

Note on Transitive Representations of Generalized Inverse $\ast$ -Semigroups¹

ISAMU INATA (稲田 勇)
TERUO IMAOKA (今岡 輝男)

Abstract

In [1], we obtained that an effective representation of a locally [generalized] inverse $\ast$ -semigroup S is the sum of a uniquely determined family of transitive representations of S . In this paper, we will determine a transitive representation of a generalized inverse $\ast$ -semigroup by using right ω -cosets. This is a generalization of Schein's result [5] for inverse semigroups.

1 Introduction

A semigroup S with a unary operation $\ast : S \rightarrow S$ is called a *regular $\ast$ -semigroup* if it satisfies (i) $(x^\ast)^\ast = x$; (ii) $(xy)^\ast = y^\ast x^\ast$; (iii) $xx^\ast x = x$. Let S be a regular $\ast$ -semigroup. An idempotent e in S is called a *projection* if $e^\ast = e$. Denote the sets of idempotents and projections of S by $E(S)$ and $P(S)$, respectively.

Let S be a regular $\ast$ -semigroup. If eSe is an inverse semigroup, for every $e \in E(S)$, S is called a *locally inverse $\ast$ -semigroup*. If $E(S)$ is a normal band, that is, it satisfies the identity $xyzx = xzyx$, S is called a *generalized inverse $\ast$ -semigroup*. A regular $\ast$ -semigroup S is a generalized inverse $\ast$ -semigroup if and only if it is a locally inverse $\ast$ -semigroup and $E(S)$ forms a band.

Result 1.1 [3] *Let S be a regular $\ast$ -semigroup. Define a relation $\leq$ on S by*

$$a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).$$

Then $\leq$ is a partial order on S satisfying that $a \leq b$ implies $a^\ast \leq b^\ast$. If S is a generalized inverse $\ast$ -semigroup, then $\leq$ is compatible.

For a subset A of a regular $\ast$ -semigroup S , the set

$$A\omega = \{x \in S : \text{there exists } a \in A \text{ such that } a \leq x\}$$

is called the *closure* of A . The following statements are easily verified.

¹This is the abstract and details will be published elsewhere.

$$(1) A \subseteq A\omega; \quad (2) A \subseteq B \Rightarrow A\omega \subseteq B\omega; \quad (3) (A\omega)\omega = A\omega.$$

We say that A is *closed* if $A\omega = A$.

Lemma 1.2 *If H is a regular $*$ -subsemigroup of a generalized inverse $*$ -semigroup S , then $H\omega$ is a closed generalized inverse $*$ -subsemigroup of S .*

Let S be a regular $*$ -semigroup and H a regular $*$ -subsemigroup of S . If an element a in S satisfies $aa^* \in H$, then $(Ha)\omega$ is called a *right ω -coset* of H .

Lemma 1.3 *Let S be a generalized inverse $*$ -semigroup, and let $(Ha)\omega$ and $(Hb)\omega$ be right ω -cosets of a regular $*$ -subsemigroup H of S . Then*

$$(Ha)\omega \subseteq (Hb)\omega \iff a \in (Hb)\omega.$$

A non-empty set X with its reflexive and symmetric relation σ is called an ι -set, and denoted by $(X; \sigma)$. If σ is transitive, that is, it is an equivalence relation, then $(X; \sigma)$ is called a *transitive ι -set*.

Let $(X; \sigma)$ be an ι -set. A subset A of X is called an ι -single subset if, for any $x \in X$, there exists at most one element $y \in A$ such that $(x, y) \in \sigma$. If $(X; \sigma)$ is a transitive ι -set, A is an ι -single subset if and only if it satisfies that

$$(a, b) \in \sigma \ (a, b \in A) \implies a = b.$$

A mapping α in the symmetric inverse semigroup $\mathcal{I}_X$ is called a *partial one-to-one ι -mapping* of $(X; \sigma)$ if $d(\alpha)$ and $r(\alpha)$ are both ι -single subsets of $(X; \sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α , respectively. Denote the set of all partial one-to-one ι -mappings of $(X; \sigma)$ by $\mathcal{LI}_{(X; \sigma)}$. If σ is transitive, we denote it by $\mathcal{GI}_{(X; \sigma)}$ instead of $\mathcal{LI}_{(X; \sigma)}$. For any $\alpha, \beta \in \mathcal{LI}_{(X; \sigma)}$, denote $\theta_{\alpha, \beta}$ by

$$\theta_{\alpha, \beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\} = (r(\alpha) \times d(\beta)) \cap \sigma.$$

Since a subset of an ι -single subset is also an ι -single subset, $\theta_{\alpha, \beta} \in \mathcal{LI}_{(X; \sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{LI}_{(X; \sigma)}\}$. Define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{LI}_{(X; \sigma)}$ as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $\mathcal{LI}_{(X; \sigma)}(\circ, *)$ by $\mathcal{LI}_{(X; \sigma)}(\mathcal{M})$ or simply by $\mathcal{LI}_{(X; \sigma)}$. In this paper, we use $\mathcal{LI}_{(X; \sigma)}$ rather than $\mathcal{LI}_{(X; \sigma)}(\mathcal{M})$.

Result 1.4 [4] *For an ι -set $(X; \sigma)$, we have the following:*

(1) *The $*$ -groupoid $\mathcal{LI}_{(X; \sigma)}$, defined above, is a locally inverse $*$ -semigroup. Moreover, any locally inverse $*$ -semigroup can be embedded (up to $*$ -isomorphism) in $\mathcal{LI}_{(X; \sigma)}$ on some ι -set $(X; \sigma)$.*

(2) *$E(\mathcal{LI}_{(X; \sigma)}) = \mathcal{M}$ and $P(\mathcal{LI}_{(X; \sigma)}) = \{1_A : A \text{ is an } \iota\text{-single subset of } (X; \sigma)\}$.*

(3) *If $(X; \sigma)$ is a transitive ι -set, then $\mathcal{LI}_{(X; \sigma)}$ is a generalized inverse $*$ -semigroup. Moreover, any generalized inverse $*$ -semigroup can be embedded (up to $*$ -isomorphism) in $\mathcal{GI}_{(X; \sigma)}$ on some transitive ι -set $(X; \sigma)$.*

(4) *If σ is the identity relation on X , then $\mathcal{LI}_{(X; \sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X .*

We call $\mathcal{LI}_{(X; \sigma)} [\mathcal{GI}_{(X; \sigma)}]$ the ι -symmetric locally [generalized] inverse $*$ -semigroup on the ι -set [the transitive ι -set] $(X; \sigma)$ with the structure sandwich set $\mathcal{M}$.

Result 1.5 [1] *Let H be a locally [generalized] inverse $*$ -subsemigroup of $\mathcal{LI}_{(X; \sigma)}$ [$\mathcal{GI}_{(X; \sigma)}$] on a [transitive] ι -set $(X; \sigma)$, and define a relation τ_H on X by*

$$(x, y) \in \tau_H \iff \text{there exists } \alpha \in H \text{ such that } x \in d(\alpha) \text{ and } x\alpha = y.$$

Then τ_H is a symmetric and transitive relation on X .

The subset $\{x \in X : (x, x) \in \tau_H\} = d(\tau_H)$, say, of X is called the *domain* of τ_H . If $d(\tau_H) = X$, that is, τ_H is an equivalence relation on X , then H is said to be *effective*. If τ_H is the universal relation on X , then H is said to be *transitive*.

A representation $\phi : S \rightarrow \mathcal{LI}_{(X; \sigma)}$ of a locally inverse $*$ -semigroup S is called a *effective [transitive] representation* if $S\phi$ is an effective [transitive] locally inverse $*$ -subsemigroup of $\mathcal{LI}_{(X; \sigma)}$. Similarly, the effectivity and the transitivity for a representation $\phi : S \rightarrow \mathcal{GI}_{(X; \sigma)}$ of a generalized inverse $*$ -semigroup S are defined.

Result 1.6 [1] *An effective representation of a locally [generalized] inverse $*$ -semigroup S is the sum of a uniquely determined family of transitive representations of S .*

The purpose of this paper is to characterize a transitive representation of a generalized inverse $*$ -semigroup. The notation and the terminology are those of [1] and [2], unless otherwise stated.

2 Transitive representations

Let S be a generalized inverse $*$ -semigroup, and let $(X; \sigma)$ be a transitive ι -set and $\psi : S \rightarrow \mathcal{GI}_{(X; \sigma)}$ ($s \mapsto \psi^s$) a transitive representation of S . Fix an element z in X and set

$$H = \{s \in S : z\psi^s = z\}.$$

Lemma 2.1 *The set H , defined above, is a closed generalized inverse $*$ -subsemigroup of S .*

Define a relation δ on S by

$$\delta = \{(a, b) \in S \times S : z\psi^a = z\psi^b\}.$$

We also assume that $(a, b) \in \delta$ if $z \notin d(\psi^a) \cup d(\psi^b)$.

Lemma 2.2 *The relation δ , defined above, is a right congruence on S satisfying the following conditions:*

- (1) $\delta \cap (H \times H) = H \times H$,
- (2) For $a \in S$ and $h \in H$, $(a, h) \in \delta$ implies $a \in H$.

Let $\mathcal{X}$ be the set of all right ω -cosets of H . Define a relation $\sim$ on $\mathcal{X}$ by

$$(Ha)\omega \sim (Hb)\omega \iff (a, b) \in \delta.$$

Lemma 2.3 *The relation $\sim$, defined above, is an equivalence relation on $\mathcal{X}$.*

Let $\mathcal{X}/\sim = \mathcal{Y}$, say, and denote the $\sim$ -class containing $(Ha)\omega$ by $(Ha)\tilde{\omega}$. For any $a \in S$, define a partial mapping ϕ_H^a on $\mathcal{Y}$ by

$$d(\phi_H^a) = \{(Hxaa^*)\tilde{\omega} : xaa^*x^* \in H\} \text{ and } \phi_H^a : (Hxaa^*)\tilde{\omega} \mapsto (Hxa)\tilde{\omega},$$

Lemma 2.4 *For any $a \in S$ and $(Ha)\tilde{\omega} \in \mathcal{Y}$, we have*

$$(Hx)\tilde{\omega} \in d(\phi_H^a) \iff (x, xaa^*) \in \delta$$

Lemma 2.5 *For any $a \in S$, $\phi_H^a \in \mathcal{I}_{\mathcal{Y}}$ and $(\phi_H^a)^{-1} = \phi_H^{a^*}$.*

Define a relation Ω on $\mathcal{Y}$ by

$$\Omega = \{((Hx)\tilde{\omega}, (Hy)\tilde{\omega}) : (Hx)\tilde{\omega}\phi_H^e = (Hy)\tilde{\omega} \text{ for some } e \in E(S)\}.$$

Lemma 2.6 *The relation Ω , defined above, is an equivalence relation on $\mathcal{Y}$, that is, $(\mathcal{Y}; \Omega)$ is a transitive ι -set.*

Now we can consider the ι -symmetric generalized inverse $*$ -semigroup $\mathcal{GI}_{(\mathcal{Y}; \Omega)}$ on the transitive ι -set $(\mathcal{Y}; \Omega)$.

Lemma 2.7 *For any $a \in S$, $d(\phi_H^a)$ and $r(\phi_H^a)$ are ι -single subsets of $(\mathcal{Y}; \Omega)$.*

Lemma 2.8 *For any $a, b \in S$, $\theta_{\phi_H^a, \phi_H^b} = \phi_H^{a^*abb^*}$.*

Lemma 2.9 *The mapping $\phi_H: S \rightarrow \mathcal{GI}_{(\mathcal{Y}; \Omega)}$ ($a \mapsto \phi_H^a$) is a transitive representation of S .*

Let $\varphi: S \rightarrow \mathcal{GI}_{(X; \sigma)}$ and $\xi: S \rightarrow \mathcal{GI}_{(Y; \tau)}$ be two representations of a generalized inverse $*$ -semigroup S . Then φ and ξ are *equivalent* if there exists a bijection $\theta: X \rightarrow Y$ such that, for $s \in S$ and $x \in X$,

$$x \in d(\varphi^s) \iff x\theta \in d(\xi^s) \text{ and } (x\varphi^s)\theta = (x\theta)\xi^s.$$

Lemma 2.10 *The transitive representation $\psi: S \rightarrow \mathcal{GI}_{(X; \sigma)}$ is equivalent to ϕ_H , defined above.*

From result 1.5, lemma 3.1 and 3.2, we obtain a following theorem.

Theorem 2.11 *Every effective representation of a generalized inverse $*$ -semigroup S is uniquely a sum of transitive representations ψ_α , each of which is equivalent to ϕ_{H_α} for some closed generalized inverse $*$ -subsemigroup H_α of S .*

References

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Department of Mathematics, Shimane University, Matsue 690-8504, Shimane, Japan

The first author's current address: Department of Information Science, Toho University, Funabashi 274-8510, Japan