

Cofinal types around $\mathcal{P}_\kappa\lambda$ and the tree property for directed sets

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Abstract

Generalizing a result of Todorćević, we prove the existence of directed sets D, E such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering. As an application, we show that the tree property for directed sets introduced by Hinnion is not preserved under products. Most of the results appear in [14].

1 Introduction

Any notion of convergence, described in terms of sequences, nets or filters, involves directed sets, or at least a particular kind of them. In general, directed sets are considered to express the type of convergence. Tukey defined an ordering on the class of all directed sets [17]. This ordering, now called Tukey ordering, was studied by Schmidt [15], Isbell [11],[12], Todorćević [16] and others. In particular, the directed sets of the form $\mathcal{P}_\kappa\lambda$ are of interest, because they possess some nice properties. In section 4 we generalize the directed sets $D(S)$ introduced by Todorćević to $D_\kappa(S)$, where κ is an arbitrary infinite regular cardinal. With these directed sets, we show (Theorem 4.8) that there exist directed sets D, E such that $D \not\geq \mathcal{P}_\kappa\lambda$ and $E \not\geq \mathcal{P}_\kappa\lambda$ but $D \times E \geq \mathcal{P}_\kappa\lambda$ in the Tukey ordering.

The notion of tree property for infinite cardinals (the nonexistence of an Aronszajn tree) is well known, and is related to a large variety of set theoretic statements. The tree property for directed sets was invented by Hinnion [10], and studied by Esser and Hinnion [8],[9]. It is a generalization of the usual tree property for infinite cardinals and especially, for $\mathcal{P}_\kappa\lambda$, it is closely related with the mild ineffability if κ is strongly inaccessible (see Corollary 7.5). By an application of the result mentioned above, we show (Theorem 8.1) that there exist two directed sets D, E for which $\text{add}(D) = \text{add}(E)$ is weakly compact, and both D and E have the tree property but $D \times E$ does not. It was an open problem whether such D, E exist [8].

2 Directed sets and cofinal types

By classifying directed sets into isomorphism types, and further identifying a directed set with its cofinal subset, we arrive at the notion of cofinal type. On the other hand, the same equivalence relation is deduced from a quasi-ordering on the class of all directed sets. First we state the definitions.

Definition 2.1 Let $\langle D, \leq_D \rangle, \langle E, \leq_E \rangle$ be directed sets. A function $f: E \rightarrow D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e [f(e') \geq_D d]$$

is called a *convergent function*. If such a function exists we write $D \leq E$ and say E is *cofinally finer than* D . $\leq$ is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g: D \rightarrow E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D [g(d') \leq_E e \rightarrow d' \leq_D d]$$

is called a *Tukey function*.

If there exists a directed set C into which D and E can be embedded cofinally, we say D is *cofinally similar with* E . In this case we write $D \equiv E$. $\equiv$ is an equivalence relation, and the equivalence classes with respect to $\equiv$ are the *cofinal types*.

The following propositions give the connection between the definitions. For the proofs, consult [16]

Proposition 2.2 For directed sets D and E , the following are equivalent.

- (a) $D \leq E$.
- (b) There exists a Tukey function $g: D \rightarrow E$.
- (c) There exist functions $g: D \rightarrow E$ and $f: E \rightarrow D$ such that

$$\forall d \in D \forall e \in E [g(d) \leq_E e \rightarrow d \leq_D f(e)].$$

Proposition 2.3 For directed sets D and E , the following are equivalent.

- (a) $D \equiv E$.
- (b) $D \leq E$ and $E \leq D$.

So we can regard $\leq$ as an ordering on the class of all cofinal types.

One should always keep in mind the distinction between the unbounded and the cofinal subsets of a directed set.

Proposition 2.4 For directed sets D and E ,

- (i) $f: E \rightarrow D$ is convergent iff $\forall C \subseteq E$ cofinal $[f[C]]$ cofinal.
- (ii) $g: D \rightarrow E$ is Tukey iff $\forall X \subseteq D$ unbounded $[g[X]]$ unbounded.

With two or more directed sets, we can form the product of these, to which we will always give the product ordering.

Proposition 2.5 For directed sets D and E , $D \times E$ is the least upper bound of $\{D, E\}$ in the Tukey ordering.

The next two cardinal functions are the most basic ones, being taken up in various contexts (mostly on a particular kind of directed sets).

Definition 2.6 For a directed set D ,

$$\begin{aligned} \text{add}(D) &\stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\}, \\ \text{cof}(D) &\stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}. \end{aligned}$$

These are the *additivity* and the *cofinality* of a directed set. $\text{add}(D)$ is only well-defined for D without maximum. In the sequel, any statement referring to $\text{add}(D)$ presupposes that D has no maximum.

Proposition 2.7 For a directed set D (without maximum),

$$\aleph_0 \leq \text{add}(D) \leq \text{cof}(D) \leq |D|.$$

Furthermore, $\text{add}(D)$ is regular and $\text{add}(D) \leq \text{cf}(\text{cof}(D))$. Here cf is the cofinality of a cardinal, which is the same as the additivity of it.

Proposition 2.8 For directed sets D and E , $D \leq E$ implies

$$\text{add}(D) \geq \text{add}(E) \quad \text{and} \quad \text{cof}(D) \leq \text{cof}(E).$$

From the above proposition we see that these cardinal functions are invariant under cofinal similarity.

Example 2.9 (see [1, chapter 2]) Let $\mathcal{M}, \mathcal{N}$ be respectively the meager ideal and the null ideal, each ordered by inclusion. $\langle {}^\omega\omega, \leq^* \rangle$ is the eventual dominance order on the reals. We have $\langle {}^\omega\omega, \leq^* \rangle \leq \mathcal{M} \leq \mathcal{N}$ in the Tukey ordering, and thus

$$\aleph_1 \leq \text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \text{cof}(\mathcal{M}) \leq \text{cof}(\mathcal{N}) \leq 2^{\aleph_0}.$$

Proposition 2.10 For directed sets D and E ,

$$\begin{aligned} \text{add}(D \times E) &= \min\{\text{add}(D), \text{add}(E)\}, \\ \text{cof}(D \times E) &= \max\{\text{cof}(D), \text{cof}(E)\}. \end{aligned}$$

3 The width of a directed set

In the following, κ is always an infinite regular cardinal. If P is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for X a subset of P and $a \in P$.

The cofinal type of $\mathcal{P}_\kappa\lambda$ is an interesting topic by itself (see [16]). As usual, $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion.

Lemma 3.1 $\text{add}(\mathcal{P}_\kappa\lambda) = \kappa$, and $\lambda \leq \text{cof}(\mathcal{P}_\kappa\lambda) \leq \lambda^{<\kappa}$. In particular, if κ is strongly inaccessible, then $\text{cof}(\mathcal{P}_\kappa\lambda) = \lambda^{<\kappa}$.

Proof For the last statement, notice that in general for a cofinal $C \subseteq \mathcal{P}_\kappa\lambda$, $\mathcal{P}_\kappa\lambda = \bigcup_{x \in C} \mathcal{P}x$, and thus $\lambda^{<\kappa} \leq 2^{<\kappa} \cdot |C|$. $\square$

Lemma 3.2 For a directed set D , if $\text{add}(D) \geq \kappa$ and $\text{cof}(D) \leq \lambda$, then $D \leq \mathcal{P}_\kappa\lambda$.

It turns out that the following cardinal function, which seems to be a natural one, gives a suitable formulation of Theorem 7.1.

Definition 3.3 The *width* of a directed set D is defined by

$$\text{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a thin subset of D ' means

$$\forall d \in D [|X_{\leq d}| < \text{add}(D)].$$

Example 3.4 Let κ, λ, μ be regular with $\lambda^{<\kappa} = \lambda$ and $\lambda \leq \mu$. Then for the directed set $\mu \times \mathcal{P}_\kappa\lambda$ ordered by

$$\langle \alpha, x \rangle \leq \langle \beta, y \rangle \iff \alpha \leq \beta \wedge x \subseteq y$$

we have

$$\begin{aligned} \text{add}(\mu \times \mathcal{P}_\kappa\lambda) &= \kappa, \\ \text{wid}(\mu \times \mathcal{P}_\kappa\lambda) &= \lambda^+, \\ \text{cof}(\mu \times \mathcal{P}_\kappa\lambda) &= \mu. \end{aligned}$$

The second equation can be verified using Proposition 4.1.

Fix D and put $\kappa := \text{add}(D)$.

Lemma 3.5 For any cardinal $\lambda \geq \kappa$, the following are equivalent.

- (a) D has a thin subset of size λ .
- (b) $D \geq \mathcal{P}_\kappa\lambda$.
- (c) There exists an order-preserving function $f: D \rightarrow \mathcal{P}_\kappa\lambda$ with $f[D]$ cofinal in $\mathcal{P}_\kappa\lambda$.

Proof (a) $\Rightarrow$ (b) Let $X \subseteq D$ be a thin subset of size λ . Define

$$\begin{array}{ccc} f: D & \rightarrow & \mathcal{P}_\kappa X \\ \Psi & & \Psi \\ d & \mapsto & X_{\leq d} \end{array}$$

Then f is (order-preserving and) convergent.

(b) $\Rightarrow$ (c) If $f : D \rightarrow \mathcal{P}_\kappa \lambda$ is convergent, define

$$\begin{array}{ccc} g : D & \rightarrow & \mathcal{P}_\kappa \lambda \\ \cup & & \cup \\ d & \mapsto & \bigcap_{d' \geq d} f(d') \end{array}$$

Then g is convergent and also order-preserving.

(c) $\Rightarrow$ (a) For such g as above, pick for each $\alpha \in \lambda$ a $d_\alpha \in D$ such that $g(d_\alpha) \ni \alpha$, and put $X := \{d_\alpha \mid \alpha \in \lambda\}$. It is readily seen that X is thin. Furthermore $|X| = \lambda$ since $\bigcup_{d \in X} g(d) = \lambda$. $\square$

Corollary 3.6

$$\begin{aligned} \text{wid}(D) &= \sup\{\lambda^+ \mid D \geq \mathcal{P}_\kappa \lambda\} \\ &= \sup\{\lambda^+ \mid \exists f : D \rightarrow \mathcal{P}_\kappa \lambda \text{ order-preserving with } f[D] \text{ cofinal in } \mathcal{P}_\kappa \lambda\}. \end{aligned}$$

The next inequality is checked easily.

Lemma 3.7

$$\text{add}(D)^+ \leq \text{wid}(D) \leq \text{cof}(D)^+.$$

Lemma 3.8 $\text{wid}(D)$ is never singular.

Proof Assume $\lambda := \text{wid}(D) > \text{cf}(\lambda)$ for a directed set D with $\text{add}(D) = \kappa$. Fix a sequence of ordinals $\langle \theta_\alpha \mid \alpha < \text{cf}(\lambda) \rangle$ converging up to λ . Then there are convergent order-preserving mappings $f_\alpha : D \rightarrow \mathcal{P}_\kappa \theta_\alpha$ for all $\alpha < \text{cf}(\lambda)$. Fix also a convergent order-preserving $g : D \rightarrow \mathcal{P}_\kappa \text{cf}(\lambda)$. Consider

$$\begin{array}{ccc} h : D & \rightarrow & \mathcal{P}_\kappa \lambda \\ \cup & & \cup \\ d & \mapsto & \bigcap_{\alpha \in g(d)} f_\alpha(d). \end{array}$$

h is order-preserving and convergent. Hence we have a contradiction. $\square$

However, the next proposition will show that $\text{wid}(D)$ can be a limit cardinal. For example, that for any strongly inaccessible λ there is a directed set D such that $\text{wid}(D) = \lambda$.

Proposition 3.9 Let κ be regular and let λ be strongly κ^+ -inaccessible (i.e. λ is regular and $\forall \mu < \lambda [\mu^\kappa < \lambda]$). Then there exists a directed set D such that $\text{add}(D) = \kappa$ and $\text{wid}(D) = \lambda$.

Proof Consider

$$D = \prod_{\kappa \leq \alpha < \lambda}^{(\kappa^+)} \mathcal{P}_\kappa \alpha.$$

I.e. D is the set of functions f such that $\text{dom}(f) \subseteq \lambda \setminus \kappa$, $|\text{dom}(f)| \leq \kappa$, and for all $\alpha \in \text{dom}(f)$, $f(\alpha) \in \mathcal{P}_\kappa \alpha$. The order is given by

$$f \leq_D g \iff \text{dom}(f) \subseteq \text{dom}(g) \wedge \forall \alpha \in \text{dom}(f) [f(\alpha) \subseteq g(\alpha)].$$

Since $\text{add}(D) = \kappa$ and $\mathcal{P}_\kappa \alpha \leq D$ for each $\alpha \in \lambda \setminus \kappa$, $\text{wid}(D) \geq \lambda$. To show that equality holds, let $\langle f_\alpha \mid \alpha < \lambda \rangle$ be a sequence of distinct elements in D . By the Δ -system lemma there are $d \subseteq \lambda \setminus \kappa$ and $A \subseteq \lambda$ such that $|A| = \lambda$ and $\text{dom}(f_\alpha) \cap \text{dom}(f_\beta) = d$ for distinct $\alpha, \beta \in A$. Then by noting that $|\prod_{\alpha \in d} \mathcal{P}_\kappa \alpha| < \lambda$, there is a $g \in D$ which bounds κ -many f_α 's. $\square$

4 The directed sets $D_\kappa(S)$

One notices at once that if $\text{add}(D) = \text{add}(E)$, then $\text{wid}(D \times E) \geq \max\{\text{wid}(D), \text{wid}(E)\}$. But unlike add and cof , the width of finite products cannot be computed easily. In this section we show that there are directed sets D, E such that $\text{add}(D) = \text{add}(E)$ and $\text{wid}(D \times E) > \max\{\text{wid}(D), \text{wid}(E)\}$.

Before that, we will take a look at the case $\text{add}(D) \neq \text{add}(E)$.

Proposition 4.1 *If $\text{add}(D) < \text{add}(E)$, then $\text{wid}(D \times E) = \text{wid}(D)$.*

This is proved by the next lemma.

Lemma 4.2 *Let $\kappa := \text{add}(D) < \text{add}(E)$. Then*

$$\mathcal{P}_\kappa \lambda \leq D \times E \iff \mathcal{P}_\kappa \lambda \leq D$$

for any cardinal $\lambda \geq \kappa$.

Proof ($\Leftarrow$) Let $X \subseteq D \times E$ be a thin subset of size λ , and let $p : D \times E \rightarrow D$ be the projection. Put $Y := p[X]$. Then Y is thin and $|Y| = \lambda$, since for each $d \in Y$, $|p^{-1}[Y_{\leq d}]| < \kappa$.

($\Rightarrow$) Trivial, using transitivity of $\leq$. $\square$

Now we turn to our main results on cofinal types.

Definition 4.3 Let κ, λ be both regular with $\kappa < \lambda$. We define the following directed set, where the ordering is given by inclusion. For $S \subseteq E_\kappa^\lambda = \{\alpha \in \lambda \mid \text{cf} \alpha = \kappa\}$,

$$D_\kappa(S) \stackrel{\text{def}}{=} \{x \subseteq S \mid |x| \leq \kappa \text{ and } \forall y \subseteq x [\text{otp } y = \kappa \rightarrow \sup y \in x]\}.$$

Here, otp denotes the order type of a set of ordinals.

Todorćević [16] defined and studied these directed sets for $\kappa = \omega$. Note that by letting $S := \{\alpha \in E_\kappa^\lambda \mid \alpha \text{ is not a limit point of elements of } E_\kappa^\lambda\}$, we have $D_\kappa(S) = \mathcal{P}_\kappa S \cong \mathcal{P}_\kappa \lambda$.

The following statements mimic Lemmas 1,2,3 and Theorems 4,6 in [16], but because of the assumption on cardinal arithmetic, they are not full generalizations.

Lemma 4.4 *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+ -inaccessible, and let $S, S' \subseteq E_\kappa^\lambda$ with S unbounded in λ . Then*

$$D_\kappa(S) \leq D_\kappa(S') \quad \text{implies} \quad S' \setminus S \text{ is nonstationary in } \lambda.$$

Proof Let $f: D_\kappa(S) \rightarrow D_\kappa(S')$ be a Tukey function. Without loss of generality, f depends only on its values for singletons, i.e. $f(x) = \bigcup_{\alpha \in x} f(\{\alpha\})$ for all nonempty $x \in D_\kappa(S)$. By the Δ -system lemma we obtain an $A \subseteq S$ of size λ and a $d \subseteq S'$ such that

$$\begin{aligned} & \forall \alpha, \beta \in A [\alpha \neq \beta \rightarrow f(\{\alpha\}) \cap f(\{\beta\}) = d], \\ & \forall \alpha \in A [\min(f(\{\alpha\}) \setminus d) > \sup d], \\ \text{and} \quad & \forall \alpha, \beta \in A [\alpha < \beta \rightarrow \sup(f(\{\alpha\}) \setminus d) < \min(f(\{\beta\}) \setminus d)]. \end{aligned}$$

Next, put

$$C_0 = \{\alpha \in \lambda \mid \text{there exists a strictly increasing sequence } \langle \alpha_\xi \mid \xi < \kappa \rangle \text{ such that} \\ \alpha = \sup\{\alpha_\xi \mid \xi < \kappa\} = \sup \bigcup_{\xi < \kappa} f(\{\alpha_\xi\})\}$$

and let C be the topological closure of C_0 in λ (with respect to the order topology). C_0 is closed for κ -sequences and also unbounded in λ , and thus C becomes a club. For our aim, we demonstrate that $C \cap (S' \setminus S) = \emptyset$. Suppose there were a $\gamma \in C \cap (S' \setminus S)$. Then $\gamma \in C_0$, so fix a sequence $\langle \alpha_\xi \mid \xi < \kappa \rangle$ witnessing it. But $\gamma \in S' \setminus S$ implies that $\{\alpha_\xi \mid \xi < \kappa\}$ is unbounded in $D_\kappa(S)$ and that $\{\gamma\} \cup \bigcup_{\xi < \kappa} f(\{\alpha_\xi\})$ is an upper bound of $\{f(\alpha_\xi) \mid \xi < \kappa\}$ in $D_\kappa(S')$. This contradicts the assumption that f is Tukey. $\square$

Theorem 4.5 *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+ -inaccessible. Denote by $\mathcal{D}(\kappa, \lambda)$ the set of cofinal types with additivity κ and cofinality λ . Then there are 2^λ many pairwise incomparable elements of $\mathcal{D}(\kappa, \lambda)$.*

Proof For $i \in \lambda \times 2$ let $A_i \subseteq E_\kappa^\lambda$ be pairwise disjoint stationary sets. For each $f \in {}^\lambda 2$, put $S_f := \bigcup_{i \in f} A_i$. Now $\langle D_\kappa(S_f) \mid f \in {}^\lambda 2 \rangle$ is a family of pairwise incomparable elements of $\mathcal{D}(\kappa, \lambda)$. $\square$

Lemma 4.6 ([14]) *Let $\omega \leq \kappa < \lambda$, where κ is regular and λ is strongly κ^+ -inaccessible, and let $S, S' \subseteq E_\kappa^\lambda$ be unbounded in λ . Then*

$$D_\kappa(S) \times D_\kappa(S') \geq \mathcal{P}_\kappa \lambda \quad \text{iff} \quad S \cap S' \text{ is nonstationary in } \lambda.$$

Proof ($\Rightarrow$) This is proved by a similar argument as in Lemma 4.4.

($\Leftarrow$) Suppose that $S \cap S'$ is nonstationary in λ . Pick a club $C \subseteq \lambda$ disjoint from $S \cap S'$. For $\xi < \lambda$ pick recursively $\alpha_\xi \in S$ and $\beta_\xi \in S'$ so that for all $\xi < \zeta < \lambda$ there is a $\gamma \in C$ such that

$$\alpha_\xi, \beta_\xi < \gamma < \alpha_\zeta, \beta_\zeta.$$

Consider

$$\begin{array}{ccc} f : \mathcal{P}_\kappa \lambda & \rightarrow & D_\kappa(S) \times D_\kappa(S') \\ \cup & & \cup \\ x & \mapsto & \langle \{\alpha_\xi \mid \xi \in x\}, \{\beta_\xi \mid \xi \in x\} \rangle. \end{array}$$

We show that this function is Tukey. First note that $X \subseteq \mathcal{P}_\kappa \lambda$ is unbounded iff $|\bigcup X| \geq \kappa$. If X is such, then

$$f[X] = \{ \langle \{\alpha_\xi \mid \xi \in x\}, \{\beta_\xi \mid \xi \in x\} \rangle \mid x \in X \}$$

is also unbounded, since there exists a $\gamma \in C$ which is a limit of two strictly increasing κ -sequences consisting of α_ξ ($\xi \in \bigcup X$) and β_ξ ($\xi \in \bigcup X$) respectively. $\square$

Corollary 4.7 ([14]) *Under the same notations and assumptions as above,*

$$D_\kappa(S) \geq \mathcal{P}_\kappa \lambda \quad \text{iff} \quad S \text{ is nonstationary in } \lambda.$$

Proof Just take $S = S'$ in Lemma 4.6. $\square$

Theorem 4.8 ([14]) *Let κ, λ be infinite regular cardinals with $\kappa^+ < \lambda$ and λ strongly κ^+ -inaccessible. Then there exist directed sets D_1 and D_2 such that*

$$D_i \not\geq \mathcal{P}_\kappa \lambda \quad \text{for } i = 1, 2$$

but

$$D_1 \times D_2 \equiv \mathcal{P}_\kappa \lambda.$$

Proof To prove the Theorem, let A be any unbounded nonstationary subset of E_κ^λ . Split $E_\kappa^\lambda \setminus A$ into two disjoint stationary sets S'_1 and S'_2 . Then apply Lemma 4.6 to $D_\kappa(S'_1 \cup A) \times D_\kappa(S'_2 \cup A)$. That $D_i \leq \mathcal{P}_\kappa \lambda$ ($i = 1, 2$) is clear from Lemma 3.2. $\square$

We will call such a pair D_1, D_2 of directed sets a *Tukey decomposition* of $\mathcal{P}_\kappa \lambda$.

Remark 4.9 We note that, in view of Lemma 4.2, the above D_1 and D_2 must satisfy $\text{add}(D_1) = \text{add}(D_2)$. Besides, D_1 and D_2 must have different cofinal types, because $D \times D \equiv D$ for any directed set D . (This follows from Proposition 2.5, or from the fact that the diagonal $\{\langle d, d \rangle \mid d \in D\}$ is cofinal in $D \times D$.)

5 The tree property for directed sets

Definition 5.1 (κ -tree) ([8]) Let D denote a directed set. A triple $\langle T, \leq_T, s \rangle$ is said to be a κ -tree on D if the following holds.

- 1) $\langle T, \leq_T \rangle$ is a partially ordered set.
- 2) $s : T \rightarrow D$ is an order preserving surjection.
- 3) For all $t \in T$, $s \restriction T_{\leq t} : T_{\leq t} \xrightarrow{\sim} D_{\leq s(t)}$ (order isomorphism).

4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level* d of T .

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists! t' \leq_T t [s(t') = d']$.

We write $t \downarrow d$ for this unique t' .

If a κ -tree $\langle T, \leq_T, s \rangle$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t [s(t') = d']$,

then it is called a κ -arbor on D .

If D is an infinite regular cardinal κ , a ' κ -tree on κ ' coincides with the classical ' κ -tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

Definition 5.2 (tree property) ([8]) Let $\langle D, \leq_D \rangle$ be a directed set and $\langle T, \leq_T, s \rangle$ a κ -tree on D . $f: D \rightarrow T$ is said to be a faithful embedding if f is an order embedding and satisfies $s \circ f = \text{id}_D$. If for each κ -tree T on D there is a faithful embedding from D to T , we say that D has the κ -tree property. If D has the $\text{add}(D)$ -tree property, we say simply D has the *tree property*.

We note that in [8] the tree property in our definition is called 'weakly ramifiable', and a κ -arbor is called κ -arborescence.

Classically, κ has the tree property (as a cardinal) if κ carries no Aronszajn tree, which is, in our words, a κ -tree on κ into which there is no faithful embedding.

Proposition 5.3 ([8]) *Let D be directed set and let $\kappa = \text{add}(D)$. D has the tree property iff for any κ -arbor on D there is a faithful embedding into it.*

In [8], Esser and Hinnion posed the question whether the tree property for directed sets with the same additivity is preserved under products. In fact, for the case $\text{add}(D) \neq \text{add}(E)$, a positive result was given.

Proposition 5.4 ([8]) *Let D, E be directed sets and $\text{add}(D) < \text{add}(E)$. If D has the tree property, then $D \times E$ also has the tree property.*

Proof Put $\kappa := \text{add}(D \times E) = \text{add}(D)$. Let $\langle T, \leq_T, s \rangle$ be an arbitrary κ -tree on $D \times E$. We have to find a faithful embedding $f: D \times E \rightarrow T$.

First, for each $d \in D$, $T_d := s^{-1}[\{d\} \times E]$ is a κ -tree on $\{d\} \times E (\cong E)$. Now we have $\kappa < \text{add}(E)$ and hence there exists a faithful embedding into T_d , and moreover the number of faithful embeddings is less than κ (see [8]). Let F_d be the set of all faithful embeddings from $\{d\} \times E$ to T_d , and let $\bar{g}: D_{\leq d} \times E \rightarrow \bigcup_{d' \leq_D d} T'_d$ denote the faithful embedding which is generated by $g \in F_d$. Define

$$\begin{aligned} T_* &:= \bigcup_{d \in D} \{\bar{g} \mid g \in F_d\}, \\ \bar{g} \leq_* \bar{g}' &\iff \bar{g} \subseteq \bar{g}', \\ s_*^{-1}\{d\} &:= \{\bar{g} \mid g \in F_d\} \end{aligned}$$

so that $\langle T_*, \leq_*, s_* \rangle$ becomes a κ -tree on D . Since we are assuming that D has the tree property, we get a faithful embedding $f_*: D \rightarrow T_*$. Define $f(d, e)$ to be $(f_*(d))(e)$, and this completes the proof. $\square$

So we may concentrate on the case $\text{add}(D) = \text{add}(E)$.

The following proposition gives the connection between our problem and the Tukey ordering. It is implicit in [10] but we give a direct proof. This has the advantage that the related statements in [10] can now be obtained as corollaries.

Proposition 5.5 *If E has the tree property, $D \leq E$ in the Tukey ordering and $\text{add}(D) = \text{add}(E)$, then D also has the tree property.*

Proof Let $\kappa := \text{add}(D) = \text{add}(E)$, and let $\langle T, \leq_T, s \rangle$ be an arbitrary κ -arbor on D . We have to construct a corresponding κ -arbor on E .

Fix a pair of functions $g: D \rightarrow E$ and $f: E \rightarrow D$ such that

$$\forall d \in D \forall e \in E [g(d) \leq_E e \rightarrow d \leq_D f(e)]$$

(see Proposition 2.2). Define a κ -arbor $\langle T', \leq', s' \rangle$ on E so that

$$\begin{aligned} s'^{-1}\{e\} &= \left\{ \langle e, T_{\leq t} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \mid t \in s^{-1}\{f(e)\} \right\} \quad \text{for } e \in E, \\ \text{and} \quad \langle e_1, A \rangle \leq' \langle e_2, B \rangle &\iff e_1 \leq_E e_2 \wedge A \subseteq B \quad \text{for } \langle e_1, A \rangle, \langle e_2, B \rangle \in T'. \end{aligned}$$

We check that $T' = \bigcup_{e \in E} s'^{-1}\{e\}$ is actually a κ -arbor on E . It is straightforward that $\leq'$ is transitive, that s' is order preserving, and that each level has size less than κ . To prove the upwards access property, fix $e_0, e \in E$ with $e_0 \leq_E e$ and $t_0 \in s^{-1}\{f(e_0)\}$ arbitrarily. Take some upper bound of $\{f(e_0), f(e)\}$ in D , say d^* . By the upwards access property of T , there is some $t^* \in s^{-1}\{d^*\}$ with $t^* \geq_T t_0$. Then by the downwards uniqueness property of T ,

$$\langle e_0, T_{\leq t_0} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle \leq' \langle e, T_{\leq t^*} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \in s'^{-1}\{e\}.$$

To prove downwards uniqueness, fix $e_0 \leq_E e$ and $t \in s^{-1}\{f(e)\}$ arbitrarily. Take an upper bound d^* of $\{f(e_0), f(e)\}$ in D . By the upwards access property of T , we have a $t^* \in s^{-1}\{d^*\}$ with $t^* \geq_T t$. Put $t_0 := t^* \downarrow f(e_0)$. Then

$$\begin{aligned} \langle e_0, T_{\leq t_0} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle &= \langle e_0, T_{\leq t^*} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle \\ &= \langle e_0, T_{\leq t} \cap s^{-1}g^{-1}[E_{\leq e_0}] \rangle = \langle e, T_{\leq t} \cap s^{-1}g^{-1}[E_{\leq e}] \rangle \downarrow e_0. \end{aligned}$$

By assumption, there exists a faithful embedding $\varphi: E \rightarrow T'$. From it we can deduce a faithful embedding from D into T , by choosing the image to be exactly $\bigcup \{A \mid \exists e \in E [\langle e, A \rangle = \varphi(e)]\}$. $\square$

Thus the tree property is a property applying to the cofinal type of a directed set.

Remark 5.6 We note that this proposition does not hold if $\text{add}(D) \neq \text{add}(E)$. $D = \omega_1$ and $E = \mathcal{P}_\omega(\omega_1)$ is a counterexample.

Corollary 5.7 ([8]) *If D has the tree property, then $\text{add}(D)$ has the tree property in the classical sense.*

By Hechler's theorem (see [4]), the eventual dominance order on the reals $\langle {}^\omega\omega, \leq^* \rangle$ can be consistently cofinally similar with any directed set which has $\text{add}(D) \geq \aleph_1$. Hence to obtain the following result we apply Hechler's theorem by taking $\langle D, \leq_D \rangle = \langle \kappa, \in \rangle$. For (1), we let $\kappa = \omega_1$, and for (2), we let κ be weakly compact.

Theorem 5.8

- (1) ZFC and ZFC + " $\langle {}^\omega\omega, \leq^* \rangle$ does not have the tree property" are equiconsistent.
- (2) ZFC + $\exists$ weakly compact and ZFC + " $\langle {}^\omega\omega, \leq^* \rangle$ has the tree property" are equiconsistent.

Since Hechler's theorem holds with $\langle {}^\omega\omega, \leq^* \rangle$ replaced by $\mathcal{M}$ [2] or $\mathcal{N}$ [5], we have analogous results for $\mathcal{M}$ and $\mathcal{N}$.

6 Mild ineffability

Mild ineffability was introduced by DiPrisco and Zwicker, and studied by Carr [6] in detail. It can be viewed as a kind of tree property for $\mathcal{P}_\kappa\lambda$. We give the definition and an overview on the basic facts. In all the statements of section 6 and 7, the possibility of taking $\kappa = \omega$ is not excluded.

Definition 6.1 (mild ineffability) ([6]) $\mathcal{P}_\kappa\lambda$ is said to be *mildly ineffable* (or κ is *mildly λ -ineffable*) iff for any given $\langle A_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ with $A_x \subseteq x$ for all x , there exists some $A \subseteq \lambda$ such that

$$\forall x \in \mathcal{P}_\kappa\lambda \exists y \in \mathcal{P}_\kappa\lambda [x \subseteq y \wedge A_y \cap x = A \cap x].$$

Proposition 6.2 ([6]) *For a cardinal κ , the following are equivalent:*

- (a) κ is mildly κ -ineffable.
- (b) κ is strongly inaccessible and has the tree property.
- (c) κ is weakly compact.

Proposition 6.3 ([6]) *If κ is mildly λ -ineffable and $\kappa \leq \lambda' \leq \lambda$, then κ is mildly λ' -ineffable.*

The relation between mild ineffability and strong compactness for pairs of cardinals κ, λ is as follows.

Proposition 6.4 ([6]) *For cardinals $\kappa \leq \lambda$,*

- (1) *If κ is mildly $2^{\lambda^{<\kappa}}$ -ineffable then κ is λ -strongly compact.*
- (2) *If κ is λ -strongly compact then κ is mildly λ -ineffable.*

Proof (1) Let $\mathcal{P}(\mathcal{P}_\kappa\lambda) = \{X_\alpha \mid \alpha < 2^{\lambda^{<\kappa}}\}$. For each $x \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$, we put

$$A_x := \{\alpha \in x \mid x \cap \lambda \in X_\alpha\}.$$

By the mild $2^{\lambda^{<\kappa}}$ -ineffability of κ , there exists an $A \subseteq 2^{\lambda^{<\kappa}}$ such that

$$\forall x \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}}) \exists y \in \mathcal{P}_\kappa(2^{\lambda^{<\kappa}}) [x \subseteq y \wedge A_y \cap x = A \cap x].$$

If we let $\mathcal{U} := \{X_\alpha \mid \alpha \in A\}$, then one can check (by applying the above formula to suitable x 's) that $\mathcal{U}$ is a κ -complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$.

(2) Assume that there exists a κ -complete fine ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$. We are given $\langle A_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ such that $A_x \subseteq x$ for all x . For each $\alpha < \lambda$, put $X_\alpha := \{x \in \mathcal{P}_\kappa\lambda \mid \alpha \in A_x\}$. Let $A := \{\alpha < \lambda \mid X_\alpha \in \mathcal{U}\}$. We check that this is the required set. Let $x \in \mathcal{P}_\kappa\lambda$ be arbitrary. Then $X_\alpha \in \mathcal{U}$ for $\alpha \in x \cap A$, and $\mathcal{P}_\kappa\lambda \setminus X_\alpha \in \mathcal{U}$ for $\alpha \in x \setminus A$. Put

$$X := \bigcap \{X_\alpha \mid \alpha \in x \cap A\} \cap \bigcap \{\mathcal{P}_\kappa\lambda \setminus X_\alpha \mid \alpha \in x \setminus A\} \in \mathcal{U}.$$

X is cofinal in $\mathcal{P}_\kappa\lambda$ since $\mathcal{U}$ is fine, so we can pick $y \in X$ with $y \supseteq x$, and thus $A_y \cap x = A \cap x$. $\square$

Corollary 6.5 (GCH) *Assume κ is not strongly compact. Let λ be the least cardinal such that κ is not λ -strongly compact, and let μ be the least cardinal such that κ is not mildly μ -ineffable. Assume that λ is regular. Then $\mu = \lambda$ or $\mu = \lambda^+$.*

Corollary 6.6 ([6]) *For a cardinal κ , κ is mildly λ -ineffable for all $\lambda \geq \kappa$ iff κ is strongly compact.*

7 Characterization of the tree property by mild ineffability

The next theorem is stated in [9, Theorem 3.3] with a different formulation. Using the cardinal width, we can state the theorem in a more convenient way.

Theorem 7.1 ([14], cf [9]) *Let D be a directed set and let $\kappa := \text{add}(D)$ be strongly inaccessible. The following are equivalent:*

- (a) D has the tree property.
- (b) For all $\lambda < \text{wid}(D)$, $\mathcal{P}_\kappa\lambda$ has the tree property.
- (c) For all $\lambda < \text{wid}(D)$, $\mathcal{P}_\kappa\lambda$ is mildly ineffable.

The proof we give here is a combination of the proofs in [14] and [7]. It enabled a good deal of simplification.

Definition 7.2 ([7]) Let $\langle T, \leq_T, s \rangle$ be an arbor on a directed set D . We define an equivalence relation on D . For $d_1, d_2 \in D$,

$$d_1 \sim d_2 \iff \forall d' \in D [d' \geq d_1, d_2 \rightarrow \forall t_1 \in s^{-1}\{d_1\} \exists! t_2 \in s^{-1}\{d_2\} \forall u \in s^{-1}\{d'\} [t_1 \leq_T u \iff t_2 \leq_T u]].$$

In the above formula, we say that the $t_1 \in s^{-1}\{d_1\}$ and the corresponding $t_2 \in s^{-1}\{d_2\}$ are *linked*. Equivalent levels give the same partial information on how to take the faithful embedding. Notice that $d_1 \sim d_2$ does not imply that they are comparable.

Lemma 7.3 For the relation defined above,

$$d_1 \sim d_2 \iff \exists d' \in D [d' \geq d_1, d_2 \wedge \forall t_1 \in s^{-1}\{d_1\} \exists! t_2 \in s^{-1}\{d_2\} \forall u \in s^{-1}\{d'\} [t_1 \leq_T u \iff t_2 \leq_T u]].$$

Thus $\sim$ is in fact an equivalence relation on D .

Proof $(\Rightarrow)$ Trivial, since D is directed.

$(\Leftarrow)$ Use upwards access and downwards uniqueness. □

Lemma 7.4 Assume that $\kappa := \text{add}(D)$ is strongly inaccessible, and let $\langle T, \leq_T, s \rangle$ be a κ -arbor on D . If $F \subseteq D$ is a set of representatives with respect to $\sim$, then F is thin.

Proof Fix an arbitrary $d_0 \in D$. For each element $d \in D_{\leq d_0}$,

$$P_d := \{ \{u \in s^{-1}\{d_0\} \mid u \geq_T t\} \mid t \in s^{-1}\{d\} \}$$

provides a partition of $s^{-1}\{d_0\}$. By Lemma 7.3, we see that $P_{d_1} = P_{d_2}$ iff $d_1 \sim d_2$ for $d_1, d_2 \in D_{\leq d_0}$. Since κ is strongly inaccessible, the number of partitions of $s^{-1}\{d_0\}$ is less than κ . □

Proof of 7.1 (a) $\Rightarrow$ (b) Let $\text{add}(D) \leq \lambda < \text{wid}(D)$. Then $\mathcal{P}_\kappa \lambda \leq D$, so by Proposition 5.5 $\mathcal{P}_\kappa \lambda$ has the tree property.

(b) $\Rightarrow$ (c) It suffices to show, for an arbitrary λ , that the tree property for $\mathcal{P}_\kappa \lambda$ implies its mild ineffability. Assume that $\mathcal{P}_\kappa \lambda$ has the tree property. Suppose we are given a family $\langle A_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ such that $A_x \in {}^x 2$ for $x \in \mathcal{P}_\kappa \lambda$. Then

$$\langle \{A_y \restriction x \mid y \supseteq x\} \mid x \in \mathcal{P}_\kappa \lambda \rangle$$

is a κ -tree on $\mathcal{P}_\kappa \lambda$ since κ is strongly inaccessible. Therefore we have a faithful embedding, which is the same as an $A \in {}^\lambda 2$ such that

$$\forall x \in \mathcal{P}_\kappa \lambda \exists y \in \mathcal{P}_\kappa \lambda [x \subseteq y \wedge A_y \restriction x = A \restriction x].$$

(c) $\Rightarrow$ (a) Let $\langle T, \leq_T, s \rangle$ be a κ -arbor on D . Our goal is to produce a faithful embedding $f: D \rightarrow T$. Fix a set of representatives $F \subseteq D$ with respect to the equivalence $\sim$ defined above.

Put $\lambda := |F|$. Then $T^* := s^{-1}[F]$ also has size λ . As we have $\lambda < \text{wid}(D)$, the assumption (c) says κ is mildly λ -ineffable.

We define a family $\langle A_x \mid x \in \mathcal{P}_\kappa T^* \rangle$ to which we will apply the mild ineffability. For each $x \in \mathcal{P}_\kappa T^*$, pick an upper bound $d \in D$ of $s[x]$, and fix $t \in s^{-1}\{d\}$. For $v \in x$ we put $A_x(v) = 1$ if $v \leq_T t$, and $A_x(v) = 0$ otherwise. Then we get an $A \in {}^{T^*}2$ such that

$$\forall x \in \mathcal{P}_\kappa T^* \exists y \in \mathcal{P}_\kappa T^* [x \subseteq y \wedge A_y \restriction x = A \restriction x].$$

It remains to derive the faithful embedding f from A . For each $d \in F$, let v_d be the unique $v \in s^{-1}\{d\}$ such that $A(v) = 1$. Then $d \mapsto v_d$ is an embedding from F to T^* . To extend this map to all of D , let $d \in D$ be arbitrary and let $d \sim d^* \in F$ be the corresponding representative. Now v_{d^*} is defined, and we can put $f(d)$ to be the unique $u \in s^{-1}\{d\}$ such that u and v_{d^*} are linked. One can verify that $f: D \rightarrow T$ is a faithful embedding. $\square$

Corollary 7.5 *Let κ be strongly inaccessible and $\lambda \geq \kappa$. Then*

$\mathcal{P}_\kappa \lambda$ has the tree property iff κ is mildly $\lambda^{<\kappa}$ -ineffable.

8 Application of the Tukey decomposition

Theorem 8.1 *Assume that κ is weakly compact but not strongly compact, and that $\lambda > \kappa^+$ is the least cardinal such that κ is not mildly λ -ineffable. Assume further that λ is strongly κ^+ -inaccessible. Then there exist directed sets D_1 and D_2 with $\text{add}(D_1) = \text{add}(D_2) = \kappa$ such that*

D_1 and D_2 have the tree property

but

$D_1 \times D_2$ does not have the tree property.

Proof By the Theorem 4.8, we have directed sets D_1 and D_2 such that $D_i \not\leq \mathcal{P}_\kappa \lambda$ for $i = 1, 2$ but $D_1 \times D_2 \equiv \mathcal{P}_\kappa \lambda$. Recalling how D_1 and D_2 were defined (or by Remark 4.9), we see that $\text{add}(D_1) = \text{add}(D_2) = \kappa$. By Theorem 7.1, D_1 and D_2 have the tree property but $D_1 \times D_2$ does not have the tree property. $\square$

At last, we discuss the consistency of the assumption in the above theorem.

We quote the following theorem.

Theorem 8.2 ([13]) *If λ is regular and κ is mildly λ -ineffable, then for each regular $\eta < \kappa$, any stationary set $S \subseteq E_\eta^\lambda$ is reflecting.*

Here we call $S \subseteq E_\eta^\lambda$ reflecting iff there is a limit ordinal $\gamma < \lambda$ such that $S \cap \gamma$ is stationary in γ . Otherwise S is called nonreflecting.

Assuming a strongly compact cardinal κ , we perform a forcing which destroys the mild λ^+ -ineffability of κ and which at the same time preserves the mild λ -ineffability. By Theorem 8.2 the standard forcing which adds a nonreflecting stationary subset (see [3, Definition 4.14]) serves our purpose. To be precise, define P to be the forcing which consists of conditions $p \in {}^{<\lambda^+}2$ (i.e. p is a characteristic function for a subset of an ordinal $< \lambda^+$) such that if we let $S_p := p^{-1}\{1\}$, then $S_p \subseteq E_\omega^{\lambda^+}$ and for all limit ordinals $\gamma < \lambda^+$, $S_p \cap \gamma$ is nonstationary in γ . For $p, q \in P$, p extends q iff $p \supseteq q$. It is known [3] that P preserves cardinals, cofinalities, and GCH, and that P is λ -strategically closed.

This completes the proof.

Theorem 8.3 *If we assume the consistency of $\text{ZFC} + \exists$ strongly compact, then $\text{ZFC} +$ "the tree property for directed sets is not always preserved under products" is consistent.*

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References

- [1] T.Bartoszyński and H.Judah, *Set Theory On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts, 1995.
- [2] T.Bartoszyński and M.Kada, *Hechler's theorem for the meager ideal*, to appear in Topology Appl. <http://front.math.ucdavis.edu/math.LO/0209086>
- [3] J.Burgess, *Forcing*, in: J. Barwise, ed., Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977, 403–452.
- [4] M.R.Burke, *A proof of Hechler's theorem on embedding $\aleph_1$ -directed sets cofinally into $(\omega^\omega, <^*)$* , Arch. Math. Logic. (6) **36** (1997), 399–403.
- [5] M.R.Burke, M.Kada, *Hechler's theorem for the null ideal*, Arch. Math. Logic (5) **43** (2004), 703–722.
- [6] D.M.Carr, *$P_{\aleph_1}$ -Generalizations of Weak Compactness*, Zeitschr. f. math. Logik und Grundlagen d. Math. **31** (1985), 393–401.
- [7] O.Esser, unpublished manuscript.
- [8] O.Esser and R.Hinnion, *Large Cardinals and Ramifiability for Directed Sets*, Math. Log. Quart. (1) **46** (2000), 25–34.
- [9] O.Esser and R.Hinnion, *Tree Properties for Orders : a survey*, A tribute to Maurice Boffa. Bull. Belg. Math. Soc. Simon Stevin (2001) suppl., 133–144.

- [10] R.Hinnion, *Ramifiable Directed Sets*, Math. Log. Quart. (1) **44** (1998), 216–228.
- [11] J.R.Isbell, *The Category of cofinal types II*, Trans. Amer. Math. Soc. **116** (1965), 394–416.
- [12] J.R.Isbell, *Seven cofinal types*, J. London Math. Soc. (2) **4** (1972), 651–654.
- [13] C.A.Johnson, *On Ideals and Stationary Reflection*, J. Symbolic Logic (2) **54** (1989), 568–575.
- [14] M.Karato, *A Tukey decomposition of $\mathcal{P}_\kappa\lambda$ and the tree property for directed sets*, to appear in Math. Log. Quart.
- [15] J.Schmidt, *Konfinalität*, Z. Math. Logik Grundlag. Math. **1** (1955), 271–303.
- [16] S.Todorčević, *Directed Sets and Cofinal Types*, Trans. Amer. Math. Soc. (2) **290** (1985), 711–723.
- [17] J.W.Tukey, *Convergence and uniformity in topology*, Ann. of Math. Studies, no.2, Princeton Univ. Press, 1940.