

F-thresholds on toric rings

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Abstract

The poster is written about F-thresholds on toric rings. We consider some questions about them (Q1) using the formula for them on toric rings.

which is the main theorem in this poster. (S1)

In general, F-thresholds are the invariants of pairs $(R, \mathfrak{a})$ of rings R of char. $p > 0$ and ideals $\mathfrak{a}$.

On regular rings, they are jumping numbers of test ideals. (S2)

In that case, F-thresholds are analogous to jump. coeff. of multiplier ideals. (S1)

Connection between char. 0 and char. $p > 0$

Notation in this section

R_0 : a $\mathbb{Q}$ -Goren. normal local f.g. $\mathbb{Q}$ -algebra.
 $\mathfrak{a}_0 \subseteq R_0$ an ideal.

Assume $\exists R_0 \subseteq R_p$, $\mathfrak{a}_p = \mathfrak{a}_0 \cap R_p$
s.t. $R_0 \otimes \mathbb{Q} \cong R_p$, $\mathfrak{a}_0 R_p = \mathfrak{a}_p$
(i.e., R_p the fibre over generic point 0.)

$p \in \mathbb{Z}$ a prime number, $R_p := R_0 \otimes \mathbb{F}_p$, $\mathfrak{a}_p := \mathfrak{a}_0 R_p$
(i.e., R_p a fibre over closed point p .)

Note and Assum. about char. 0

$f: X' \rightarrow X := \text{Spec } R_0$: log resolution of $\mathfrak{a}_0$, that is,

f a proper birational, X' smooth,

$\exists D$: an effective div. on X' s.t.
 $\mathfrak{a}_0 \mathcal{O}_{X'} = \mathcal{O}_{X'}(-D)$, $\text{Supp}(D) = \text{SVC}$.

Definition 1 (multiplier ideals). $i \in \mathbb{R}_{\geq 0}$. Define the multiplier ideal $\mathfrak{a}_0^{(i)}$ by

$$J(\mathfrak{a}_0^{(i)}) := f_* \mathcal{O}_{X'}((K_{X'} - f^* K_X - iD)).$$

use $[D] = \sum [D_i]$, for $D = \sum d_i D_i$.

Definition 2 (jumping coefficients).

$$0 < \tau^1(\mathfrak{a}_0) < \tau^2(\mathfrak{a}_0) < \dots < \tau^r(\mathfrak{a}_0) < \dots$$

the jumping coefficients of $\mathfrak{a}_0$ if

$$J(\mathfrak{a}_0^{(i)}) = J(\mathfrak{a}_0^{(j)}), \forall i \in [\tau^r(\mathfrak{a}_0), \tau^{r+1}(\mathfrak{a}_0)).$$

Note and Assum. about char. $p > 0$

$F: R_p \rightarrow R_p$: the absolute Frobenius map,

R_p : the ring R_p viewed as an R_p -module via F^ ,

$S := R_p$: injective hull of the residue field of R_p .

$F_p: S \xrightarrow{F^{(p)}} S \otimes S$.

Definition 3 (test ideal). $i \in \mathbb{R}_{\geq 0}$. Define the test ideal of $\mathfrak{a}_p^{(i)}$ by

$$\tau(\mathfrak{a}_p^{(i)}) := \text{Ann}_{S_p} \mathfrak{a}_p^{(i)}.$$

we see $\mathfrak{a}_p^{(i)} \subseteq S \iff \exists n \neq 0 \in R_p$ s.t. $\mathfrak{a}_p^{(i)} \mathfrak{a}_p^{(n)} = 0$, $\forall n \geq 0$.

Theorem 4 ([HY03]). For fixed $i \in \mathbb{R}_{\geq 0}$, $\exists U \subseteq \text{Spec } S$ open into a t.

$$J(\mathfrak{a}_0^{(i)}) \cong \tau(\mathfrak{a}_p^{(i)}) \subseteq R_p, \forall p \in U.$$

We can define F-jumping coefficients as follows. p analog of jump. coeff.

Definition 5 (F-jumping coefficients).

$$0 < \tau^1(\mathfrak{a}_p) < \tau^2(\mathfrak{a}_p) < \dots < \tau^r(\mathfrak{a}_p) < \dots$$

are the F-jumping coefficients of $\mathfrak{a}_p$ if

$$\tau(\mathfrak{a}_p^{(i)}) = \tau(\mathfrak{a}_p^{(j)}), \forall i \in [\tau^r(\mathfrak{a}_p), \tau^{r+1}(\mathfrak{a}_p)).$$

Corollary 6 ([MTW06]).

$$\lim_{p \rightarrow \infty} \tau^i(\mathfrak{a}_p) = \tau^i(\mathfrak{a}_0), \forall i.$$

Question 1. $\tau^i(\mathfrak{a}_0) \in \mathbb{Q}$ is clear, since $K_{X'}$, $F^* K_X$, D : $\mathbb{Q}$ -div. Then $\tau^i(\mathfrak{a}_p) \in \mathbb{Q}$? (We consider this question at the end of this poster.)

Remark. In char. p , an another definition of F-jump. coeff. is given as the F-threshold in some rings. We give its def. and observations in the next section.

2 Connection between F-thresholds and F-jumping coefficients

Note and Assum.

R : Noether comm. reduced ring of char. $p > 0$, $\tau(R) = R$.

R : F -pure ($\mathfrak{a}^{[p]} \subseteq \mathfrak{a}$) F splits as R -module map.)

R : F -finite ($\mathfrak{a}^{[p]} \subseteq \mathfrak{a}$) R is a f.g. R -module.)

$\mathfrak{a} \subseteq R$: $\mathfrak{a}$ ideals s.t. $0 \neq \mathfrak{a} \subseteq \sqrt{R} \subseteq R$.

Definition 7. Define the F-threshold of $\mathfrak{a}$ w.r.t. J by

$$\tau^J(\mathfrak{a}) := \lim_{p \rightarrow \infty} \frac{\tau^J(\mathfrak{a}_p)}{p},$$

where $\tau^J(\mathfrak{a}_p) := \max\{i \in \mathbb{N} \mid \mathfrak{a}^{[p^i]} \subseteq J\}$

Example. $R = k[[X, Y]]$, $\mathfrak{a} = (X^2 + Y^2)$, $J = (X, Y)$. Then,

$$\tau^J(\mathfrak{a}) = \begin{cases} \frac{1}{2} & p \equiv 1 \pmod{4}, \\ \frac{1}{4} & p \equiv 3 \pmod{4}. \end{cases}$$

Theorem 8 ([MTW05], [BM06]). R : a regular ring. Then,

$$\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \leq i, \forall i \in \mathbb{R}_{\geq 0}.$$

$$\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \subseteq J, \forall J \subseteq R.$$

In particular,

$$\tau^J(\mathfrak{a}) = \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}).$$

Question 2. If R non-regular, then $\exists$ inequality bet. τ^J and $\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a})$?

Question 3. In particular, in which condition $\tau^J = \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a})$?

We consider the questions on toric rings at the next section.

3 On toric rings

Note and Assum.

$N = \mathbb{Z}^d$, $M = \text{Hom}(N, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$
 $(-, -): M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$

$\sigma = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_d$, a str. conv. rat. poly. cone of $N_{\mathbb{R}}$.

(v_i) : primitive

$\sigma^\vee := \{(u \in M_{\mathbb{R}}) \mid (u, v_i) \geq 0, \forall v_i \in \sigma\}$

$R = k[\sigma^\vee \cap M]$ a toric ring, k : a perfect field

$\mathfrak{a} = (X^{u_1}, \dots, X^{u_n})$ a monomial ideal of R

$\mathfrak{a}^\vee := \{(u \in \sigma^\vee \cap M) \mid (u, v_i) < 1\} \subseteq \sigma^\vee$

$P(\mathfrak{a}) := \text{Conv}\{(u \in \sigma^\vee \cap M) \mid X^u \in \mathfrak{a}\} \subseteq \sigma^\vee$

$Q(\mathfrak{a}) := \bigcup_{X^u \in \mathfrak{a}} u + \sigma^\vee \subseteq \sigma^\vee$

The following is the main theorem of this poster. Watanabe suggested the idea of the theorem.

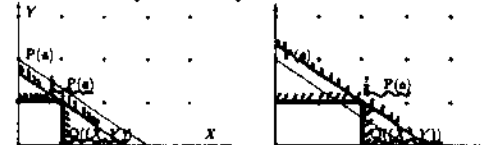
Theorem 9 (Main Theorem).

$$\tau^J(\mathfrak{a}) = \sup_{u \in \sigma^\vee \cap M} \lambda(u),$$

where $\lambda(u) := \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid u \in \lambda \cdot P(\mathfrak{a})\}$

Example. $R = k[X, Y]$, $\mathfrak{a} = (X^2, Y^2)$.

Then $\tau^J(\mathfrak{a}) = \frac{1}{2}$, $\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = \frac{1}{4}$.

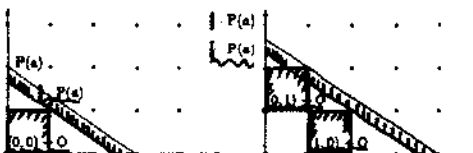


Theorem 10 (Howald type ([HY03], [B04])).

$$\tau^J(\mathfrak{a}) \geq \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \iff (u + \sigma) \cap (i \cdot P(\mathfrak{a})) \neq \emptyset$$

Example. $R = k[X, Y]$, $\mathfrak{a} = (X^2, Y^2)$. Then,

$$\tau^J(\mathfrak{a}) = \begin{cases} \frac{1}{2}, & i < \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} \leq i < \frac{3}{4}, \\ \frac{1}{4}, & i \geq \frac{3}{4}. \end{cases}$$



This implies a formula of F-jump. coeff.

Corollary 11.

$$\tau^J(\mathfrak{a}) = \sup_{u \in \sigma^\vee \cap M} \lambda(u) = \inf_{X^u \in \mathfrak{a}} \mu(u),$$

where $\mu(u) := \sup_{v \in \sigma^\vee \cap M} \lambda(u + v)$. In general, $i \in \mathbb{Z}_{\geq 0}$.

$$\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = \inf_{X^u \in \mathfrak{a}^{(i)}} \mu(u).$$

Answer of Q. 2

Corollary 12. R : a toric ring, $\mathfrak{a}$: a monomial ideal

$$\tau^J(\mathfrak{a}) \leq \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}), \forall i.$$

Sketch of pf. $\exists X^u \in \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \setminus \tau^J(\mathfrak{a})$, $u + \sigma \subseteq \sigma^\vee \setminus Q(\mathfrak{a}^{(i)})$. Then, $\tau^J(\mathfrak{a}) \leq \sup_{u \in \sigma^\vee \cap M} \lambda(u) \leq \sup_{u \in \sigma^\vee \cap M} \lambda(u) = \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a})$. $\square$

4 Applications

Part of answer of Q. 3

Proposition 13. R : a τ -Goren. toric ring ($\tau > 1$). Then,

$$\tau^J(\mathfrak{a}) < \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}), \forall \mathfrak{a} \text{ monomial ideal.}$$

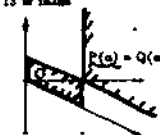
Sketch of pf. $\exists u \in \sigma^\vee \cap M$ s.t. $(u, v_i) = 1, \forall v_i \in \sigma$ and $u \notin M$. Then, $\tau^J(\mathfrak{a}) = \lambda(u) < (1 + \varepsilon)\lambda(u) \leq \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a})$. $\square$

Example. If R : Goren. ring, then Prop. 13 is false.

$R = k[X, Y, Z, Y^2 Y^{-1}]$: A_1 -Sing.

$\mathfrak{a} = (X)$. Then,

$$\tau^J(\mathfrak{a}) = 1, \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = (X), \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = 1$$



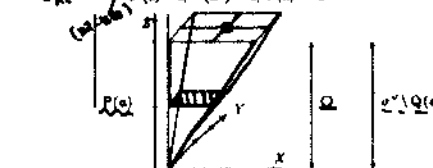
Part of answer of Q. 3

Proposition 14. R : a toric ring defined by a simplicial cone.

$\exists$ m -primary ideal $\mathfrak{a}$ s.t. $\tau^J(\mathfrak{a}) = \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \iff R$: regular ring

Example. If R defined by non-simplicial cone, then Prop. 14 is false. $R = k[X, Y, Z, XY^2, Z]$. $\mathfrak{a} = (X, Y, Z, XY^2, Z)$. Then,

$$\tau^J(\mathfrak{a}) = \frac{1}{2}, \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = \frac{1}{2}, \tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) = \frac{1}{2}$$



Remark. Known results about rationality of F-thresholds and F-jump. coeff.

[BM90]: R : a regular ring of essentially of finite type of k

$\implies \forall \mathfrak{a}$, $\tau^J(\mathfrak{a})$, $\tau^{\mathfrak{a}^{(i)}}(\mathfrak{a}) \in \mathbb{Q}$

[HM90]: R : a regular ring

$\implies \forall f \in R$, $\tau^J((f))$, $\tau^{\mathfrak{a}^{(i)}}((f)) \in \mathbb{Q}$

Part of Answer of Q. 1

Proposition 15. R : a toric ring defined by a simplicial cone,

$\mathfrak{a}$: a monomial ideal,

J : an m -primary monomial ideal.

$\implies \tau^J(\mathfrak{a}) \in \mathbb{Q}$

Sketch of pf. $\exists X^u \in \tau^J(\mathfrak{a})$, $u + \sigma \subseteq \sigma^\vee$.

$\implies \sigma^\vee \setminus Q(J)$: bounded,

$\implies \partial Q(J)$: bounded.

In general,

$$u, u' \in \sigma^\vee \implies \lambda(u) \leq \lambda(u + u').$$

Then,

$$\tau^J(\mathfrak{a}) = \max_{u \in \sigma^\vee \cap M} \lambda(u)$$

$\sigma^\vee$: simplicial $\implies$ the following set is a finite set of rational pt. as

