

ON $\tilde{H}$ -COBORDISMS BETWEEN
THREE DIMENSIONAL HOMOLOGY HANDLES

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The present note will introduce a cobordism theory, called $\tilde{H}$ -cobordism, to the class of 3-dimensional homology oriented handles and to the class of 3-dimensional homology non-orientable handles. These classes modulo $\tilde{H}$ -cobordism relations will form groups $\Omega(S^1 \times S^2)$, $\Omega(S^1 \times_{\tau} S^2)$, called $\tilde{H}$ -cobordism groups, respectively.

We will discuss about the properties of the invariants of $\Omega(S^1 \times S^2)$ and $\Omega(S^1 \times_{\tau} S^2)$. Then we will know that $\Omega(S^1 \times S^2)$ is so related to the Fox-Milnor's classical knot cobordism group C^1 and the Levine's matrix cobordism group G_- , and that $\Omega(S^1 \times_{\tau} S^2)$ is isomorphic to the direct sum of infinite countable copies of the cyclic group of order 2.

Section 1 will construct the oriented $\tilde{H}$ -cobordism group $\Omega(S^1 \times S^2)$. In Section 2, we will discuss about the properties of the invariants of $\Omega(S^1 \times S^2)$ and compare $\Omega(S^1 \times S^2)$ with the Fox-Milnor's knot cobordism group C^1 and with the Levine's matrix cobordism group G_- . Section 3 will describe the non-orientable $\tilde{H}$ -cobordism group $\Omega(S^1 \times_{\tau} S^2)$ and determine its group structure. Section 4 will propose further discussions and questions.

Throughout this note, the space will be considered in the

piecewise-linear category.

§ 1

A CONSTRUCTION OF THE ORIENTED $\tilde{H}$ -COBORDISM GROUP $\Omega(S^1 \times S^2)$

A 3-dimensional homology orientable handle M is a compact 3-manifold having the homology of the orientable handle $S^1 \times S^2$: $H_*(M; \mathbb{Z}) \approx H_*(S^1 \times S^2; \mathbb{Z})$. A homology orientable handle M is oriented if one generator of $H_3(M; \mathbb{Z})$ is distinguished. The class of all homology oriented handles is denoted by $\mathcal{O}(S^1 \times S^2)$. If M is in $\mathcal{O}(S^1 \times S^2)$, then $-M$, which is the same manifold as M but has the opposite orientation, also lies in $\mathcal{O}(S^1 \times S^2)$.

1.1 DEFINITION. Two homology oriented handles M_0, M_1 are $\tilde{H}$ -cobordant and denoted by $M_0 \sim M_1$, if there exists a compact connected oriented 4-manifold W with the boundary ∂W being the disjoint union $M_0 \cup (-M_1)$ and such that there is an infinite cyclic connected covering $(\tilde{W}; \tilde{M}_0, \tilde{M}_1) \longrightarrow (W; M_0, M_1)$ of the triad $(W; M_0, M_1)$ with $H_*(\tilde{W}; \mathbb{Q})$ being finitely generated over $\mathbb{Q}$.

As usual, the triad $(W; M_0, M_1)$ is called an $\tilde{H}$ -cobordism.

For any $M \in \mathcal{O}(S^1 \times S^2)$, note that $H_*(\tilde{M}; \mathbb{Q})$ is finitely generated over $\mathbb{Q}$. Then the use of the Mayer-Vietoris sequence clearly yields the following.

1.2 LEMMA. The $\tilde{H}$ -cobordism relation $\sim$ is an equivalence relation.

If $M \sim S^1 \times S^2$ then we write $M \sim 0$. Note that $M \sim 0$ if

and only if there exists a compact connected oriented 4-manifold W with the boundary ∂W being M and such that there is an infinite cyclic connected covering $(\tilde{W}, \tilde{M}) \rightarrow (W, M)$ of the pair (W, M) with $H_*(\tilde{W}; Q)$ being finitely generated over Q . In this case the notation $(W; M, \emptyset)$ may be adapted.

1.3 DEFINITION. The set $\Omega(S^1 \times S^2)$ is defined to be the set of $\vec{\mathcal{C}}(S^1 \times S^2)$ modulo the $\tilde{H}$ -cobordism relation.

For any $M \in \vec{\mathcal{C}}(S^1 \times S^2)$ $[M]$ denotes the element of $\Omega(S^1 \times S^2)$ having M as the representative.

To show that the set $\Omega(S^1 \times S^2)$ forms a non-trivial abelian group, we introduce a sum operation, called a circle union.

Let $M_0, M_1 \in \vec{\mathcal{C}}(S^1 \times S^2)$ and choose polyhedral simple closed curves $\omega_0 \subset M_0, \omega_1 \subset M_1$ which represent generators of $H_1(M_0; \mathbb{Z}), H_1(M_1; \mathbb{Z})$, respectively. Then there exist closed connected orientable surfaces $F_0 \subset M_0, F_1 \subset M_1$ such that $F_0 \cap \omega_0, F_1 \cap \omega_1$ consist of single points, respectively. [To see this, first note that the identity map $\omega_0 \subset \omega_0$ can be extended to a piecewise-linear map $f_0: M_0 \rightarrow \omega_0$ by means of the elementary obstruction theory. Second, note that there is a point $p_0 \in \omega_0$ such that the preimage $f_0^{-1}(p_0)$ is a closed (not necessarily connected) orientable surface. Now choose the component of $f_0^{-1}(p_0)$ containing p_0 as F_0 . Similarly, the desired F_1 exists.]

Consider the solid torus $S^1 \times B^2$ and piecewise-linear embeddings

$$\begin{aligned} h_0: S^1 \times B^2 \times 0 &\rightarrow M_0 \\ h_1: S^1 \times B^2 \times 1 &\rightarrow M_1 \end{aligned}$$

such that

(1) there exist points $s \in S^1, b \in \text{Int} B^2$ with $h_0(s \times B^2 \times 0) \subset F_0$, $h_0(S^1 \times b \times 0) = \omega_0$, $h_1(s \times B^2 \times 1) \subset F_1$ and $h_1(S^1 \times b \times 1) = \omega_1$,

(2) the orientations of $S^1 \times B^2 \times 0$ and $S^1 \times B^2 \times 1$ are induced from some orientation of $S^1 \times B^2 \times [0, 1]$ and h_0 is orientation-preserving and h_1 is orientation-reversing.

1.4 DEFINITION. The oriented manifold

$$M_0 \circ M_1 = M_0 \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1 - S^1 \times \text{Int} B^2 \times [0, 1]$$

is called a circle union of M_0 and M_1 .

From construction, easily we have $M_0 \circ M_1 \in \mathcal{C}(S^1 \times S^2)$.

1.5 REMARK. In general, the homeomorphism type of $M_0 \circ M_1$ depends upon the choices of ω_0 and ω_1 . For example let $\omega \subset S^1 \times S^2$ be a simple closed curve of geometrical index 1 and $T(\omega)$ be the tubular neighborhood of ω in $S^1 \times S^2$. If the circle union $S^1 \times S^2 \circ S^1 \times S^2$ is defined to be the double of $\text{cl}(S^1 \times S^2 - T(\omega))$, then $S^1 \times S^2 \circ S^1 \times S^2$ is clearly piecewise-linearly homeomorphic to $S^1 \times S^2$.

On the other hand, consider for example a simple closed curve

$\omega' \subset S^1 \times S^2$ of geometrical index 3 and algebraic index 1 (See figure 1.) and let $T(\omega')$ be the tubular neighborhood of ω' in $S^1 \times S^2$. If the circle union $S^1 \times S^2 \circ' S^1 \times S^2$ is defined to be the double of $\text{cl}(S^1 \times S^2 - T(\omega'))$, then $S^1 \times S^2 \circ' S^1 \times S^2$ is not homeomorphic to $S^1 \times S^2 \approx S^1 \times S^2 \circ S^1 \times S^2$, because the natural inclusion $\partial T(\omega') \rightarrow S^1 \times S^2 \circ' S^1 \times S^2$ induces the monomorphism $\pi_1(\partial T(\omega')) \rightarrow \pi_1(S^1 \times S^2 \circ' S^1 \times S^2)$ by the loop theorem and hence $\pi_1(S^1 \times S^2 \circ' S^1 \times S^2) \neq \mathbb{Z}$.

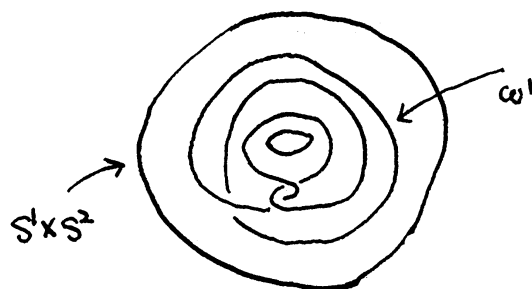


figure 1.

In spite of REMARK 1.5, for arbitrary two circle unions $M_0 \circ M_1, M_0 \circ' M_1$ we have the following.

1.6 LEMMA. $M_0 \circ M_1 \sim M_0 \circ' M_1$.

Proof. Let $M_0 \circ M_1 = M_0 \times 0 \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1 \times 0 - S^1 \times \text{Int} B^2 \times [0, 1]$ and $-(M_0 \circ' M_1) = M_0 \times 1 \cup_{h'_0} S^1 \times B^2 \times [0, 1] \cup_{h'_1} M_1 \times 1 - S^1 \times \text{Int} B^2 \times [0, 1]$ and

$$W = M_0 \times [0, 1] \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1 \times [0, 1] \cup_{h'_0} S^1 \times B^2 \times [0, 1] \cup_{h'_1} M_1 \times [0, 1] \quad (\text{See figure 2.}).$$

Clearly we have $\partial W = M_0 \circ M_1 \cup -(M_0 \circ' M_1)$. Further, the infinite cyclic connected covering $\widetilde{M_0 \circ M_1} \rightarrow M_0 \circ M_1$ can be easily extended to an infinite cyclic covering $\widetilde{W} \rightarrow W$. From construction the restriction to $M_0 \circ' M_1$ gives the infinite cyclic connected covering $-\widetilde{M_0 \circ' M_1} \rightarrow -(M_0 \circ' M_1)$. Using the Mayer-Vietoris sequence we obtain that $H_*(\widetilde{W}; \mathbb{Q})$ is finitely generated over $\mathbb{Q}$. Thus, the triad $(W; M_0 \circ M_1, M_0 \circ' M_1)$ gives an $\widetilde{H}$ -cobordism and the proof is completed.

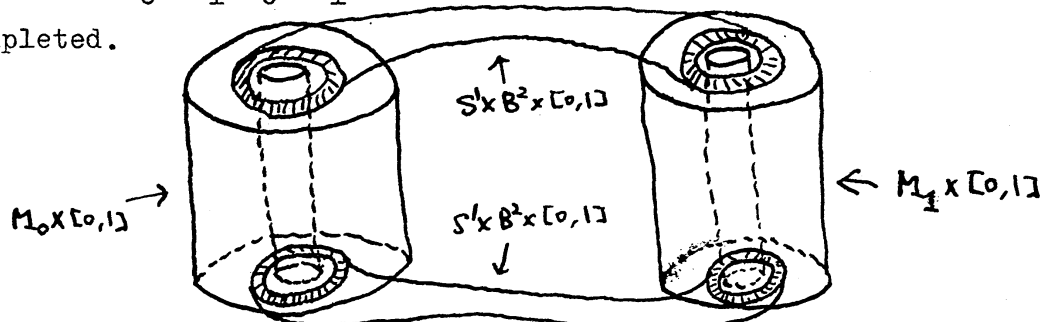


figure 2.

1.7 LEMMA. $M_0 \sim M_1$ is equivalent to $M_0 \circ M_1 \sim 0$.

Proof. Assume $M_0 \sim M_1$. Then there is a compact connected oriented 4-manifold W with $\partial W = M_0 \cup -M_1$ and such that for some infinite cyclic connected covering $(\tilde{W}; \tilde{M}_0, \tilde{M}_1) \rightarrow (W; M_0, M_1)$ $H_*(\tilde{W}; Q)$ is finitely generated over Q . Let $M_0 \circ M_1 = M_0 \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} (-M_1) - S^1 \times \text{Int} B^2 \times [0, 1]$ and $W' = W \cup_{h_0, h_1} S^1 \times B^2 \times [0, 1]$ (See figure 3.). Clearly, $\partial W' = M_0 \circ M_1$ and the infinite cyclic covering $\tilde{W} \rightarrow W$ is extended to an infinite cyclic covering $\tilde{W}' \rightarrow W'$ and $H_*(\tilde{W}'; Q)$ is finitely generated over Q . Therefore $M_0 \circ M_1 \sim 0$. Conversely, assume $M_0 \circ M_1 \sim 0$. Then there is a compact connected oriented 4-manifold W'' with $\partial W'' = M_0 \circ M_1$ and such that for some infinite cyclic connected covering $(\tilde{W}'', \tilde{M}_0 \circ \tilde{M}_1) \rightarrow (W'', M_0 \circ M_1)$, $H_*(\tilde{W}''; Q)$ is finitely generated over Q . Note that by the definition of the circle union there is a natural injection $j: S^1 \times \partial B^2 \times [0, 1] \rightarrow M_0 \circ M_1$. Now we let $W''' = W'' \cup_j S^1 \times B^2 \times [0, 1]$. It is easy to see that $\partial W'''$ is equal to the disjoint union $M_0 \cup -M_1$ and that the triad $(W'''; M_0, M_1)$ gives an $\tilde{H}$ -cobordism between M_0 and M_1 . This completes the proof.

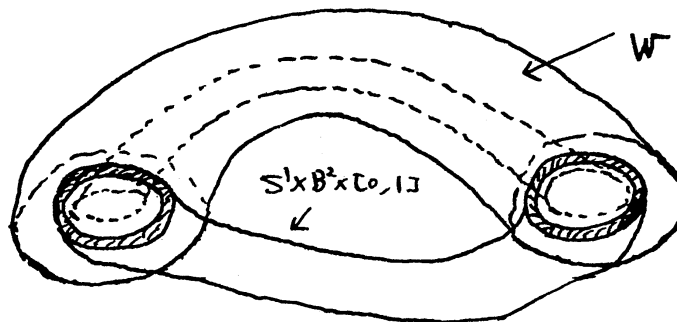


figure 3.

1.8 LEMMA. If $M_0 \sim 0$ and $M_1 \sim 0$, then $M_0 \circ M_1 \sim 0$.

Proof. Let $(W; M_0, \emptyset)$, $(W'; M_1, \emptyset)$ be $\tilde{H}$ -cobordisms, and let $M_0 \circ M_1 = M_0 \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1 - S^1 \times \text{Int} B^2 \times [0, 1]$. If we let $W'' = W \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} W'$ (See figure 4.), then the triad $(W''; M_0 \circ M_1, \emptyset)$ gives an $\tilde{H}$ -cobordism, which completes the proof.

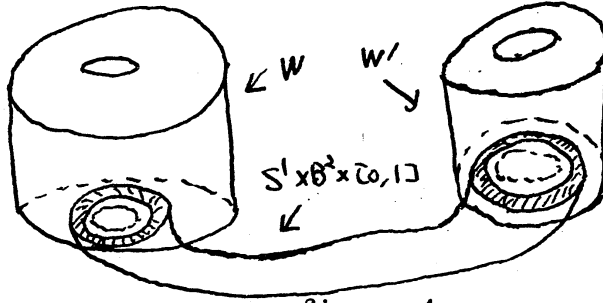


figure 4.

Now we state the main theorem of this section.

1.9 THEOREM. The set $\Omega(S^1 \times S^2)$ forms an abelian group under the sum $[M_0] + [M_1] = [M_0 \circ M_1]$. The zero element of this group is $[S^1 \times S^2]$. The inverse of any element $[M]$ is the element $[-M]$.

Proof. To show that the sum $[M_0] + [M_1] = [M_0 \circ M_1]$ is well-defined, we let $M_0 \sim M'_0$ and $M_1 \sim M'_1$. By LEMMA 1.7, $M_0 \circ -M'_0 \sim 0$ and $M_1 \circ -M'_1 \sim 0$. Then by LEMMA 1.8

$(M_0 \circ -M'_0) \circ (M_1 \circ -M'_1) \sim 0$. On the other hand, clearly

$$(M_0 \circ M_1) \circ -(M'_0 \circ M'_1) \sim (M_0 \circ -M'_0) \circ (M_1 \circ -M'_1)$$

(Use LEMMA 1.6.). Hence again by LEMMA 1.7 $M_0 \circ M_1 \sim M'_0 \circ M'_1$. Thus,

$[M_0] = [M'_0]$ and $[M_1] = [M'_1]$ imply $[M_0] + [M_1] = [M'_0] + [M'_1]$. It is clear that $([M_0] + [M_1]) + [M_2] = [M_0] + ([M_1] + [M_2])$ and $[M_0] + [M_1] = [M_1] + [M_0]$. Also, we have $[M] + [S^1 \times S^2] = [M \circ S^1 \times S^2] = [M]$ and, by LEMMA 1.7, $[M] + [-M] = [S^1 \times S^2]$. This completes the proof.

The group $\Omega(S^1 \times S^2)$ is called the (oriented) $\tilde{H}$ -cobordism

group between 3-dimensional homology oriented handles. The zero element is denoted by 0 and the inverse of $[M]$ is $-[M]$.

§ 2

GEOMETRIC AND ALGEBRAIC STRUCTURES OF $\Omega(S^1 \times S^2)$

Let $p: \tilde{M} \rightarrow M$ be the infinite cyclic orientation-preserving covering projection and t be a generator of the covering transformation group. Since M is oriented, one fundamental class $[M] \in H_3(M; \mathbb{Z})$ is specified. Then the choice of t determines a finite fundamental class $\mu \in H_2(\tilde{M}; \mathbb{Z}) (\approx \mathbb{Z})$ of $\tilde{M}$. In fact, we let $\mu = p_*^{-1}(\omega \cap [M])$, where $\omega \in H^1(M; \mathbb{Z})$ is a cocycle identified with the covering transformation t and $p_*: H_2(\tilde{M}; \mathbb{Z}) \approx H_2(M; \mathbb{Z}) (\approx \mathbb{Z})$ (See Kawauchi [6, Remark 2.4]). Note that the dual isomorphism $\cap \mu: H^1(\tilde{M}; \mathbb{Q}) \approx H_1(\tilde{M}; \mathbb{Q})$ holds (See Kawauchi [5, Theorem 2.3]). Equivalently, the skew-symmetric cup product pairing

$$H^1(\tilde{M}; \mathbb{Q}) \times H^1(\tilde{M}; \mathbb{Q}) \xrightarrow{U} H^2(\tilde{M}; \mathbb{Q}) \xrightarrow{\cap \mu} H_0(\tilde{M}; \mathbb{Q}) = \mathbb{Q}$$

is non-singular (See Milnor [10, pl27]).

2.1 DEFINITION. The bilinear form $\langle \cdot, \cdot \rangle: H^1(\tilde{M}; \mathbb{Q}) \times H^1(\tilde{M}; \mathbb{Q}) \rightarrow \mathbb{Q}$ defined by the equality $\langle x, y \rangle = (x \cup y) \cap \mu + (y \cup x) \cap \mu$ is called the quadratic form of M .

It is easy to check that the quadratic form is uniquely determined by the oriented M (in particular, it does not depend upon the choice of t), and that it is a symmetric bilinear form

with the identity $\langle tx, ty \rangle = \langle x, y \rangle$. Furthermore, this form $\langle \cdot, \cdot \rangle$ is always non-singular. To see this, it is convenient to introduce the concept of the Alexander polynomial.

2.2 DEFINITION. The (rational) Alexander polynomial $A(t)$ of M is the characteristic polynomial of the isomorphism $t : H_1(\tilde{M}; \mathbb{Q}) \longrightarrow H_1(\tilde{M}; \mathbb{Q})$ (See Kawauchi[6] for details and the definition of the integral Alexander polynomial.).

$A(t)$ is the invariant of M up to units $ct^i \in Q[t, t^{-1}]$ and has the properties $A(\pm 1) \neq 0$ and $A(t) \doteq A(t^{-1})$.

Suppose, for all x , $\langle x, y \rangle = [xU(t-t^{-1})y] \cap \mu = 0$. Then $(t-t^{-1})y = -t^{-1}(t-1)(t+1)y = 0$, which implies $y = 0$, since $A(+1) \neq 0$. Thus, the quadratic form $\langle, \rangle$ is non-singular.

Since $\langle , \rangle$ is non-singular and symmetric, with a suitable basis of $H_1(\tilde{M}; \mathbb{Q})$, $\langle , \rangle$ represents a rational diagonal matrix

$$\begin{pmatrix} a_1 & \dots & a_m & & \\ & & & -b_1 & \dots & \\ & & & & \ddots & \\ 0 & & & & & -b_n \end{pmatrix} \quad (a_i > 0, b_j > 0).$$

2.3 DEFINITION. The integer $\sigma(M) = m - n$ is called the signature of the oriented M .

It is not difficult to see that $\mathfrak{S}(M)$ is even (, since $\dim_{\mathbb{Q}} H_1(\tilde{M}; \mathbb{Q}) = \deg A(t)$ is even) and that $\mathfrak{S}(-M) = -\mathfrak{S}(M)$.

2.4 THEOREM. If $M \sim 0$, then the signature $\sigma(M)$ is 0 and the Alexander polynomial $A(t)$ of M has the form $f(t)f(t^{-1})$ for some rational polynomial $f(t)$.

One may note an analogy of the signature and the Alexander polynomial between the oriented $\tilde{H}$ -cobordism group $\Omega(S^1 \times S^2)$ and the Fox-Milnor's knot cobordism group C^1 (See Fox-Milnor [2]). This relation will be clarified in this section.

2.5 DEFINITION. The reduced Alexander polynomial $\tilde{A}(t)$ of M is the rational polynomial obtained from the Alexander polynomial $A(t)$ by cancelling the factors of the type $f(t)f(t^{-1})$.

Let $\tilde{A}(t), \tilde{A}'(t)$ be the reduced Alexander polynomials of M, M' , respectively. The following is a direct consequence of THEOREM 2.4.

2.6 THEOREM. If $M \sim M'$, then $\sigma(M) = \sigma(M')$ and $\tilde{A}(t) \doteq \tilde{A}'(t)$. [Note that there is a canonical isomorphism $H_1(\tilde{M} \otimes M') \approx H_1(\tilde{M}) + H_1(-\tilde{M}')$]

2.7 PROOF OF THEOREM 2.4. Since $M \sim 0$, there exists an $\tilde{H}$ -cobordism $(W; M, \emptyset)$. Then for an infinite cyclic connected covering $(\tilde{W}, \tilde{M}) \rightarrow (W, M)$, $H_*(\tilde{W}; Q)$ is finitely generated over Q .

Now we consider the following diagram

$$\begin{array}{ccccc} H^1(\tilde{W}; Q) & \xrightarrow{i^*} & H^1(\tilde{M}; Q) & \xrightarrow{\delta} & H^2(\tilde{W}, \tilde{M}; Q) \\ & & \downarrow \eta_\mu & & \downarrow \eta_\mu \\ & & H_1(\tilde{M}; Q) & \xrightarrow{i_*} & H_1(\tilde{W}; Q) \end{array}$$

Here, the top sequence is exact and the vertical maps are isomorphisms and $\bar{\mu} \in H_3(\tilde{W}, \tilde{M}; Z) (\approx Z)$ is a finite fundamental class (See Kawauchi [5, Theorem 2.3].) obtained from the finite fundamental class $\mu \in H_2(\tilde{M}; Z)$ by the boundary-isomorphism $\partial: H_3(\tilde{W}, \tilde{M}; Z) \approx H_2(\tilde{M}; Z)$. And the square is commutative.

Because the sequence $0 \rightarrow \text{Im } i^* \rightarrow H^1(\tilde{M}; Q) \rightarrow \text{Im } \delta \rightarrow 0$ is exact, the equality $A(t) \doteq B(t)C(t)$ holds, where $B(t), C(t)$ are the characteristic polynomials of the linear isomorphisms $t: \text{Im } i^* \rightarrow \text{Im } i^*$ and $t: \text{Im } \delta \rightarrow \text{Im } \delta$, respectively.

By the commutativity of the above square, we have the isomorphism $\cap \bar{\mu}: \text{Im } \delta \rightarrow \text{Im } i_*$. This asserts that the equality $C(t^{-1}) \doteq B(t)$ holds. [Use the identities $(tu) \cap \bar{\mu} = t^{-1}(u \cap \bar{\mu})$ and $\text{Im } i^* = \text{Hom}(\text{Im } i_*, Q)$.] Thus, we have $A(t) \doteq C(t)C(t^{-1}) \doteq B(t^{-1})B(t)$.

Next, for all $u \in H^1(\tilde{W}; Q)$, suppose $\langle i^*(u), y \rangle = 0$. This situation is equivalent to $\delta(t-t^{-1})y = 0$, that is, $(t-t^{-1})y \in \text{Im } i^*$, because $\langle i^*u, y \rangle = i^*u \cup (t-t^{-1})y = u \cup \delta(t-t^{-1})y$. [Use the above square is commutative.] Using $(t-t^{-1})\text{Im } i^* \subset \text{Im } i^*$ and the isomorphism $t-t^{-1}: H^1(\tilde{M}; Q) \rightarrow H^1(\tilde{M}; Q)$, $(t-t^{-1})y \in \text{Im } i^*$ is equivalent to $y \in \text{Im } i^*$. Thus, we showed that the orthogonal complement of $\text{Im } i^*$ is $\text{Im } i^*$ itself. Then, a familiar process implies $\mathfrak{C}(M) = 0$ (See for example Milnor-Husemoller [11, p13].). This completes the proof.

Let $\mathcal{K}$ be the set of knot types of tame oriented 1-knots in the oriented S^3 . We shall construct a function $m: \mathcal{K} \rightarrow \mathcal{C}(S^1 \times S^2)$. (Now we regard the class $\mathcal{C}(S^1 \times S^2)$ as the set of orientation-

preserving homeomorphism types of homology oriented handles.)

Let $T(k) \subset S^3$ be the tubular neighborhood of a knot $k \subset S^3$. By Schubert [12], the knotted torus $T(k)$ in S^3 has unique meridian and longitude curves (up to isotopies of $\partial T(k)$). Define $m(k)$ to be the oriented manifold obtained from the surgery of S^3 along $T(k)$ by using the unique meridian and longitude curves: $m(k) = S^3 - T(k) \cup B^2 \times S^1$. (The orientation of $m(k)$ will adapt the orientation induced from $S^3 - T(k)$.)

This assignment clearly implies a function $m: K \rightarrow \mathcal{C}(S^1 \times S^2)$ from the knot types to the homeomorphism types.

Two knot types k_1, k_2 are (knot) cobordant if for representative knots $k_1 \in k_1$ and $k_2 \in k_2$ the sum $k_1 \# k_2 \subset S^3$ bounds a locally flat 2-cell in the 4-cell B^4 . Such a concept is called the knot cobordism. The set K modulo the knot cobordism relation forms an abelian group C^1 , called the knot cobordism group (See Fox-Milnor [2] for details.).

Note that the function $m: K \rightarrow \mathcal{C}(S^1 \times S^2)$ induces a homomorphism $m: C^1 \rightarrow \mathcal{C}(S^1 \times S^2)$. In fact, easily we have $m(k_1 \# k_2) = m(k_1) \circ m(k_2)$, and if k is cobordant to a trivial knot then $m(k) \sim 0$ [To see this, let $D^2 \subset B^4$ be a locally flat 2-cell with $k = \partial D^2 \subset S^3$. By using an embedding $\phi: \partial B^2 \times B^2 \rightarrow S^3$, giving a tubular neighborhood of $k = \phi(\partial B^2 \times p)$ such that a circle $\phi(\partial B^2 \times p')$ is the longitude curve, we construct a 4-manifold $W = B^4 \cup_{\phi} B^2 \times B^2$. Then $\partial W = m(k)$ and a 2-sphere $\Sigma = D^2 \cup B^2 \times p \subset W$ is locally flat. By performing a surgery along the tubular neighborhood of Σ , we obtain a 4-manifold W' with $\partial W' = m(k)$

*) This function is not injective. A non-invertible knot would provide such an example.

and $H_*(W'; \mathbb{Z}) \approx H_*(S^1; \mathbb{Z})$. The triad $(W'; m(k), \emptyset)$ gives an $\tilde{H}$ -cobordism.].

2.8 LEMMA. The homomorphism $m: C^1 \rightarrow \Omega(S^1 \times S^2)$ satisfies
 $\sigma\langle k \rangle = \sigma[m(k)]$ and $A_{\langle k \rangle}(t) \doteq A_{[m(k)]}(t)$ for all $\langle k \rangle \in C^1$.

The proof will be given later.

By LEMMA 2.8, the known results of C^1 also imply the following two corollaries.

2.9 COROLLARY. For any integer i , there exists $M \in \Omega(S^1 \times S^2)$
with $\sigma(M) = 2i$.

2.10 COROLLARY. The oriented $\tilde{H}$ -cobordism group $\Omega(S^1 \times S^2)$ has
the free part of infinite rank and contains a torsion element.

For example, for the figure eight knot 4_1 (See figure 5.), the element $[m(4_1)] \in \Omega(S^1 \times S^2)$ gives an element of order 2, because the element $\langle 4_1 \rangle \in C^1$ has order 2 and the reduced Alexander polynomial of $m(4_1)$ is $t^2 - 3t + 1$ which implies $[m(4_1)] \neq 0$ by THEOREM 2.4. — .

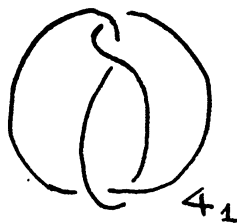


figure 5.

Now we need a concept of Seifert matrices. A Seifert matrix V is an integral square matrix with $\det(V-V') = \pm 1$. (V' is the transepose of V .)

For two Seifert matrices V and W , if the block sum $V \oplus -W$ is congruent (over the integers) to a matrix of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ (B, C, D are square matrices of the same size.) then V is said to be (matrix) cobordant to W . Such a concept is called the matrix cobordism. The set of Seifert matrices modulo the matrix cobordism relation forms a group G_- , called the matrix cobordism group (See Levine[8] for details. Note that only Seifert matrices with sign -1 are considered.). By Levine [9], G_- is isomorphic to the direct sum $\sum_{i=1}^{\infty} \mathbb{Z}^i \oplus \sum_{i=2}^{\infty} \mathbb{Z}^i \oplus \sum_{i=1}^{\infty} \mathbb{Z}^i$.

For a knot k in S^3 , denote $M(k)$ to be the knot exterior, or the closed knot complement of k , and $\tilde{M}(k)$ to be its infinite cyclic covering space. By a Seifert matrix of the knot k we will mean a Seifert matrix which is S -equivalent to a Seifert matrix associated with a Seifert surface of k . (See Trotter[13] for recent results of S -equivalences.)

The quadratic form $\langle \cdot, \cdot \rangle : H^1(\tilde{M}(k), \partial\tilde{M}(k); \mathbb{Q}) \times H^1(\tilde{M}(k), \partial\tilde{M}(k); \mathbb{Q}) \rightarrow \mathbb{Q}$ of the oriented knot k in the oriented S^3 is defined by the equality $\langle x, y \rangle = (x \cup_t y) \cap \mu + (y \cup_t x) \cap \mu$ (See Milnor [10] and Erle [11].), which is a complete analogue of DEFINITION 2.1. [Note that since k and S^3 are oriented, both t and μ are specified uniquely.] (Here, $\mu \in H_2(\tilde{M}(k), \partial\tilde{M}(k); \mathbb{Z})$ is a finite fundamental class [5].)

Erle [11] then showed that, with a suitable basis of

$H^1(\tilde{M}(k), \partial\tilde{M}(k); Q)$, the linear isomorphism $t : H^1(\tilde{M}(k), \partial\tilde{M}(k); Q) \rightarrow H^1(\tilde{M}(k), \partial\tilde{M}(k); Q)$ represents the matrix $V'^{-1}V$ and the quadratic form $\langle , \rangle$ represents the matrix $V + V'$ for some non-singular Seifert matrix V .

The same assertion also applies for the homology oriented handles.

By Kawauchi [5, Corollary 1.3], there is a piecewise-linear map $f : M \rightarrow S^1$ such that $F = f^{-1}(p)$ is a closed ^{connected} surface. Clearly, the homology class $[F] \in H_2(\tilde{M}; Z)$ coincides with $\pm\mu \in H_2(\tilde{M}; Z)$. If t is specified, then μ is also specified and hence we may orient F so that $[F] = \mu$. Let M^* be a manifold obtained from M by splitting along F . Note that a duality $H^1(F; Z) \approx H_1(M^*; Z)$ holds. Let $\partial M^* = FU - F$ (Here we identify the component of ∂M^* with the orientation compatible with F). With dual bases of $H_1(F; Z)$ and $H_1(M^*; Z)$, the canonical homomorphism $H_1(F; Z) \rightarrow H_1(M^*; Z)$ represents a square matrix V_0 . To show that V_0 is a Seifert matrix, let V_0^- be another matrix representing the canonical homomorphism $H_1(-F; Z) \rightarrow H_1(M^*; Z)$. By an analogy of Levine [7] it is not difficult to see that the matrix $tV_0 - V_0^-$ is a relation matrix of $H_1(\tilde{M}; Z)$ and that V_0^- is in fact the transpose V_0' of V_0 . Thus, $tV_0 - V_0'$ is a relation matrix of $H_1(\tilde{M}; Z)$. Using $H_1(M; Z) = Z$, $\det(V_0 - V_0') = \pm 1$. Thus, V_0 is a Seifert matrix.

2.11 DEFINITION. A Seifert matrix V which is S-equivalent to V_0 is called a Seifert matrix of M (with a specified generator of $H_1(M; Z)$).

Note that if another generator of $H_1(M; \mathbb{Z})$ is specified then the transpose V' of V is considered as a Seifert matrix of M .

A technique of Erle [1] then implies the following :

2.12 LEMMA. With a suitable basis of $H^1(\tilde{M}; \mathbb{Q})$, the linear isomorphism $t : H^1(\tilde{M}; \mathbb{Q}) \rightarrow H^1(\tilde{M}; \mathbb{Q})$ represents the matrix $V'^{-1}V$ and the quadratic form $\langle , \rangle : H^1(\tilde{M}; \mathbb{Q}) \times H^1(\tilde{M}; \mathbb{Q}) \rightarrow \mathbb{Q}$ represents the matrix $V + V'$ for some non-singular Seifert matrix V of M .

Using LEMMA 2.12, we obtain a well-defined homomorphism $\psi : \Omega(S^1 \times S^2) \rightarrow G_-$ sending homology oriented handles to the Seifert matrices (See Levine [9, p101]). [Note that the Seifert matrix V is always cobordant to the transpose V' , although V is in general not S-equivalent to V' (See Trotter [3]).]

Thus, we sketched the following.

2.13 THEOREM. There is the commutative triangle

$$\begin{array}{ccc}
 C^1 & \xrightarrow{m} & \Omega(S^1 \times S^2) \\
 \searrow \phi & & \swarrow \psi \\
 & G_- & \\
 & \parallel & \\
 & \sum_{i=1}^{\infty} Z^i \oplus \sum_{i=1}^{\infty} Z^i_2 \oplus \sum_{i=1}^{\infty} Z^i_4 &
 \end{array}$$

, where $\phi : C^1 \rightarrow G_-$ is a canonical epimorphism defined by Levine [8] and $\psi : \Omega(S^1 \times S^2) \rightarrow G_-$ is an epimorphism defined as above and m satisfies $\tilde{A}_{\langle k \rangle}(t) \doteq \tilde{A}_{[m(k)]}(t)$ and $\sigma_{\langle k \rangle} = \sigma_{[m(k)]}$ for

all $\langle k \rangle \in C^1$.

2.14 PROOF OF LEMMA 2.7. The inclusion map $i : \tilde{M}(k) \rightarrow \tilde{m}(k)$ induces an isomorphism $i_* : H_1(\tilde{M}(k); \mathbb{Q}) \approx H_1(\tilde{m}(k); \mathbb{Q})$. From this, it follows that $A_k(t) \doteq A_{m(k)}(t)$, and hence $\tilde{A}_{\langle k \rangle}(t) \doteq \tilde{A}_{[m(k)]}(t)$.

Next, since the following triangle

$$\begin{array}{ccc} H^1(\tilde{M}(k); \mathbb{Q}) \times H^1(\tilde{M}(k); \mathbb{Q}) & \xrightarrow{\quad} & \mathbb{Q} \\ \approx \downarrow i_* \times i_* & & \\ H^1(\tilde{m}(k); \mathbb{Q}) \times H^1(\tilde{m}(k); \mathbb{Q}) & \xrightarrow{\quad} & \mathbb{Q} \end{array}$$

is commutative, we obtain that $\mathfrak{Q}(k) = \mathfrak{Q}(m(k))$. This completes the proof.

The general problem of deciding a geometrical condition of $\tilde{H}$ -cobordism seems difficult, but a partial result is presented here.

2.15 THEOREM. If $M \in \mathcal{C}(S^1 \times S^2)$ is embeddable in a homology 4-sphere $\bar{S}^4$, then M is $\tilde{H}$ -cobordant to 0.

Proof. Assume $M \subset \bar{S}^4$. By an easy computation of the homology, we obtain that M separates $\bar{S}^4$ into two manifolds, say, W_1, W_2 and that one of W_1, W_2 has the homology of the circle, say, $H_*(W_1; \mathbb{Z}) \approx H_*(S^1; \mathbb{Z})$. Then $(W_1; M, \emptyset)$ gives an $\tilde{H}$ -cobordism. This proves THEOREM 2.14.

Here are a few examples, whose somewhat analogous properties were also noticed by Kato[4].

2.16 EXAMPLES. First we consider a trefoil 3_1 (figure 6).

Using $\mu(m(3_1)) = \pm 2$ or $\tilde{A}(t) = t^2 - t + 1$, we see that $m(3_1)$ is not $\tilde{H}$ -cobordant to 0. Hence $m(3_1)$ is not embeddable to the 4-sphere S^4 and the minimal embedding dimension of $m(3_1)$ into a sphere is five [In fact, Hirsch[3] showed that every compact orientable manifold is locally flatly embeddable to the 5-sphere.].



figure 6.

On the other hand, a stevedore's knot 6_1 (figure 7) is a slice knot and hence $m(6_1) \sim 0$.

Note that a slice knot k can be realized as a local knot type of a 2-sphere $S(k)$ in S^4 with only one locally knotted point (See Fox-Milnor[2].).

Let $N(S(6_1); S^4)$ be the regular neighborhood of $S(6_1)$ in S^4 . It is not hard to see that $\partial N(S(6_1); S^4) = m(6_1)$. Thus, $m(6_1)$ is embeddable in the 4-sphere S^4 .

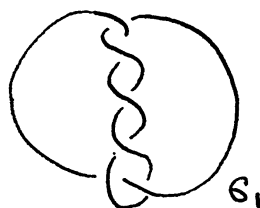


figure 7.

Similar arguments also applies for a granny knot $3_1 \# 3_1$ and a square knot $3_1 \# -3_1$ (See figure 8.). In fact, $m(3_1 \# 3_1)$ is not

embeddable to S^4 , although $m(3_1 \# -3_1)$ is embeddable to S^4 , since $\sigma(m(3_1 \# -3_1)) = 2\sigma(3_1) = \pm 4$ and $3_1 \# -3_1$ is a slice knot.

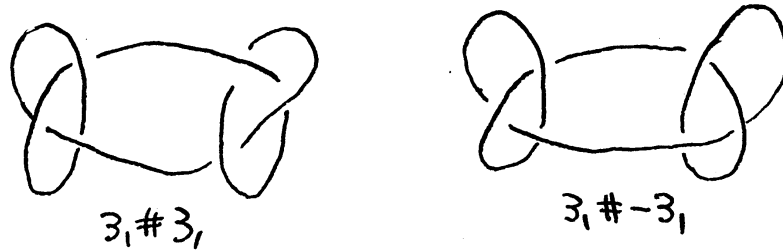


figure 8.

§ 3

THE NON-ORIENTABLE $\tilde{H}$ -COBORDISM GROUP $\Omega(S^1 \times S^2)$

A 3-dimensional homology non-orientable handle M is a compact 3-manifold having the homology of the non-orientable handle $S^1 \times S^2$: $H_*(M; \mathbb{Z}) \approx H_*(S^1 \times S^2; \mathbb{Z})$, and let $\mathcal{C}(S^1 \times S^2)$ be the class of the homology non-orientable handles.

In $\mathcal{C}(S^1 \times S^2)$, an $\tilde{H}$ -cobordism relation is defined as an analogy of DEFINITION 1.1.

3.1 DEFINITION. Two homology handles M_0, M_1 in $\mathcal{C}(S^1 \times S^2)$ are $\tilde{H}$ -cobordant and denoted by $M_0 \sim M_1$ if there exists a compact connected (non-orientable) 4-manifold W with the boundary ∂W being the disjoint union $M_0 \cup M_1$ and such that there is an infinite cyclic connected covering $(\tilde{W}; \tilde{M}_0, \tilde{M}_1) \rightarrow (W; M_0, M_1)$ with $\tilde{W}$ being orientable and with $H_*(\tilde{W}; \mathbb{Q})$ being finitely generated over $\mathbb{Q}$. [Note that $\tilde{M}$ is always orientable (See Kawauchi[6, Lemma 2.3].).]

We say that M is $\tilde{H}$ -cobordant to O if M is $\tilde{H}$ -cobordant to $S^1 \times_{\tau} S^2$.

For $M_0, M_1 \in \mathcal{C}(S^1 \times_{\tau} S^2)$, choose polyhedral simple closed curves $\omega_0 \subset M_0$, $\omega_1 \subset M_1$ which represent generators of $H_1(M_0; \mathbb{Z})$, $H_1(M_1; \mathbb{Z})$, respectively. It is not difficult to see that the tubular neighborhoods $T(\omega_0) \subset M_0$ of ω_0 and $T(\omega_1) \subset M_1$ of ω_1 are both piecewise-linear homeomorphic to the solid Klein bottle $S^1 \times_{\tau} B^2$.

Let $F_0 \subset M_0$, $F_1 \subset M_1$ be closed connected orientable surfaces transversally intersecting ω_0 , ω_1 in single points, respectively.

Consider two piecewise-linear embeddings

$$h_0: S^1 \times_{\tau} B^2 \times 0 \longrightarrow M_0$$

$$h_1: S^1 \times_{\tau} B^2 \times 1 \longrightarrow M_1$$

such that there exist points $s \in S^1$, $b \in \text{Int} B^2$ with $h_0(S^1 \times_{\tau} b \times 0) = \omega_0$, $h_0(s \times_{\tau} B^2 \times 0) \subset F_0$, $h_1(S^1 \times_{\tau} b \times 1) = \omega_1$ and $h_1(s \times_{\tau} B^2 \times 1) \subset F_1$.

As an analogy of DEFINITION 1.4, we may have DEFINITION 3.2.

3.2 DEFINITION. The homology non-orientable handle

$$M_0 \circ M_1 = M_0 \cup_{h_0} S^1 \times_{\tau} B^2 \times [0, 1] \cup_{h_1} M_1 - S^1 \times_{\tau} \text{Int} B^2 \times [0, 1]$$

is called a circle union of M_0 and M_1 .

It is not difficult to check that for two circle unions $M_0 \circ M_1$, $M_0 \circ' M_1$, $M_0 \circ M_1 \sim M_0 \circ' M_1$. Further, we can prove that $M_0 \sim M_1$ if and only if $M_0 \circ M_1 \sim O$ as an analogy of LEMMA 1.7.

Thus, we sketched that the set $\Omega(S^1 \times_{\tau} S^2) = \mathcal{C}(S^1 \times_{\tau} S^2) / \sim$

forms an abelian group under the sum $[M_0] + [M_1] = [M_0 \circ M_1]$. This group is called the non-orientable $\tilde{H}$ -cobordism group of 3-dimensional homology non-orientable handles.

Every non-zero element of $\Omega(S^1 \times \tau S^2)$ has order 2, by construction.

Further, $\Omega(S^1 \times \tau S^2)$ is not finitely generated. Actually, the following is obtained.

3.3 THEOREM. $\Omega(S^1 \times \tau S^2) \approx \sum_{i=1}^{\infty} \mathbb{Z}_2^i$.

To prove THEOREM 3.3, the Alexander polynomial is useful.

The Alexander polynomial $A(t)$ of $M \in \mathcal{C}(S^1 \times \tau S^2)$ is simply defined to be the characteristic polynomial of the linear isomorphism $t : H_1(\tilde{M}; \mathbb{Q}) \rightarrow H_1(\tilde{M}; \mathbb{Q})$. (See Kawauchi [6].)

Then THEOREM 3.3 follows from LEMMA 3.4 (, which is somewhat analogous to THEOREM 2.4).

3.4 LEMMA. If $M \in \mathcal{C}(S^1 \times \tau S^2)$ is $\tilde{H}$ -cobordant to 0 then the Alexander polynomial $A(t)$ of M has a type of $f(t)f(-t^{-1})$ for some rational polynomial $f(t)$.

3.5 PROOF OF THEOREM 3.3. By Kawauchi [6], the irreducible integral polynomial $A_n(t) = nt^2 + t - n$ ($n = 1, 2, 3, \dots$) is realized as the Alexander polynomial of some $M_n \in \mathcal{C}(S^1 \times \tau S^2)$. Then it is easy to see that $M_1, M_2, M_3, \dots$ represent a set of linearly independent elements of $\Omega(S^1 \times \tau S^2)$. This completes the proof.

3.6 PROOF OF LEMMA 3.3. Since $M \sim 0$, there exists a compact connected 4-manifold W with $\partial W = M$ and such that for some infinite cyclic connected covering $(\tilde{W}, \tilde{M}) \rightarrow (W, M)$, $\tilde{W}$ is orientable and $H_*(\tilde{W}; Q)$ is finitely generated over Q . Then from the exact sequence $H^1(\tilde{W}; Q) \xrightarrow{i^*} H^1(\tilde{M}; Q) \xrightarrow{\delta} H^2(\tilde{W}, \tilde{M}; Q)$ we obtain the short exact sequence $0 \rightarrow \text{Im } i^* \rightarrow H^1(\tilde{M}; Q) \rightarrow \text{Im } \delta \rightarrow 0$. Thus we have $A(t) \doteq B(t)C(t)$, where $B(t)$, $C(t)$ are the characteristic polynomials of $t : \text{Im } i^* \rightarrow \text{Im } i^*$, $t : \text{Im } \delta \rightarrow \text{Im } \delta$, respectively. Since the square

$$\begin{array}{ccc} H^1(\tilde{M}; Q) & \xrightarrow{\delta} & H^2(\tilde{W}, \tilde{M}; Q) \\ \approx \downarrow \cap \mu & & \approx \downarrow \cap \mu \\ H_1(\tilde{M}; Q) & \xrightarrow{i_*} & H_1(\tilde{W}; Q) \end{array}$$

is commutative, we obtain the dual isomorphism $\cap \bar{\mu} : \text{Im } \delta \approx \text{Im } i_*$. Using the identities $(tu) \cap \bar{\mu} = -t^{-1}(u) \cap \bar{\mu}$ and $\text{Im } i^* = \text{Hom}(\text{Im } i_*, Q)$, this dual isomorphism gives the equality $C(-t^{-1}) \doteq B(t)$. This proves LEMMA 3.3.

§ 4

FURTHER DISCUSSIONS AND QUESTIONS

The most basic and interesting problem on this paper is the following question.

4.1 QUESTION. Whether or not are the homomorphisms m, ϕ, ψ in THEOREM 2.13 isomorphic ?

This question also asks the difference between $\tilde{H}$ -cobordism and

H-cobordism.

Usually, for compact closed oriented n -manifolds N_1^n, N_2^n , if there exists an oriented compact $(n+1)$ -manifold H^{n+1} with $\partial H^{n+1} = N_1^n \cup -N_2^n$ and $H_*(H^{n+1}, N_1^n; \mathbb{Z}) = 0$ ($= H_*(H^{n+1}, -N_2; \mathbb{Z})$), then N_1 is said to be H-cobordant to N_2 . Also, such a concept is called H-cobordism.

4.2 QUESTION. In $\mathcal{C}(S^1 \times S^2)$, are H-cobordism and $\tilde{H}$ -cobordism strictly distinct?

For example, it is not difficult to see that two H-cobordant homology oriented handles are $\tilde{H}$ -cobordant.

In $\mathcal{C}(S^1 \times S^2)$ or a class of more general manifolds it seems difficult to define a non-trivial H-cobordism group. However, for the class of homology oriented n -spheres, the H-cobordism group $\mathcal{H}(S^n)$ is defined in the natural way. In the piecewise-linear category, it is not so hard to see that $\mathcal{H}(S^n) = 0$ for $n \geq 5$. At $n = 4$, the author does not know whether $\mathcal{H}(S^4)$ vanishes or not. At $n = 3$, Kato pointed out that $\mathcal{H}(S^3)$ is non-trivial, that is, there exists a homology 3-sphere which is not the boundary of any homology 4-ball. In fact, the dodecahedral space $\bar{S}^3 = S^3/SL(2, 5)$ is such an example.

4.3 QUESTION. For any homology 3-sphere $\bar{S}^3$, is the connected sum $S^1 \times S^2 \# \bar{S}^3$ $\tilde{H}$ -cobordant to $S^1 \times S^2$?

Note that for the dodecahedral space $\bar{S}^3$, $S^1 \times S^2 \# \bar{S}^3$ is not H-cobordant to $S^1 \times S^2$.

Let $\mathcal{C}^{3,n}$ be the class of compact oriented 3-manifolds having the integral homology of the connected sum $\#^n S^1 \times S^2$ of n copies of $S^1 \times S^2$. Similarly, let $\mathcal{C}_Q^{3,n}$ be the class of compact oriented 3-manifolds having the rational homology of $\#^n S^1 \times S^2$.

4.4 QUESTION. In $\mathcal{C}^{3,n}$ or $\mathcal{C}_Q^{3,n}$ ($n \geq 2$), can a $\tilde{H}$ -cobordism theory be developed ?

It seems that for $n \geq 2$ all things would become extremely difficult.

In $\mathcal{C}_Q^{3,1}$, the $\tilde{H}$ -cobordism group $\Omega_Q^{3,1}$ is actually defined as an analogy of $\Omega(S^1 \times S^2)$. (This group is so related to the Levine's rational matrix cobordism group G_-^Q .)

Now suppose the $\tilde{H}$ -cobordism groups $\Omega^{3,n} = \mathcal{C}^{3,n}/\sim$ and $\Omega_Q^{3,n} = \mathcal{C}_Q^{3,n}/\sim$ are obtained. Let $\Omega^{3,0} = \mathcal{K}(S^3)$ and let $\Omega_Q^{3,0}$ be the rational H -cobordism group of rational homology 3-spheres. The direct sums $\Omega^3 = \Omega^{3,0} \oplus \Omega^{3,1} \oplus \Omega^{3,2} \oplus \dots$ and $\Omega_Q^3 = \Omega_Q^{3,0} \oplus \Omega_Q^{3,1} \oplus \Omega_Q^{3,2} \oplus \dots$ would have ring structures under the connected sum operation.

In the higher dimensional case, we can also define the $\tilde{H}$ -cobordism groups $\Omega(S^1 \times S^{n-1})$ and $\Omega(S^1 \times_{\tau} S^{n-1})$ of n -dimensional homology oriented and non-orientable handles, respectively.

The following seems not so difficult for $n \geq 5$.

4.5 QUESTION. Is $\Omega(S^1 \times S^{n-1})$ isomorphic to the piecewise-linear $(n-2)$ -knot cobordism group C_{PL}^{n-2} ? Also, is $\Omega(S^1 \times_{\tau} S^{n-1})$ isomorphic to $\sum_{i=1}^{\infty} \mathbb{Z}_2^i$ if n is even, or to 0 if n is odd ?

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