

Colloque E.D.P.
Saint-Jean de Monts
JUIN 1987

PROPAGATION OF ANALYTIC SINGULARITIES

UP TO NON SMOOTH BOUNDARIES

Pierre SCHAPIRA

1.- Propagation for sheaves

We shall follow the notations of [K-S 1]. In particular if X is a real manifold, we denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology, and if $F \in D^b(X)$ we denote by $SS(F)$ its microsupport. Recall that $SS(F)$ is a closed conic involutive subset of T^*X . We shall also make use of the bifunctor μhom , from $D^b(X)^0 \times D^b(X)$ to $D^b(T^*X)$, a slight generalization of the functor of Sato's microlocalization.

Let h be a real C^2 -function defined on an open subset U of T^*X , H_h its hamiltonian vector field. If $(x; \xi)$ is a system of homogeneous symplectic coordinates, with $\omega_X = \sum_j \xi_j dx_j$, then:

$$(1.1) \quad H_h = \sum_j \left(\frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

If $p \in U$ we denote by b_p^+ the positive half integral curve of H_h issued at p . We define similarly b_p^- and $b_p = b_p^- \cup b_p^+$. We also set for $* = 0, +, -$:

$$(1.2) \quad V_* = \{p \in U ; h(p) \geq 0 \quad (* = +) \quad \text{or} \\ h(p) \leq 0 \quad (* = -) \quad \text{or} \quad h(p) = 0 \quad (* = 0)\}.$$

The following result is easily deduced from [K-S 1, Th. 5.2.1].

Theorem 1.1. Let F and G belong to $D^b(X)$ with
 $SS(G) \cap U \subset V_-$, $SS(F) \cap U \subset V_+$. Let $j \in \mathbb{Z}$ and let
 u be a section of $H^j(\text{phom}(G, F))$ on U . Then
 $p \in \text{supp}(u)$ implies $b_p^+ \subset \text{supp}(u)$.

(Remark that $\text{supp}(u)$ is contained in V_0).

2.- Wave front sets at the boundary [S 1]

Let M be a real analytic manifold of dimension n , X a complexification of M , Ω an open subset of M . We introduce :

$$(2.1) \quad C_{\Omega|X} = \text{phom}(\mathcal{I}_{\Omega}, \mathcal{O}_X) \otimes \omega_{M/X}[n]$$

where $\omega_{M/X}$ is the relative orientation sheaf.

Let π denote the projection $T^*X \longrightarrow X$, and let

$B_M = R\Gamma_M(\mathcal{O}_X) \otimes \omega_{M/X}[n]$ denote the sheaf of Sato's hyperfunctions on M . There is a natural isomorphism :

$$(2.2) \quad \alpha : \Gamma_{\Omega}(B_M) \xrightarrow{\sim} \pi_* H^0(C_{\Omega|X}).$$

Hence a hyperfunction u on Ω defines a section $\alpha(u)$ of $H^0(C_{\Omega|X})$ all over T^*X . We set :

$$(2.3) \quad SS_{\Omega}(u) = \text{supp}(\alpha(u)).$$

Since $H^0(C_\Omega|_X)$ is supported by the conormal bundle T_M^*X , $SS_\Omega(u)$ is a closed conic subset of T_M^*X . It coincides with the classical analytical wave front set above Ω , but it may be strictly larger than its closure in T_M^*X (cf. [S 1]).

Now let P be a differential operator defined on X , and assume for simplicity that the principal symbol $\sigma(P)$ never vanishes identically. Let $\mathcal{O}_X^P$ denote the sheaf of holomorphic solutions of the equation $Pf = 0$. Replacing $\mathcal{O}_X$ by $\mathcal{O}_X^P$ in the preceding discussion, we define :

$$(2.4) \quad C_\Omega^P|_X = \text{phom}(\mathbb{Z}_\Omega, \mathcal{O}_X^P) \otimes \omega_{M/X} [n] .$$

Let B_M^P denote the sheaf of hyperfunction solutions of the equation $Pu = 0$. There is a natural isomorphism :

$$(2.5) \quad \alpha : \Gamma_\Omega(B_M^P) \xrightarrow{\sim} \pi_* H^0(C_\Omega^P|_X) .$$

If u is a hyperfunction on Ω solution of the equation $Pu = 0$, we set :

$$(2.6) \quad SS_\Omega^P(u) = \text{supp}(\alpha(u)) .$$

Remark that

$$(2.7) \quad SS_\Omega^P(u) \subset SS(\mathbb{Z}_\Omega) \cap \text{char}(P)$$

(where $\text{char}(P) = \sigma(P)^{-1}(0)$), but in general $SS_\Omega^P(u)$ is no more contained in T_M^*X .

I don't know if $SS_\Omega^P(u) \cap T_M^*X = SS_\Omega(u)$, but this is true when $M \setminus \Omega$ is convex (locally, up to analytic diffeomorphisms).

Of course the preceding discussion extends to solutions of general systems of differential equations (cf. [S 1]).

Now assume $\partial\Omega = N$ is a real analytic hypersurface and let Y be a complexification of N in X . Assume P of order m , Y is non characteristic for P , and a normal vector field to N in M is given, so that the induced system $(D_X/D_X P)_Y$ is isomorphic to D_Y^m ; (as usual, D_X denotes the ring of differential operators).

Let ρ and $\bar{w}$ denote the natural maps associated to $Y \longrightarrow X$:

$$(2.8) \quad T^*Y \xleftarrow[\rho]{} Y \times_{\substack{X \\ T^*X}} T^*X \xrightarrow[\bar{w}]{} T^*X .$$

Let $u \in \Gamma(\Omega; B_M^P)$ be a hyperfunction on Ω solution of $Pu = 0$, and let $b(u) \in \Gamma(N; B_N^m)$ be its traces. Recall (cf. [S 1], [S 2]) :

Theorem 2.1. In the preceding situation, one has :

$$SS_N(b(u)) = \rho \bar{w}^{-1} SS_\Omega^P(u) .$$

In other words, the analytic wave front set of $b(u)$ is exactly the projection of $SS_\Omega^P(u)$. Remark that if $\text{char}(P) \cap SS(\mathbb{R}_\Omega)$ is contained in T_M^*X , $SS_\Omega^P(u)$ may be replaced by $SS_\Omega(u)$ in Theorem 2.1.

Remark moreover that $b(u)$ does not make sense when $\partial\Omega$ is not smooth, but $SS_\Omega(u)$ always does.

3.- Transversal propagation for non smooth boundaries

Let M be a real analytic manifold, X a complexification of M , Ω an open subset of M .

If $x \in M$, the cone $N_x(\Omega)$ is defined in [K-S 1]. Recall that $N_x(\Omega)$ is an open convex cone of T_x^*M , and $\theta \in N_x(\Omega)$, $\theta \neq 0$ implies that there exists a convex open cone γ (in a system of local coordinates around x) such that $\theta \in \gamma$ and $\Omega + \gamma \subset \Omega$.

We shall have to consider the real underlying structure of T^*X . Recall that if ω_X is the complex canonical 1-form on T^*X , this real symplectic structure is defined by $2\text{Re } \omega_X$.

If h is a real C^2 -function on T^*X , we denote by H_h^{IR} its real Hamiltonian vector field.

If (z, ζ) is a system of homogeneous holomorphic symplectic coordinates on T^*X , such that $\omega_X = \sum_j \zeta_j dz_j$, and $z = x + iy$, $\zeta = \xi + i\eta$, then

$$(3.1) \quad \mathcal{L}_{H_h^{\text{IR}}} = \sum_j \left(\frac{\partial h}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j} \frac{\partial}{\partial \xi_j} + \frac{\partial h}{\partial y_j} \frac{\partial}{\partial \eta_j} - \frac{\partial h}{\partial \eta_j} \frac{\partial}{\partial y_j} \right).$$

Now let P be a differential operator on X , u a hyperfunction on Ω , solution of the equation $Pu = 0$. Let $p \in T_M^*X$, $x_0 = \pi(p)$.

Theorem 3.1. Assume :

- a) $\text{Im } \sigma(P) \big|_{T_M^*X} = 0$
- b) $\pi(H_{\text{Im } \sigma(P)}^{\text{IR}}(p)) \in N_{x_0}(\Omega)$.

Let b_p^+ be the positive half integral curve of $H_{\text{Im } \sigma(P)}^{\text{IR}}$ issued at p . Then $p \in \text{SS}_\Omega(u)$ implies $b_p^+ \subset \text{SS}_\Omega(u)$.

Remark that

$$\mathcal{L}_{H_{\text{Im } \sigma(P)}^{\text{IR}}} \big|_{T_M^*X} = \sum_j \left(\frac{\partial \text{Re } \sigma(P)}{\partial x_j} \frac{\partial}{\partial \xi_j} - \frac{\partial \text{Re } \sigma(P)}{\partial \xi_j} \frac{\partial}{\partial x_j} \right).$$

Proof

We may assume X is open in $\mathbb{C}^n$ and $M = X \cap \mathbb{R}^n$. Then there exists a convex open cone γ such that $\Omega + \gamma \subset \Omega$ (in a neighborhood of x_0) and $\pi(H_{\text{Im } \sigma(P)}^{\mathbb{R}}(p)) \in \gamma$. This last condition implies :

$$\langle d_{\xi} \text{Im } \sigma(P)(x, i\eta), \xi \rangle \geq c|\xi|$$

for some $c > 0$ and all $\xi \in \gamma^0$ (γ^0 is the polar set to γ). Hence :

$$(3.2) \quad \text{Im } \sigma(P)(x, \xi + i\eta) \leq 0$$

for $(x, \xi + i\eta)$ in a neighborhood of p , $\xi \in \gamma^{\text{oa}}$, where $\gamma^{\text{oa}} = -\gamma^0$.

Since $\Omega + \gamma \subset \Omega$, we have (cf. [K-S 1]) :

$$SS(\mathbb{Z}_{\Omega}) \subset T_M^*X + \gamma^{\text{oa}}.$$

Thus :

$$(3.3) \quad \text{Im } \sigma(P) \leq 0 \text{ on } SS(\mathbb{Z}_{\Omega})$$

in a neighborhood of p .

Now let $u \in \Gamma(\Omega; B_M)$ be a solution of the equation $Pu = 0$.

Then u defines a section $\alpha(u) \in \Gamma(T^*X; H^n(\mu_{\text{hom}}(\mathbb{Z}_{\Omega}, \theta_X^P)))$ and $p \in SS_{\Omega}(u)$ implies $p \in SS_{\Omega}^P(u)$, that is, $p \in \text{supp}(\alpha(u))$.

Since $SS(\theta_X^P) = \text{char}(P) \subset \{\text{Im } \sigma(P) = 0\}$, we may apply Theorem 1.1 and we obtain :

$$b_p^+ \subset SS_{\Omega}^P(u).$$

But $b_p^+ \setminus \{p\}$ is contained in $\pi^{-1}(\Omega)$ and

$SS_{\Omega}^P(u) = SS_{\Omega}(u) = SS_M(u)$ above Ω . Thus $b_p^+ \subset SS_{\Omega}(u)$,

which is the desired result.

4.- Diffraction

We keep the notations of §3, but we assume :

$$(4.1) \quad \Omega = \{x \in M ; x_1 > 0\}$$

$$(4.2) \quad \sigma(P) = \zeta_1^2 - g(z, \zeta')$$

where $z = (z_1, z')$, $\zeta = (\zeta_1, \zeta')$.

Moreover we assume :

$$(4.3) \text{ a) } \frac{\partial}{\partial x_1} g < 0 \text{ at } p \quad \text{or} \quad \text{b) } \frac{\partial}{\partial x_1} g \equiv 0 .$$

Theorem 4.1. Under these hypotheses, if $p \in SS_{\Omega}(u)$ then b_p^+
 or b_p^- is contained in $SS_{\Omega}(u)$, in a neighborhood of p .

The idea of the proof is the following.

If $\zeta_1 \neq 0$ at p , the result is a particular case of Theorem 3.1 . Otherwise define for $* = 0, 1, -$:

$$\Omega_* = \{z \in X ; x_1 > 0, y' = 0, y_1 \in \mathbb{R} (* = 0) \\ \text{or } y_1 \geq 0 (* = +) \text{ or } y_1 \leq 0 (* = -)\}$$

Thus $\text{Im } \sigma(P)$ is negative (resp. positive) on $SS(\mathbb{Z}_{\Omega+})$ (resp. $SS(\mathbb{Z}_{\Omega-})$) in a neighborhood of p . Then one can apply Theorem 1.1 to $\mu_{\text{hom}}(\mathbb{Z}_{\Omega*}, \mathcal{O}_X^P)$, $* = +$ or $-$, and one obtains that if $u|_{b_p}$ has compact support, then $u \in H^{n-1}(\mu_{\text{hom}}(\mathbb{Z}_{\Omega_0}, \mathcal{O}_X^P))$ and it is not difficult to conclude using the holomorphic parameter z_1 (cf. [S 2]) .

Remark that Theorem 4.1 has been first obtained by Kataoka [Ka] (under hypothesis (4.3) a)) then refined by G. Lebeau [Le].

An application : Let $(x_1, \dots, x_n)$ be the coordinates on $\mathbb{R}^n$, and let Ω_1 and Ω_2 be two open half spaces. Set $\Omega = \Omega_1 \cup \Omega_2$ and let u be a hyperfunction on Ω . One can easily prove :

$$(4.4) \quad SS_{\Omega}(u) = SS_{\Omega_1}(u) \cup SS_{\Omega_2}(u) .$$

Now assume $\Omega_i = \mathbb{R} \times \Omega'_i$, $(i = 1, 2)$ and u satisfies the wave equation $Pu = 0$, where $P = D_1^2 - \sum_{j=2}^n D_j^2$.

Applying Theorem 4.1 we get that $p \in SS_{\Omega}(u) \implies b_p^+$ or b_p^- is contained in $SS_{\Omega}(u)$, where b_p^+ and b_p^- are the half bicharacteristic curves of $\text{Im } \sigma(P)$.

Problem : to extend this result to the case where

$\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$, and A is any convex closed subset of $\mathbb{R}^{n-1}$.

BIBLIOGRAPHY

- [K-S 1] KASHIWARA, M. and SCHAPIRA, P. : Microlocal study of sheaves. Astérisque 128 (1985).
- [Ka] KATAOKA, K : Microlocal theory of boundary value problems. J. Fac. Sc. Univ. Tokyo Sect. A. I : 27, 355-399 (1980), II : 28, 31-56 (1981).
- [Le] LEBEAU, G. : Deuxième microlocalisation sur les sous-variétés isotropes. Ann. Inst. Fourier, Grenoble 35,2, 145-216 (1985).
- [S 1] SCHAPIRA, P. : Front d'onde analytique au bord II. Sem. E.D.P. Ecole Polytechnique, Exp. 13 (1986).
- [S 2] SCHAPIRA, P. : Microfunctions for boundary value problems. Volume dedicated to Pr. M. Sato. To appear.