Relative $\mathbb{A}^1$-homotopy and its applications

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What's $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$- homology?

Let k be a field. $\mathbb{A}^1$-homotopy theory is a homotopy theory of smooth schemes over k introduced by Morel-Voevodsky [MV]. In this theory, the affine line $\mathbb{A}^1$ plays the role of unit interval $[0,1]$ in the ordinary homotopy theory. An analogue of homotopy groups in the $\mathbb{A}^1$-homotopy theory, called $\mathbb{A}^1$-homotopy sheaf, is defined as a sheaf on the Nisnevich topology on the category of smooth schemes over k, where the Ninsnevich topology is a Grothendieck topology stronger than Zariski and weaker than étale. In Morel's book [Mo], he introduced an $\mathbb{A}^1$-version of the homology groups, called $\mathbb{A}^1$-homology sheaves, as Nisnevich sheaves, and prove an analogue of the Hurewicz theorem relates $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology sheaves.

Ordinary homotopy	$\mathbb{A}^1$-homotopy
Topological spaces	Smooth schemes
Unit interval [0,1]	Affine line $\mathbb{A}^1$
Homotopy group $\pi_i(X)$	$\mathbb{A}^1$-homotopy sheaf $\pi_i^{\mathbb{A}^1}(X)$
Homology group $H_i(X)$	$\mathbb{A}^1$-homology sheaf $\mathbf{H}_i^{\mathbb{A}^1}(X)$

$\mathbb{A}^1$-Whitehead theorem

Whitehead theorem on homology groups states that a continuous map of simply connected CW-complexes which induces isomorphisms for all homology groups is a homotopy equivalence. Our first result in [Sh] is an analogue of this result with a degree bound. A smooth k-scheme is called $\mathbb{A}^1$-n-connected, if the $\mathbb{A}^1$-homotopy sheaves are trivial (*i.e.* isomorphic to the constant sheaf of a point) in degree $\leq n$. Our theorem as follows.

Theorem 1 ([Sh]). Let $f: X \to Y$ be a morphism of smooth schemes over a perfect field, and we write

$$d = \max\{\dim X + 1, \dim Y\}.$$

Suppose that X and Y are $\mathbb{A}^1$-1-connected and that f induces an isomorphism

$$\mathbf{H}_i^{\mathbb{A}^1}(X) \xrightarrow{\cong} \mathbf{H}_i^{\mathbb{A}^1}(Y)$$

for all $2 \leq i < d$ and an epimorphism

$$\mathbf{H}_d^{\mathbb{A}^1}(X) \twoheadrightarrow \mathbf{H}_d^{\mathbb{A}^1}(Y).$$

Then the morphism f is a homotopy equivalence in the sense of $\mathbb{A}^1$-homotopy theory.

$\mathbb{A}^1$-excision theorem

Our secnd theorem is excision results for $\mathbb{A}^1$-homology and $\mathbb{A}^1$-homotopy sheaves. That is, our theorem says that Zariski open embeddings induce isomorphisms for the $\mathbb{A}^1$-homology and the $\mathbb{A}^1$-homotopy sheaves in low degrees. The precise statement as follows.

Theorem 2 ([Sh]). Let X be a smooth scheme over a perfect field and $U \subseteq X$ be an open set whose complement has codimension r.
(1) The induced morphism

$$\mathbf{H}_i^{\mathbb{A}^1}(U) \to \mathbf{H}_i^{\mathbb{A}^1}(X)$$

is an isomorphism for every $0 \leq i < r-1$ and an epimorphism for $i = r-1$.
(2) If X and U are $\mathbb{A}^1$-1-connected, then the induced morphism

$$\pi_i^{\mathbb{A}^1}(U) \to \pi_i^{\mathbb{A}^1}(X)$$

is an isomorphism for every $0 \leq i < r-1$ and an epimorphism for $i = r-1$.

Relative $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$ -homology

For proving Theorems 1 and 2, we introduce a relative version of $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology sheaves which satisfice an analogue of relative Hurewicz theorem of topological spaces. By using the language of relative $\mathbb{A}^1$- theory, our two theorems are described by the vanishing of the relative $\mathbb{A}^1$-homotopy and the relative $\mathbb{A}^1$-homology. Moreover, we compute the relative $\mathbb{A}^1$-homology of the hyperplane embedding $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.

Theorem 3 ([Sh]). When the base field is perfect, we have

$$\mathbf{H}_i^{\mathbb{A}^1}(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong \begin{cases} \mathbf{K}_n^{MW} & (i = n) \\ 0 & (i < n) \end{cases}$$

for all $n > 0$ and $0 \leq i \leq n$, where $\mathbf{K}_n^{MW}$ is the sheaf of the unramified Milnor-Witt K-theory introduced by Morel [Mo].

References

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