q, [\frac{n}{q}]+[2\sigma]+6}. \end{aligned}$$

Case 4: Finally it remain to consider the case that $n = 3$ and $q = r = \frac{3}{2}$. We choose ρ so that $1 < \rho < \frac{3}{2}$. Then we have

$$\begin{aligned} \|v_6\|_{\frac{3}{2}} &\leq C \int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}} \|P_0 h(s)\|_{\rho, 2} ds \\ &\leq C \int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}} \|h(s)\|_{\rho, 2} ds \\ &\leq C \int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}} (1+s)^{-1} ds \|u_0\|_{\Omega, q, [\frac{n}{q}]+6} \\ &\leq C(1+t)^{-\frac{3}{2\rho}+\frac{1}{2}} \log(1+t) \|u_0\|_{\Omega, q, [\frac{n}{q}]+6} \leq C(1+t)^{-\frac{1}{2}} \|u_0\|_{\Omega, q, [\frac{n}{q}]+6}. \end{aligned}$$

Summing up, we have prove (6.27). ■

References

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