

Variable exponent version of Hedberg-Wolff inequalities

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Introduction.

Hedberg-Wolff gave the following inequalities in [HW]:

$$C^{-1} \int_{\mathbf{R}^N} [(G_\alpha * \mu)(x)]^{p'} dx \leq \int_{\mathbf{R}^N} \mathcal{W}_{\alpha,p}^\mu(x, 1) d\mu(x) \leq C \int_{\mathbf{R}^N} [(G_\alpha * \mu)(x)]^{p'} dx \quad (1)$$

for every nonnegative measure μ on $\mathbf{R}^N$ with a positive constant C independent of μ , where G_α is the Bessel kernel of order α ($0 < \alpha < N$) on $\mathbf{R}^N$, $1 < p < \infty$, $1/p + 1/p' = 1$ and

$$\mathcal{W}_{\alpha,p}^\mu(x, R) = \int_0^R \left(\frac{\mu(B(x, r))}{r^{N-\alpha p}} \right)^{p'-1} \frac{dr}{r} \quad (R > 0).$$

The function $\mathcal{W}_{\alpha,p}^\mu(\cdot, R)$ is called the *Wolff-potential* of μ of order (α, p) . Inequalities (1) imply

$$\mu \in (\mathcal{L}^{\alpha,p}(\mathbf{R}^N))^* \Leftrightarrow \int_{\mathbf{R}^N} \mathcal{W}_{\alpha,p}^\mu(x, 1) d\mu(x) < \infty, \quad (2)$$

where

$$\mathcal{L}^{\alpha,p}(\mathbf{R}^N) = \{u = G_\alpha * f; f \in L^p(\mathbf{R}^N)\}$$

with the norm $\|u\|_{\alpha,p} = \|f\|_p$. Since $\mathcal{L}^{m,p}(\mathbf{R}^N) = W^{m,p}(\mathbf{R}^N)$ for $m \in \mathbf{N}$ (A.P. Calderón), (2) shows that

$$\mu \in (W^{m,p}(\mathbf{R}^N))^* \Leftrightarrow \int_{\mathbf{R}^N} \mathcal{W}_{m,p}^\mu(x, 1) d\mu(x) < \infty, \quad (2')$$

for $m \in \mathbf{N}$.

In [AH], the proof of (1) is given via the following inequalities

$$C^{-1} \|M_{\alpha,R} \mu\|_q \leq \|G_\alpha * \mu\|_q \leq C \|M_{\alpha,R} \mu\|_q \quad (3)$$

for $0 < q < \infty$ and $R > 0$ with a positive constant C independent of μ , where

$$(M_{\alpha,R} \mu)(x) = \sup_{0 < r < R} r^{\alpha-N} \mu(B(x, r)).$$

These results have been generalized to the case where G_α is replaced by a general convolution kernel satisfying certain conditions (cf. [JPW], [AE, Part II]).

In the present paper, we consider variable exponents $p(x)$ on $\mathbf{R}^N$ and show that inequalities (1) and (3) hold in some restricted forms, and relations (2) and (2') still hold true for μ with finite total mass when we replace p by $p(x)$ satisfying certain conditions. We discuss these for convolution kernels.

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1. Definitions

As a potential kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a nonnegative nonincreasing lower semicontinuous function $k(r)$ on $(0, \infty)$ such that

(k.1) there is $R_0 > 0$ such that $k(r)$ is positive and satisfies the doubling condition on $(0, R_0)$, i.e., $k(r) \leq C_d k(2r)$ for $0 < r < R_0/2$;

$$(k.2) \quad \int_0^1 k(r) r^{N-1} dr < \infty.$$

By (k.2), $k(x) \in L^1_{loc}(\mathbf{R}^N)$. The k -potential of a nonnegative measure μ on $\mathbf{R}^N$ is defined by

$$(k * \mu)(x) = \int k(x - y) d\mu(y).$$

For $R > 0$, the (k, R) -maximal function of μ is defined by

$$(M_{k,R}\mu)(x) = \sup_{0 < r < R} k(r) \mu(B(x, r)).$$

We consider a variable exponent $p(x)$ on $\mathbf{R}^N$ such that

(P1) $1 < p^- := \inf p(\cdot) \leq p^+ := \sup p(\cdot) < \infty$;

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(1/|x - y|)} \quad \text{for } |x - y| \leq \frac{1}{2}$$

with a constant $C_p \geq 0$, which is referred to as the constant of log-Hölder continuity.

We refer to [KR] for the definition of the $p(\cdot)$ -norm $\|f\|_{p(\cdot)}$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^N)$ and the variable exponent Sobolev space $W^{m,p(\cdot)}(\mathbf{R}^N)$ ($m \in \mathbf{N}$).

For $R > 0$, we define the $(k, p(\cdot))$ -Wolff potential of μ by

$$\mathcal{W}_{k,p(\cdot)}^\mu(x, R) = \int_0^R k(r)^{p(x)} \mu(B(x, r))^{p(x)-1} r^{N-1} dr.$$

Example. For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = 1/|x|^{N-\alpha}$ and the Bessel kernel G_α of order α are typical examples of $k(x)$. For these kernels, we can take R_0 any positive value.

$$\mathcal{W}_{\alpha,p(\cdot)}^\mu(x, R) := \mathcal{W}_{I_\alpha,p(\cdot)'}^\mu(x, R) = \int_0^R \left(\frac{\mu(B(x, r))}{r^{N-\alpha p(x)}} \right)^{p(x)'-1} \frac{dr}{r}$$

and

$$\mathcal{W}_{G_\alpha,p(\cdot)}^\mu(x, R) \sim \mathcal{W}_{\alpha,p(\cdot)'}^\mu(x, R).$$

For a nonnegative measure μ and $R > 0$, let

$$M(\mu, R) := \sup_{x \in \mathbf{R}^N} \mu(B(x, R)).$$

It is easy to see that if $M(\mu, R) < \infty$ holds for some $R > 0$, then so holds for all $R > 0$.

Lemma 1. *If either $k * \mu \in L^{p(\cdot)}(\mathbf{R}^N)$ or $M_{k,R}\mu \in L^{p(\cdot)}(\mathbf{R}^N)$ or*

$$\int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x) < \infty,$$

then $M(\mu, R) < \infty$ for all $R > 0$.

Proof. Suppose that $M(\mu, R) = \infty$ for some $R > 0$. As remarked above, we may assume $0 < R < R_0$ and $M(\mu, R/3) = \infty$. Then, for every $n \in \mathbf{N}$, there exists $\xi_n \in \mathbf{R}^N$ such that $\mu(B(\xi_n, R/3)) \geq n$. If $x \in B(\xi_n, R/3)$, then $\mu(B(x, 2R/3)) \geq n$, so that

$$(k * \mu)(x) \geq \int_{B(x, 2R/3)} k(x - y) d\mu(y) \geq k(R)n$$

and

$$(M_{k,R}\mu)(x) \geq k(2R/3)\mu(B(x, 2R/3)) \geq k(R)n.$$

Thus

$$\int [(k * \mu)(x)]^{p(x)} dx \geq \int_{B(\xi, R/3)} [(k * \mu)(x)]^{p(x)} dx \geq C_1 n^{p^-}$$

with a constant $C_1 > 0$ independent of n . This shows that $k * \mu \notin L^{p(\cdot)}(\mathbf{R}^N)$. Similarly, we see that $M_{k,R}\mu \notin L^{p(\cdot)}(\mathbf{R}^N)$.

Also, if $x \in B(\xi_n, R/3)$, then

$$\begin{aligned} \mathcal{W}_{k,p(\cdot)}^\mu(x, R) &\geq \int_{2R/3}^R k(r)^{p(x)} \mu(B(x, r))^{p(x)-1} r^{N-1} dr \\ &\geq \int_{2R/3}^R k(R)^{p(x)} \mu(B(x, 2R/3))^{p(x)-1} r^{N-1} dr \geq C_2 n^{p^- - 1} \end{aligned}$$

with a constant $C_2 > 0$ independent of n , so that

$$\int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x) \geq \int_{B(\xi_n, R/3)} \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x) \geq C_2 n^{p^-}$$

for all $n \in \mathbf{N}$.

We call $\int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x)$ the $(k, p(\cdot))$ -energy of μ .

2. Estimate of $(k, p(\cdot))$ -energy by $p(\cdot)$ -integral of k -potential

Theorem 1. *Let $M_0 \geq 1$, $0 < R < R_0/2$. Then*

$$\int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x) \leq C \left(\mu(\mathbf{R}^N) + \int [(k * \mu)(x)]^{p(x)} dx \right)$$

for all nonnegative measure μ such that $M(\mu, R) \leq M_0$, with a constant $C > 0$ depending only on N , C_d , p^+ , C_p , M_0 and $K_R := \int_0^R k(r) r^{N-1} dr$.

Proof. We consider a nonlinear potential

$$V_{k,p(\cdot)}^\mu = k * (k * \mu)^{p(\cdot)-1}.$$

Since

$$\int [(k * \mu)(x)]^{p(x)} dx = \int V_{k,p(\cdot)}^\mu(x) d\mu(x),$$

it suffices to show

$$\mathcal{W}_{k,p(\cdot)}^\mu(x, R) \leq C(1 + V_{k,p(\cdot)}^\mu(x)). \quad (2.1)$$

for $0 < R < R_0/2$.

Since $k(r)$ is nonincreasing and $K_R < \infty$, $k(r) \leq NK_R r^{-N}$ for $0 < r < R$. Hence, (P2) implies that

$$(k(y)\mu(B(x, |y|)))^{p(x)} \leq C(k(y)\mu(B(x, |y|)))^{p(x-y)} \quad (2.2)$$

for $|y| \leq R$ whenever $M(\mu, R) \leq M_0$ and $k(y)\mu(B(x, |y|)) \geq 1$, with a constant $C = C(N, K_R, C_p, M_0) > 0$.

If $|y| \leq R$, then $|x - y - \xi| \leq 2|y|$ for $\xi \in B(x, |y|)$, so that

$$k(y)\mu(B(x, |y|)) \leq C_d k(2y)\mu(B(x, |y|)) \leq C_d (k * \mu)(x - y).$$

Hence, using (2.2) we have

$$\begin{aligned} \mathcal{W}_{k,p(\cdot)}^\mu(x, R) &= \frac{1}{\sigma_N} \int_{\{|y| < R\}} k(y) (k(y)\mu(B(x, |y|)))^{p(x)-1} dy \\ &\leq \frac{1}{\sigma_N} \int_{\{|y| < R\}} k(y) dy + C \int_{\{|y| < R\}} k(y) (k(y)\mu(B(x, |y|)))^{p(x-y)-1} dy \\ &\leq K_R + C \int_{\{|y| < R\}} k(y) [(k * \mu)(x - y)]^{p(x-y)-1} dy \\ &\leq K_R + C V_{k,p(\cdot)}^\mu(x), \end{aligned}$$

with constants $C = C(N, C_d, K_R, p^+, C_p, M_0) > 0$, which shows (2.1). (Here, σ_N denotes the surface area of the unit sphere in $\mathbf{R}^N$.)

Remark. In Theorem 3, it is not known whether the term $\mu(\mathbf{R}^N)$ is really necessary.

On the other hand, for non-constant exponent $p(\cdot)$, the following inequality *does not hold* even if $M(\mu, R) \leq M_0$:

$$\int [(k * \mu)(x)]^{p(x)} dx \leq C \int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x).$$

In fact, if $p(\cdot)$ is continuous and non-constant in $\mathbf{R}^N$, then we can find nonnegative measures $\{\mu_j\}$ such that $M(\mu_j, R) \rightarrow 0$ ($j \rightarrow \infty$) and

$$\frac{\int \mathcal{W}_{G_\alpha, p(\cdot)}^{\mu_j}(x, R) d\mu_j(x)}{\int [(G_\alpha * \mu_j)(x)]^{p(x)} dx} \rightarrow 0 \quad (2.3)$$

as $j \rightarrow \infty$ for every $0 < \alpha < N$ and every $R > 0$.

Proof. We can choose two compact sets K_1 and K_2 and $1 < p_1 < p_2 < \infty$ such that $|K_1| > 0$, $|K_2| > 0$,

$$p(x) \leq p_1 \quad \text{for } x \in K_1 \quad \text{and} \quad p(x) \geq p_2 \quad \text{for } x \in K_2.$$

Let $\mu_j = (1/j)\chi_{K_2}dx$, $j = 1, 2, \dots$. Obviously, $M(\mu_j, R) \rightarrow 0$.

Since $\mu_j(B(x, r)) \leq (1/j)c_N r^N$ for any $x \in \mathbf{R}^N$ and $r > 0$,

$$\begin{aligned} \mathcal{W}_{G_\alpha, p(\cdot)}^{\mu_j}(x, R) &\leq C(\alpha, N, p^+) \int_0^R (r^{\alpha-N})^{p(x)} [(1/j)r^N]^{p(x)-1} r^{N-1} dr \\ &= C(\alpha, N, p^+) (1/j)^{p(x)-1} \int_0^R r^{\alpha p(x)-1} dr \leq C(\alpha, N, p^+, p^-, R) (1/j)^{p(x)-1} \end{aligned}$$

with constants $C(\dots) > 0$. If $x \in K_2$, then $(1/j)^{p(x)-1} \leq (1/j)^{p_2-1}$, so that

$$\int \mathcal{W}_{G_\alpha, p(\cdot)}^{\mu_j}(x, R) d\mu_j(x) \leq C(\alpha, N, p^+, p^-, R) (1/j)^{p_2} |K_2|. \quad (2.4)$$

On the other hand, since G_α is positive continuous on $\mathbf{R}^N$,

$$A = A(\alpha, K_1, K_2) := \inf\{G_\alpha(x-y); x \in K_1, y \in K_2\} > 0.$$

If $x \in K_1$, then

$$(G_\alpha * \mu_j)(x) = (1/j) \int_{K_2} G_\alpha(x-y) dy \geq (1/j) A |K_2|.$$

Thus,

$$\begin{aligned} \int (G_\alpha * \mu_j)^{p(x)} dx &\geq \int_{K_1} [(1/j) A |K_2|]^{p(x)} dx \\ &\geq (1/j)^{p_1} \min(A |K_2|, 1)^{p_1} |K_1|. \end{aligned} \quad (2.5)$$

In view of (2.4) and (2.5), we obtain (2.3), since $p_2 > p_1$.

3. Estimate of $p(\cdot)$ -integral of (k, R) -maximal function by $(k, p(\cdot))$ -energy

Theorem 2. Let $M_0 \geq 1$ and $0 < R < R_0/3$. Then

$$\int [(M_{k,R} \mu)(x)]^{p(x)} dx \leq C \left(\mu(\mathbf{R}^N) + \int \mathcal{W}_{k, p(\cdot)}^\mu(x, 3R) d\mu(x) \right)$$

for all nonnegative measure μ such that $M(\mu, 3R) \leq M_0$, with a constant $C > 0$ depending only on N , C_d , K_R , p^+ , C_p and M_0 .

Proof. Let $0 < R < R_0/3$. For $0 < r < R$,

$$\begin{aligned} \int_0^{3R/2} k(t)^{p(x)} \mu(B(x, t))^{p(x)} \frac{dt}{t} &\geq \int_r^{3r/2} k(2r)^{p(x)} \mu(B(x, r))^{p(x)} \frac{dt}{t} \\ &\geq \log(3/2) C_d^{-p^+} [k(r) \mu(B(x, r))]^{p(x)}. \end{aligned}$$

Hence

$$[(M_{k,R}\mu)(x)]^{p(x)} \leq 3C_d^{p^+} \int_0^{3R/2} k(t)^{p(x)} \mu(B(x,t))^{p(x)} \frac{dt}{t},$$

and so

$$\int [(M_{k,R}\mu)(x)]^{p(x)} dx \leq 3C_d^{p^+} \int_0^{3R/2} \left(\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx \right) \frac{dt}{t}.$$

Now,

$$\begin{aligned} & \int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx \\ &= \int k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} \left(\int \chi_{B(x,t)}(y) d\mu(y) \right) dx \\ &= \int \left(\int \chi_{B(y,t)}(x) k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} dx \right) d\mu(y) \\ &= \int \left(\int_{B(y,t)} k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} dx \right) d\mu(y). \end{aligned}$$

As in the proof of Theorem 1, we have

$$[k(t)\mu(B(x,t))]^{p(x)-1} \leq C [k(t)\mu(B(x,t))]^{p(y)-1} \leq C [k(t)\mu(B(y,2t))]^{p(y)-1}$$

whenever $|x - y| < t < 3R/2$, $M(\mu, 3R) \leq M_0$ and $k(t)\mu(B(x,t)) \geq 1$, where constants C depend only on N , C_d , K_{3R} , p^+ , C_p and M_0 . Thus,

$$\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} dx \leq |B(0,t)| \left(k(t)\mu(\mathbf{R}^N) + C \int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} d\mu(y) \right).$$

Therefore

$$\begin{aligned} & \int [(M_{k,R}\mu)(x)]^{p(x)} dx \\ & \leq C \left(\mu(\mathbf{R}^N) \int_0^{3R/2} k(t)t^{N-1} dt + \int_0^{3R/2} t^{N-1} \left(\int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} d\mu(y) \right) dt \right) \\ & \leq C \left(\mu(\mathbf{R}^N) + \int \left(\int_0^{3R/2} k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} t^{N-1} dt \right) d\mu(y) \right) \\ & \leq C \left(\mu(\mathbf{R}^N) + \int \mathcal{W}_{k,p(\cdot)}^\mu(y, 3R) d\mu(y) \right) \end{aligned}$$

with constants C depending only on N , C_d , K_{3R} , p^+ , C_p and M_0 .

4. Estimate of $p(\cdot)$ -norm of convolution potential by $p(\cdot)$ -norm of k -maximal function

The example given in the Remark in section 2 also shows that the following (modular) inequality *does not hold* whenever $p(\cdot)$ is continuous and non-constant:

$$\int_{\mathbf{R}^N} (G_\alpha * \mu)^{p(x)} dx \leq C \int_{\mathbf{R}^N} (M_{\alpha,R} \mu)^{p(x)} dx.$$

However, we obtain norm inequality under an additional conditions on $p(x)$:

Theorem 3. *Suppose $k(r)$ in addition satisfies*

$$(k.3) \quad \int_1^\infty k(r) r^{N-1} dr < \infty;$$

$$(k.4) \quad \text{There is a constant } C_k > 0 \text{ such that}$$

$$\int_0^r k(t) t^{N-1} dt \leq C_k r^N k(r) \quad \text{for } 0 < r < R_0;$$

and suppose $p(x)$ in addition satisfies (P3) (log-Hölder continuity at ∞)

$$|p(x) - p(y)| \leq \frac{C_\infty}{\log(e + |x|)} \quad \text{for } |y| > |x|.$$

Then, for $0 < R < R_0/2$,

$$\|k * \mu\|_{p(\cdot)} \leq C \|M_{k,R} \mu\|_{p(\cdot)}$$

with a constant $C > 0$ depending only on $N, C_d, C_k, k(R), K, p^+, p^-, C_{lh}, C_\infty$ and R .

Note that the Bessel kernel G_α satisfies (k.3) and (k.4).

To prove Theorem 3, given $R > 0$, let

$$k_R(r) = k(r) \chi_{(0,R)}(r) \quad \text{and} \quad \tilde{k}_R(r) = k(r) \chi_{[R,\infty)}(r).$$

We treat $k_R * \mu$ and $\tilde{k}_R * \mu$ separately. First, we show

Proposition 1. *Suppose $k(r)$ satisfies (k.1), (k.2) and (k.4), and suppose $p(x)$ satisfies (P1), (P2) and (P3). Then, for $0 < R < R_0/2$,*

$$\|k_R * \mu\|_{p(\cdot)} \leq C \|M_{k,R} \mu\|_{p(\cdot)}$$

with a constant $C > 0$ depending only on $N, C_d, C_k, k(R), p^+, p^-, C_{lh}, C_\infty$ and R .

We prove this proposition applying the following theorem due to D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez [CFMP]:

C-F-M-P Theorem. *Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions on $\mathbf{R}^N$. Suppose that for some $p_0, 0 < p_0 < \infty$,*

$$\int_{\mathbf{R}^N} f(x)^{p_0} w(x) dx \leq C_0 \int_{\mathbf{R}^N} g(x)^{p_0} w(x) dx$$

for all $(f, g) \in \mathcal{F}$ and for all A_1 -weights w , where C_0 depends only on p_0 and the A_1 -constant of w . Let $p(\cdot)$ satisfy (P1), (P2) and (P3), and assume further that $p^- > p_0$. Then

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}$$

for all $(f, g) \in \mathcal{F}$.

Remark. In [CFMP], the last phrase in the above theorem is “for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbf{R}^N)$ ”. By examining its proof, we see that $g \in L^{p(\cdot)}(\mathbf{R}^N)$ (i.e., $\|g\|_{p(\cdot)} < \infty$) implies $f \in L^{p(\cdot)}(\mathbf{R}^N)$, and hence we do not need “such that $f \in L^{p(\cdot)}(\mathbf{R}^N)$ ”.

Thus the proof of Proposition 1 is reduced to the verification of

Proposition 1'. Let $1 < q < \infty$. Under the assumptions on k in Proposition 1, for $0 < R < R_0/2$,

$$\int_{\mathbf{R}^N} (k_R * \mu)^q w \, dx \leq C \int_{\mathbf{R}^N} (M_{k,R} \mu)^q w \, dx$$

for all A_1 -weights w , where C depends only on N , q , C_d , C_k , R and the A_1 -constant of w .

In the case $k(x) = G_\alpha$, this proposition is given in [T]. For general kernels k , we can prove this proposition by combining the arguments given in [T] and [AE, Part II]. Since our setting is different from either of them, we here give details of a proof.

First we recall some properties of A_1 -weights w . w is, by definition, a nonnegative locally integrable function on $\mathbf{R}^N$ such that

$$\int_B w(x) \, dx \leq A_1 |B| \operatorname{ess\,inf}_B w$$

for every ball (or cube) B . The constant A_1 is called the A_1 -constant of w . For a measurable set E in $\mathbf{R}^N$, we write $w(E) = \int_E w(x) \, dx$. An A_1 -weight satisfies the A_∞ -condition:

$$w(E) \leq C_w \left(\frac{|E|}{|Q|} \right)^\sigma w(Q) \quad (4.1)$$

for every cube Q and every measurable subset E of Q , where $C_w > 0$ and $\sigma > 0$ are constants depending only on N and the A_1 -constant of w (see, e.g., [T; Theorem 1.2.9] or [HKM; Chap.15]).

The following is the key lemma (cf. [T; Lemma 3.1.3] and [AE; Lemma 4.3.2]):

Lemma 2. Suppose $k(r)$ satisfies (k.1), (k.2) and (k.4). Let $0 < R < R_0/2$ and w be an A_1 -weight. Set $a = 4C_d^2$. Then for every $\eta > 0$ there exists $\varepsilon \in (0, 1]$, depending only on N , the A_1 -constant of w , R , C_d , C_k and η , such that

$$\begin{aligned} & w(\{x; (k_R * \mu)(x) > a\lambda\}) \\ & \leq \eta w(\{x; (k_R * \mu)(x) > \lambda\}) + w(\{x; (M_{k,R} \mu)(x) > \varepsilon\lambda\}) \end{aligned}$$

for all $\lambda > 0$.

Proof. For $\lambda > 0$, let

$$E_\lambda = \{x; (k_R * \mu)(x) > \lambda\}.$$

It is an open set, since $k_R * \mu$ is lower-semicontinuous. Let $\{Q_j\}$ be the Whitney decomposition of E_λ into closed dyadic cubes; namely, the interiors of Q_j and $Q_{j'}$ are disjoint for $j \neq j'$, $E_\lambda = \bigcup_j Q_j$ and

$$\text{diam } Q_j \leq \text{dist}(Q_j, E_\lambda^c) \leq 4 \text{diam } Q_j$$

for each j . If $\text{diam } Q_j > R/8$, then subdivide it into dyadic cubes with diameter $\leq R/8$ but $> R/16$. We denote this modified decomposition by $\{Q_j\}$ again.

Let $Q \in \{Q_j\}$, $d = \text{diam } Q$ and let $B = B(x_Q, 6d)$, where x_Q be the center of Q . Note that $8d \leq R$. Let $\mu_1 = \mu|_B$ and $\mu_2 = \mu - \mu_1$. For every $x \in Q$, $B \subset B(x, 7d)$. Hence,

$$\begin{aligned} \int_Q (k_R * \mu_1)(\xi) d\xi &= \int_Q \left(\int_B k_R(\xi - y) d\mu(y) \right) d\xi = \int_B \left(\int_Q k_R(\xi - y) d\xi \right) d\mu(y) \\ &\leq \int_{B(x, 7d)} \left(\int_{B(0, 7d)} k(\xi) d\xi \right) d\mu(y) \\ &= C(N) \left(\int_0^{7d} k(t) t^{N-1} dt \right) \mu(B(x, 7d)) \\ &\leq C_1(N, C_k) |Q| k(7d) \mu(B(x, 7d)) \quad (\text{by (k.4)}) \\ &\leq C_1(N, C_k) |Q| (M_{k,R} \mu)(x). \end{aligned}$$

Let an A_1 -weight w and $\eta > 0$ be given. Then, by (4.1), we can find $\varepsilon \in (0, 1]$ depending only on η , N , C_d , C_k and the A_1 -constant of w such that if $E \subset Q$ and $|E| \leq C_1(N, C_k)(2\varepsilon/a)|Q|$ then $w(E) \leq \eta w(Q)$. If there exists $x \in Q$ such that $(M_{k,R} \mu)(x) \leq \varepsilon \lambda$, then the above inequalities imply

$$\begin{aligned} &|\{\xi \in Q; (k_R * \mu_1)(\xi) > \frac{a}{2} \lambda\}| \\ &\leq \frac{2}{a\lambda} \int_Q (k_R * \mu_1)(\xi) d\xi \leq C_1(N, C_k)(2\varepsilon/a)|Q|, \end{aligned}$$

so that

$$w(\{\xi \in Q; (k_R * \mu_1)(\xi) > \frac{a}{2} \lambda\}) \leq \eta w(Q).$$

Thus,

$$w(\{x \in Q; (k_R * \mu_1)(x) > \frac{a}{2} \lambda, (M_{k,R} \mu)(x) \leq \varepsilon \lambda\}) \leq \eta w(Q) \quad (4.2).$$

Next, we show

$$\begin{aligned} &\{x \in Q; (k_R * \mu)(x) > a\lambda, (M_{k,R} \mu)(x) \leq \varepsilon \lambda\} \\ &\subset \{x \in Q; (k_R * \mu_1)(x) > \frac{a}{2} \lambda, (M_{k,R} \mu)(x) \leq \varepsilon \lambda\} \end{aligned} \quad (4.3)$$

If Q is one of undivided Whitney cubes, then $\text{dist}(Q, E_\lambda^c) \leq 4d$, so that $B \cap E_\lambda^c \neq \emptyset$. Let $x' \in B \cap E_\lambda^c$. Note that $d \leq R/8$, so that $12d < R_0$. If $x \in Q$ and $y \in B^c$, then

$|x - x'| \leq 7d$ and $|x - y| \geq 5d$, so that $|x - y| \geq (5/12)|x' - y|$. Hence, if $x \in Q$ and $(M_{k,R}\mu)(x) \leq \varepsilon\lambda$, then

$$\begin{aligned} (k_R * \mu_2)(x) &= \int_{B(x,R)} k(x-y) d\mu_2(y) \leq \int_{B(x,R)} k((5/12)(x' - y)) d\mu_2(y) \\ &\leq C_d^2 \int_{B(x,R)} k(x' - y) d\mu_2(y) \\ &\leq C_d^2 \int_{B(x',R)} k(x' - y) d\mu(y) + C_d^2 \int_{B(x,R) \setminus B(x',R)} k(x' - y) d\mu(y) \\ &\leq C_d^2 (k_R * \mu)(x') + C_d^2 k(R) \mu(B(x, R)) \\ &\leq C_d^2 (1 + \varepsilon) \lambda \leq 2C_d^2 \lambda \leq \frac{a}{2} \lambda. \end{aligned}$$

Thus we have (4.3) in this case.

Next let Q be one of divided cubes. Recall that $R/16 < d \leq R/8$. If $x \in Q$ and $y \in B^c$, then $|y - x| \geq 5d > (5/16)R > R/4$. Hence,

$$\begin{aligned} (k_R * \mu_2)(x) &\leq \int_{\{R/4 < |y-x| < R\}} k(x-y) d\mu(y) \\ &\leq k(R/4) \mu(B(x, R)) \leq C_d^2 k(R) \mu(B(x, R)) \leq \frac{a}{2} (M_{k,R}\mu)(x). \end{aligned}$$

Hence, if $(M_{k,R}\mu)(x) \leq \varepsilon\lambda$, then

$$(k_R * \mu_2)(x) \leq \frac{a}{2} \varepsilon \lambda \leq \frac{a}{2} \lambda,$$

which implies (4.3).

Now, from (4.2) and (4.3) we see that

$$\begin{aligned} w(\{x \in Q; (k_R * \mu)(x) > a\lambda\}) \\ \leq \eta w(Q) + w(\{x \in Q; (M_{k,R}\mu)(x) > \varepsilon\lambda\}). \end{aligned}$$

for every $Q \in \{Q_j\}$. Summing up over all Q , we obtain Lemma 2.

Proof of Proposition 1': Let a be as in the above lemma and E_λ be as in the above proof, i.e., $E_\lambda = \{x; (k_R * \mu)(x) > \lambda\}$ ($\lambda > 0$). First assume that μ has compact support. Then, $k_R * \mu$ has compact support, too, and hence $\lambda \mapsto w(E_\lambda)$ is a bounded function on $(0, \infty)$. Applying Lemma 2 with $\eta = a^{-q}/2$, we have, for any $L > 0$,

$$\begin{aligned} \int_0^{aL} w(E_\lambda) \lambda^{q-1} d\lambda &= a^q \int_0^L w(E_{a\lambda}) \lambda^{q-1} d\lambda \\ &\leq \frac{1}{2} \int_0^L w(E_\lambda) \lambda^{q-1} d\lambda + a^q \int_0^L w(\{x; (M_{k,R}\mu)(x) > \varepsilon\lambda\}) \lambda^{q-1} d\lambda \end{aligned}$$

with $\varepsilon > 0$ depending only on N , the A_1 -constant of w , R , C_d and C_k . Hence,

$$\int_0^{aL} w(E_\lambda) \lambda^{q-1} d\lambda \leq 2a^q \varepsilon^{-q} \int_0^{\varepsilon L} w(\{x; (M_{k,R}\mu)(x) > \lambda\}) \lambda^{q-1} d\lambda.$$

Now, letting $L \rightarrow \infty$, we have

$$\int_{\mathbf{R}^N} (k_R * \mu)^q w \, dx \leq 2a^q \varepsilon^{-q} \int_{\mathbf{R}^N} (M_{k,R} \mu)^q w \, dx.$$

If μ does not have compact support, let $\mu_m = \chi_{B(0,m)} \mu$ and apply the above result to μ_m , and then let $m \rightarrow \infty$. Since $k_R * \mu_m \uparrow k_R * \mu$, the required result follows by the monotone convergence theorem.

To treat $\tilde{k}_R * \mu$, we prepare another lemma. For nonnegative measure μ on $\mathbf{R}^N$ and $R > 0$, let

$$\widetilde{M}_R \mu(x) = \sup_{r \geq R} r^{-N} \mu(B(x, r)).$$

Lemma 3. *If $k(R) > 0$, then*

$$\widetilde{M}_R \mu \leq C(N, R, k(R)) \mathcal{M}(M_{k,R} \mu),$$

where, $\mathcal{M}(f)$ denotes the Hardy-Littlewood maximal function of f .

Proof. Fix $x \in \mathbf{R}^N$ and let $r \geq R > 0$. We can find a finite number of $y_j \in B(x, r)$ such that

$$B(x, r) \subset \bigcup_j B(y_j, R/2) \quad \text{and} \quad \sum_i \chi_{B(y_j, R/2)} \leq A(N) < \infty.$$

If $y \in B(y_j, R/2)$, then $B(y_j, R/2) \subset B(y, R)$, so that

$$\mu(B(y_j, R/2)) \leq \mu(B(y, R)) \leq \frac{(M_{k,R} \mu)(y)}{k(R)}.$$

Since $B(y_j, R/2) \subset B(x, r + R/2) \subset B(x, 2r)$,

$$\begin{aligned} \mu(B(x, r)) &\leq \sum_j \mu(B(y_j, R/2)) \\ &\leq \frac{1}{k(R)|B(0, R/2)|} \sum_j \int_{B(y_j, R/2)} (M_{k,R} \mu)(y) \, dy \\ &\leq \frac{2^N A(N)}{k(R)|B(0, R)|} \int_{B(x, 2r)} (M_{k,R} \mu)(y) \, dy \\ &\leq \frac{4^N A(N)}{k(R)R^N} r^N \mathcal{M}(M_{k,R} \mu)(x), \end{aligned}$$

so that

$$r^{-N} \mu(B(x, r)) \leq C(N, R, k(R)) \mathcal{M}(M_{k,R} \mu)(x)$$

for $r \geq R$. Thus, we obtain the required estimate.

Proposition 2. *Suppose $k(r)$ satisfies (k.1), (k.2) and (k.3), and suppose $p(x)$ satisfies (P1), (P2) and (P3). Then, for $0 < R < R_0$,*

$$\|\tilde{k}_R * \mu\|_{p(\cdot)} \leq C \|M_{k,R} \mu\|_{p(\cdot)}$$

with a constant $C > 0$ depending only on $N, R, k(R), C_{lh}, C_\infty, p^+, p^-$ and

$$K := \int_0^\infty k(r)r^{N-1} dt.$$

Proof.

$$\begin{aligned} (\tilde{k}_R * \mu)(x) &= \int_{\mathbf{R}^N \setminus B(x, R)} k(x - y) d\mu(y) \\ &= \int_{[R, \infty)} k(r) d[\mu(B(x, \cdot))](r) \\ &\leq \limsup_{r \rightarrow \infty} k(r)\mu(B(x, r)) + \int_{(R, \infty)} \mu(B(x, r)) d(-k)(r). \end{aligned}$$

Note that (k.3) implies that $k(r) \leq r^{-N}$ for $r \geq r_0$. Thus, if $r > \max(r_0, R)$, we have

$$k(r)\mu(B(x, r)) \leq r^N k(r)(\widetilde{M}_R \mu)(x) \leq (\widetilde{M}_R \mu)(x).$$

Hence,

$$\limsup_{r \rightarrow \infty} k(r)\mu(B(x, r)) \leq (\widetilde{M}_R \mu)(x).$$

On the other hand,

$$\begin{aligned} \int_{(R, \infty)} \mu(B(x, r)) d(-k)(r) &\leq \left(\int_{(R, \infty)} r^N d(-k)(r) \right) (\widetilde{M}_R \mu)(x) \\ &\leq \left(R^N k(R) + N \int_R^\infty k(r)r^{N-1} dr \right) (\widetilde{M}_R \mu)(x) \\ &\leq (R^N k(R) + NK) (\widetilde{M}_R \mu)(x). \end{aligned}$$

Hence

$$(\tilde{k}_R * \mu)(x) \leq C(N, R, k(R), K)(\widetilde{M}_R \mu)(x).$$

Thus, by Lemma 2, we have

$$(\tilde{k}_R * \mu)(x) \leq C\mathcal{M}(M_{k, R} \mu)(x)$$

with a constant $C = C(N, R, k(R), K) > 0$, which implies

$$\|\tilde{k}_R * \mu\|_{p(\cdot)} \leq C\|\mathcal{M}(M_{k, R} \mu)\|_{p(\cdot)}$$

with $C = C(N, R, k(R), K, p^+) > 0$. Now, under our assumptions on $p(\cdot)$, we know (see [CFN]) that

$$\|\mathcal{M}(f)\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)},$$

and hence we obtain the required estimate.

Combining Propositions 1 and 2, we obtain Theorem 3.

From Theorems 1, 2 and 3, we derive

Corollary 1. Suppose $p(\cdot)$ satisfies (P1), (P2) and (P3), and $k(r)$ satisfies (k.1), (k.2), (k.3) and (k.4). Then, for nonnegative measures μ in $\mathbf{R}^N$ with $\mu(\mathbf{R}^N) < \infty$,

$$k * \mu \in L^{p(\cdot)}(\mathbf{R}^N) \quad \text{if and only if} \quad \int \mathcal{W}_{k,p(\cdot)}^\mu(x, R) d\mu(x) < \infty.$$

It is known (see [GHN]) that if $p(\cdot)$ satisfies (P1), (P2) and (P3), then

$$W^{m,p(\cdot)}(\mathbf{R}^N) = \{u = G_m * f; f \in L^{p(\cdot)}(\mathbf{R}^N)\}$$

for $m \in \mathbf{N}$. Thus we can state

Corollary 2. If $p(\cdot)$ satisfies (P1), (P2) and (P3), then for nonnegative measures μ on $\mathbf{R}^N$ with $\mu(\mathbf{R}^N) < \infty$,

$$\mu \in (W^{m,p(\cdot)}(\mathbf{R}^N))^* \quad \text{if and only if} \quad \int \mathcal{W}_{m,p(\cdot)}^\mu(x, R) d\mu(x) < \infty$$

for $m \in \mathbf{N}$ with $0 < m < N$.

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