

Existence of Bounded Solutions to Linear Differential Equations (I)

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Abstract

We deal with linear differential equations of the form $dx/dt = Ax(t) + f(t)$ in a Banach space X , where A is the generator of a C_0 -semigroup on X and f is a periodic function. In this paper, we give a method to show the existence of bounded solutions and a structure of them. As results, we can obtain criteria for the existence of quasi-periodic, periodic, asymptotically periodic solutions.

1 Introduction

Let X be a Banach space and $\mathbb{R}$ the real line. In this paper we investigate criteria on the existence of bounded solutions to the linear differential equation of the form

$$\frac{d}{dt}u(t) = Au(t) + f(t). \quad (1)$$

Throughout the present paper we make the following assumption.

Assumption : $A : \mathcal{D}(A) \subset X \rightarrow X$ is the generator of a C_0 -semigroup $U(t)$, and $f : \mathbb{R} \rightarrow X$ is a τ -periodic function.

If $x(t)$ is a continuous function which satisfies the following equation

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)ds, \quad t \in \mathbb{R}_+ := [0, \infty), \quad (2)$$

then it is called a (mild) solution to Equation (1).

The purpose of this paper is to give criteria for the existence of bounded solutions and a structure of bounded solutions to Equation (1). The relationship between the existence of bounded solutions and the existence of τ -periodic solutions is characterized by the Massera type theorem. To complete the Massera type theorem, it is practically and theoretically important to show the existence of bounded solutions.

2 The existence of bounded solutions

In this section, we give criteria on the existence of bounded solutions to Equation (1). For a solution $x(t)$ of Equation (1) such that $x(0) = x_0$, $x(n\tau)$ is expressed as

$$x(n\tau) = U(n\tau)x_0 + S_n(U(\tau))b_f,$$

where

$$S_n(U(\tau)) = \sum_{k=0}^{n-1} U(k\tau), \quad b_f = \int_0^\tau U(\tau-s)f(s)ds.$$

The solution $x(t)$ of Equation (1) is bounded on $\mathbb{R}_+$ if and only if $x(n\tau)$, $n = 1, 2, \dots$, are bounded. If we take $x_0 = b_f$ (resp. $x_0 = 0$), then $x(n\tau) = S_{n+1}(U(\tau))b_f$ (resp. $x(n\tau) = S_n(U(\tau))b_f$). Hence a solution $x(t)$ of Equation (1) such that $x(0) = b_f$ (or $x(0) = 0$) is bounded on $\mathbb{R}_+$ if and only if

$$\limsup_{n \rightarrow \infty} \|S_n(U(\tau))b_f\| < \infty. \quad (3)$$

Based on this relation, we give criteria on the existence of bounded solutions to Equation (1). The following result can be found in [2].

Theorem 2.1 *Let Z be a subset of X . Assume that for any $x \in Z$ there exists a positive number $\alpha_x > 0$ such that $\|U_Z(n\tau)x\| \leq \alpha_x$ for all $n \in \mathbb{N}$. Then the following three statements are equivalent :*

- 1) *Every solution $x(t)$ of Equation (1) such that $x(0) \in Z$ is bounded on $\mathbb{R}_+$.*
- 2) *Condition (3) holds.*
- 3)

$$\limsup_{t \rightarrow \infty} \left\| \int_0^t U(t-s)f(s)ds \right\| < \infty.$$

2.1 The case of finite dimension

We will check Condition (3) for the case where $X = \mathbb{C}^m$, $A = (a_{ij})$, an $m \times m$ matrix. Let the characteristic polynomial of A be factorized as follows :

$$\Phi(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_\ell)^{m_\ell},$$

where $\lambda_1, \dots, \lambda_\ell$ are the distinct roots of $\Phi(\lambda)$, and $m_1 + \dots + m_\ell = m$. Put $\lambda_p =: a_p + ib_p$, $a_p, b_p \in \mathbb{R}$. Denote by $P_p : \mathbb{C}^m \rightarrow M_p$ the projection corresponding to the direct sum decomposition $\mathbb{C}^m = M_1 \oplus \dots \oplus M_\ell$, where $M_p := \mathcal{N}((A - \lambda_p I)^{n_p})$ is the generalized eigenspace corresponding to λ_p .

Theorem 2.2 *For $\tau > 0$, $b \in \mathbb{C}^m$, the vector sequence $\{S_n\}$, given as*

$$S_n := S_n(e^{\tau A})b = \sum_{k=0}^{n-1} e^{k\tau A}b,$$

is bounded if and only if for every $p = 1, \dots, \ell$, the following conditions hold :

- (i) If $a_p > 0$, then $P_p b = 0$.
- (ii) The case where $a_p = 0$;
 - (a) if $\tau b_p \in 2\pi\mathbb{Z}$, then $P_p b = 0$.
 - (b) if $\tau b_p \notin 2\pi\mathbb{Z}$, then $P_p b \in \mathcal{N}(A - \lambda_p I)$.
- (iii) If $a_p < 0$, then $P_p b$ is arbitrary.

To prove the theorem, the following lemma is needed.

Lemma 2.3 Let $Q(t)$ be a vector in $\mathbb{C}^n$, whose component is a polynomial of t , and $\lambda = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$. The vector sequence $\{R_n\}$, given as

$$R_n = \sum_{j=1}^n e^{j\lambda} Q(j),$$

is bounded if and only if the following conditions hold :

- (i) In the case where $a > 0$, $Q(t) \equiv 0$.
- (ii) In the case where $a = 0$, if $b \in 2\pi\mathbb{Z}$, then $Q(t) \equiv 0$; if $b \notin 2\pi\mathbb{Z}$, then $Q(t) = c$ (a constant vector).
- (iii) In the case where $a < 0$, $Q(t)$ is arbitrary.

Proof Set $z = e^\lambda$. Then $R_n = \sum_{j=1}^n z^j Q(j)$. If $\{R_n\}$ is bounded, the sequence $\{R_n - R_{n-1}\}_{n=2}^\infty$ is also bounded and

$$\|R_n - R_{n-1}\| = \|z^n Q(n)\| = e^{na} \|Q(n)\|.$$

Hence in the case where $a > 0$, $Q(t) \equiv 0$ if and only if $\{R_n\}$ is bounded. If $a < 0$, then $Q(t)$ is arbitrary if and only if $\{R_n\}$ is bounded. So, we see the case where $a = 0$. We note that

$$\|R_n - R_{n-1}\| = \|Q(n)\|.$$

From the definition of $Q(t)$ it follows that $\{Q(n)\}$ is bounded if and only if $Q(t) = c$ (a constant vector). If $b \in 2\pi\mathbb{Z}$, then $z = 1$, and so, $R_n = nc$. Namely, $c = 0$ if and only if $\{R_n\}$ is bounded. If $b \notin 2\pi\mathbb{Z}$, then $z \neq 1$. Hence we have

$$\|R_n\| = \left\| \frac{1 - z^{n+1}}{1 - z} c \right\| \leq \frac{2}{1 - |z|} \|c\|,$$

which implies that $\{R_n\}$ is bounded. Therefore the proof of the lemma is finished.

□

The proof of Theorem 2.2 $\mathbb{C}^m$ is decomposed as

$$\mathbb{C}^m = M_1 \oplus \dots \oplus M_\ell.$$

Take a circle C_p centered at λ_p , whose radius is sufficiently small and its disk does not contain the other points $\lambda_q, q \neq p$. Then the projection P_p is expressed as

$$P_p = \frac{1}{2\pi} \int_{C_p} (\lambda I - A)^{-1} d\lambda.$$

Then P_p is a bounded operator having the following properties :

$$P_p \mathbb{C}^m = M_p, \quad AP_p = P_p A, \quad P_p P_q = 0 \quad (p \neq q), \quad P_p^2 = P_p, \quad P_1 + P_2 + \cdots + P_\ell = I.$$

Furthermore, e^{tA} is decomposed as follows :

$$e^{tA} = \sum_{p=1}^{\ell} e^{\lambda_p t} Q_p(t) P_p, \quad Q_p(t) = \sum_{k=0}^{n_p-1} \frac{t^k}{k!} (A - \lambda_p I)^k.$$

Using those facts, we have

$$S_n(e^{\tau A}) = \sum_{j=0}^{n-1} e^{j\tau A} = \sum_{j=0}^{n-1} \sum_{p=1}^{\ell} e^{j\tau \lambda_p} Q_p(j\tau) P_p = \sum_{p=1}^{\ell} \sum_{j=0}^{n-1} e^{j\tau \lambda_p} Q_p(j\tau) P_p.$$

Since $P_p P_q = 0 (p \neq q)$, $P_p^2 = P_p$, it follows that

$$P_p S_n(e^{\tau A}) b = \sum_{j=0}^{n-1} e^{j\tau \lambda_p} Q_p(j\tau) P_p b := R_n^p.$$

Hence, the sequence $\{S_n(e^{\tau A})b\}_{n \in \mathbb{N}}$ is bounded if and only if for every $p = 1, \dots, \ell$, the sequence $\{R_n^p\}_{n \in \mathbb{N}}$ is bounded. Since

$$Q_p(j\tau) P_p b = \sum_{k=0}^{n_p-1} j^k \frac{\tau^k}{k!} (A - \lambda_p I)^k P_p b, \quad p \in \{1, 2, \dots, \ell\},$$

we have, by Lemma 2.3, the following facts.

- i) If $a_p > 0$, then $Q_p(j\tau) P_p b \equiv 0$, from which we have $P_p b = 0$.
- ii) If $a_p < 0$, then $Q_p(j\tau) P_p b$ is arbitrary ; that is, $P_p b$ is also arbitrary.
- iii) Let $a_p = 0$. If $\tau b_p \in 2\pi\mathbb{Z}$, then $Q_p(j\tau) P_p b \equiv 0$; that is, $P_p b = 0$. If $\tau b_p \notin 2\pi\mathbb{Z}$, then $Q_p(j\tau) P_p b = c$ (constant vector). Notice that $(A - \lambda_p I) P_p b = 0$ if and only if $Q_p(j\tau) P_p b = P_p b$. This means that $c = P_p b$; namely, $P_p b \in (A - \lambda_p I)$. Hence we obtain the conclusion of the theorem. $\square$

2.2 The case of infinite demension

We consider the condition 2) in Theorem 2.1 from the point of view of the spectrum of A in Equation (1).

Suppose that $\omega_e(U) := \lim_{t \rightarrow \infty} t^{-1} \log \alpha(U(t)) < 0$. Then $\exp(t\omega_e(U)) < 1$, ($t > 0$). This implies that there exists a $\gamma > 0$ such that $\sigma(U(t)) \cap \{z : |z| \geq e^{-\gamma t}\}$ and $\sigma(A) \cap \{z : \Re z \geq -\gamma\}$ consist of finite number of normal eigenvalues and that $\sigma(A) \cap \{z : -\gamma < \Re z < 0\} = \emptyset$. Hence $\sigma(A) \cap \{z : \Re z > -\gamma\}$ consists of finite number of normal eigenvalues $\lambda_j, j = 1, 2, \dots, r$, with nonnegative real parts. Set $a_j = \Re \lambda_j, b_j = \Im \lambda_j$. Assume that $a_j = 0$ for $1 \leq j \leq q$ ($\leq r$) and that $a_j > 0$ for $q < j \leq r$. Thus 1 is a normal eigenvalue of $U(\tau)$. Set $\sigma_0(A) = \{\lambda_j : 1 \leq j \leq q\}$ and $\sigma_+(A) = \{\lambda_j : q+1 \leq j \leq r\}$. We understand that $\sigma_+(A) = \emptyset$, provided $q = r$.

Let M_j be the generalized eigenspace of A corresponding to λ_j . Since λ_j is a normal eigenvalue of A , $n_j := \dim M_j$ is finite and there exists a positive integer m_j such that $M_j = \mathcal{N}((\lambda_j I - A)^{m_j})$. The space X is decomposed as follows:

$$X = Y \oplus Z, \quad Z = M_0 \oplus M_+, \quad Y = \bigcup_{j=1}^r \mathcal{R}((\lambda_j I - A)^{m_j}),$$

$$M_0 = M_1 \oplus \dots \oplus M_q, \quad M_+ = M_{q+1} \oplus \dots \oplus M_r$$

The subspaces Y and M_j are closed in X and $\dim Z = n_1 + n_2 + \dots + n_r =: d$. If we define P_j as in Section 2.1, where $(\lambda I - A)^{-1}$ is understood as the resolvent operator $R(\lambda, A)$, then $P_j : X \rightarrow M_j$ are projections such that $P_j P_k = \delta_{jk} P_j$ and $AP_j x = P_j Ax$ for $x \in D(A)$. If we set

$$P = P_1 + P_2 + \dots + P_r, \quad P_0 = I - P, \quad (4)$$

then $P : X \rightarrow Z$ and $P_0 : X \rightarrow Y$ are projections. Y, M_j and Z are invariant subspaces of $U(t)$.

Since $U(t)x = U(t)P_0 x + U(t)Px$, we have

$$\|S_n(U(\tau))x\| \leq \|S_n(U(\tau))P_0 x\| + \|S_n(U(\tau))Px\|.$$

It follows from Proposition 4.15 in [6] that there are an $\varepsilon_0 > 0$ and a constant $K \geq 1$ such that

$$\|U(t)P_0 x\| \leq K e^{-\varepsilon_0 t} \|P_0 x\| \quad \text{for all } x \in X, t \geq 0.$$

Hence we have

$$\|S_n(U(\tau))P_0 x\| \leq K \sum_{k=0}^{n-1} e^{-\varepsilon_0 k \tau} \|P_0 x\| \leq \frac{K}{1 - e^{-\varepsilon_0 \tau}} \|P_0 x\| < \infty.$$

As a result, $\|S_n(U(\tau))x\|, n = 1, 2, \dots$, are bounded if and only if $\|S_n(U(\tau))Px\|, n = 1, 2, \dots$, are bounded.

Since $d = \dim Z < \infty$, A_Z , the restriction of A to Z , is regarded as a $d \times d$ matrix with eigenvalues $\lambda_j, 1 \leq j \leq r$, and $U(t)Px = \exp(tA_Z)Px$ for all $x \in X$. Thus we have the following result from Theorem 2.2.

Theorem 2.4 Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then $S_n(U(\tau))x$, $n = 1, 2, \dots$, are bounded if and only if the following conditions hold:

- (i) For $q < j \leq r$, $P_j x = 0$.
- (ii) For $1 \leq j \leq q$,
 - (a) if $\tau b_j \in 2\pi\mathbb{Z}$, $P_j x = 0$;
 - (b) if $\tau b_j \notin 2\pi\mathbb{Z}$, $P_j x \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.

Corollary 2.5 Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then the solution $x(t)$ of Equation (1) such that $x(0) = b_f$ is bounded on $\mathbb{R}_+$ if and only if the following conditions hold :

- (i) For $q < j \leq r$, $P_j b_f = 0$.
- (ii) For $1 \leq j \leq q$,
 - (a) if $\tau b_j \in 2\pi\mathbb{Z}$, $P_j b_f = 0$;
 - (b) if $\tau b_j \notin 2\pi\mathbb{Z}$, $P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.

Combining Theorem 2.4 with Theorem 2.1, we obtain the following result.

Corollary 2.6 Suppose $U(t)$ is a bounded C_0 -semigroup such that $\omega_e(U) < 0$. Then every solution of Equation (1) is bounded on $\mathbb{R}_+$ if and only if for $j = 1, \dots, q$ the following conditions hold :

- (a) If $\tau b_j \in 2\pi\mathbb{Z}$, then $P_j b_f = 0$;
- (b) If $\tau b_j \notin 2\pi\mathbb{Z}$, then $P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.

Using Corollary 2.5, we obtain the following result on the existence of a τ -periodic solution to Equation (1).

Theorem 2.7 Assume that $\omega_e(U) < 0$ and that b_f satisfies the conditions (i) and (ii) in Corollary 2.5. Then Equation (1) has a τ -periodic solution.

Proof Since $\{S_n(U(\tau))b_f\}_n$ is bounded, it follows from Corollary 2.5 that the solution $x(t)$ of Equation (1) such that $x(0) = b_f$ is bounded on $\mathbb{R}_+$. Since 1 is a normal point of $U(\tau)$, we see that $\mathcal{R}(I - U(\tau))$ is a closed subspace of $\mathbf{X}$. Therefore the fixed point theorem by Chow and Hale implies that Equation (1) has a τ -periodic solution. $\square$

3 A structure of bounded solutions

In this section we will give a structure of bounded solutions obtained in Section 2. Throughout this section, we assume the following conditions :

- 1) $\omega_e(U) < 0$,
- 2) 1 is a normal eigenvalue of $U(\tau)$,
- 3)

$$\limsup_{n \rightarrow \infty} \|S_n(U(\tau))b_f\| < \infty.$$

We use the same notations for the points in $\sigma(A) \cap \{z : \Re z \geq 0\}$ as in Section 2. Denote by $\mathbf{SP}$ and $\mathbf{SP}_X$ the set of all τ -periodic solutions of Equation (1) and the set of all solutions of the equation $(I - U(\tau))x = b_f$, respectively. They are affine spaces. If we take an vector $x_0 \in \mathbf{SP}_X$, then $\mathbf{SP}_X = x_0 + \mathcal{N}(I - U(\tau))$. Set

$$\mathbf{N}_0 = \mathcal{N}(A - ib_1 I) \oplus \cdots \oplus \mathcal{N}(A - ib_q I).$$

Suppose that $\tau b_j \in 2\pi\mathbb{Z}$ for $1 \leq j \leq p(\leq q)$ and that $\tau b_j \notin 2\pi\mathbb{Z}$ for $p+1 \leq j \leq q$. Since 1 is a normal eigenvalue of $U(\tau)$, it follows that, $p \geq 1$ and

$$\mathcal{N}(I - U(\tau)) = \mathcal{N}(A - ib_1 I) \oplus \cdots \oplus \mathcal{N}(A - ib_p I) =: \mathbf{N}_{00} \subset \mathbf{N}_0, \quad (5)$$

cf. Proposition 4.13 in [6]. Hence $\mathbf{SP}_X = x_0 + \mathbf{N}_{00}$. Let P and P_0 be defined by (4).

Proposition 3.1 *The following results hold.*

- 1) $U(t)x$ is bounded for $t \geq 0$ if and only if $x \in \mathbf{Y} \oplus \mathbf{N}_0$; that is,
 - (i) $P_j x \in \mathcal{N}(A - \lambda_j I)$ for $1 \leq j \leq q$.
 - (ii) $P_j x = 0$ for $q+1 \leq j \leq r$.
- 2) $U(t)x$ is τ -periodic if and only if $x \in \mathbf{N}_{00}$; that is,
 - (iii) $P_0 x = 0$
 - (iv) $P_j x \in \mathcal{N}(A - ib_j I)$ for $1 \leq j \leq p$.
 - (v) $P_j x = 0$ for $p+1 \leq j \leq r$.

Proof $U(t)x$ is bounded if and only if $P_0 U(t)x = U(t)P_0 x$ and $PU(t)x = U(t)Px$ are bounded. Since $P_0 U(t)x$ is bounded for all $x \in \mathbf{X}$, it suffices to check the boundedness of $U(t)Px$. Since $P = P_1 + \cdots + P_r$, $\{U(n\tau)Px\}_n$ is bounded if and only if $\{U(n\tau)P_j x\}_n, j = 1, 2, \dots, r$, are bounded. Since

$$\|U(t)P_j x\| = \|e^{(a_j + ib_j)t} \sum_{m=0}^{m_j-1} \frac{t^m}{m!} (A - \lambda_j I)^m P_j x\| = e^{a_j t} \left\| \sum_{m=0}^{m_j-1} \frac{t^m}{m!} (A - \lambda_j I)^m P_j x \right\|$$

for $1 \leq j \leq r$, the assertion 1) is easily derived from this relation.

Similarly $U(t)x$ is τ -periodic if and only if $P_0 U(t)x = U(t)P_0 x$ and $PU(t)x = U(t)Px$ are τ -periodic. If $U(t)P_0 x$ is τ -periodic, we have $P_0 x = U(n\tau)P_0 x$ for all $n = 1, 2, \dots$. Since $U(t)P_0 x \rightarrow 0$ as $t \rightarrow \infty$, we have $P_0 x = 0$.

If $U(t)Px$ is τ -periodic, $U(t)P_j x$ is τ -periodic for $1 \leq j \leq r$. It follows at first that $P_j x \in \mathcal{N}(A - ib_j I)$ for $1 \leq j \leq q$ and that $P_j x = 0$ for $q+1 \leq j \leq r$. If $p+1 \leq j \leq q$, and if $P_j x \neq 0$, then $U(t)P_j x = e^{ib_j t} P_j x$ is not τ -periodic. Consequently, $x \in \mathbf{N}_{00}$. Clearly, if $x \in \mathbf{N}_{00}$, $U(t)x$ is τ -periodic. $\square$

Theorem 3.2 *A solution $x(t)$ of Equation (1) is bounded on $\mathbb{R}_+$ if and only if $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$.*

Proof The solution $x(t)$ is written as Equation (2) in Introduction. Notice that the integral in this equation is bounded if and only if Condition 3) holds. Hence $x(t)$ is bounded if and only if $U(t)x(0)$ is bounded. From Proposition 3.1 we have the result in the theorem. $\square$

The following result follows from Condition 3) and Theorem 3.2.

Corollary 3.3 *The following assertions hold true.*

- 1) $\text{SP}_{\mathbf{X}} \neq \emptyset$, $\text{SP}_{\mathbf{X}} \subset \mathbf{Y} \oplus \mathbf{N}_0$.
- 2) $\mathbf{M}(b_f) \subset \mathbf{Y} \oplus \mathbf{N}_0$, where $\mathbf{M}(b_f)$ is the linear space generated by $\{U(n\tau)b_f\}_{n=0}^{\infty}$.

Theorem 3.4 *Take a τ -periodic solution $u_0(t)$ of Equation (1). Then the following statements are valid.*

- 1) *Any bounded solution $x(t)$ of Equation (1) on $\mathbb{R}_+$ is written as*

$$x(t) = u_0(t) + \sum_{j=1}^p e^{ib_j t} x_j + \sum_{j=p+1}^q e^{ib_j t} x_j + U_{\mathbf{Y}}(t)y_0,$$

with some vectors $x_j \in \mathcal{N}(A - ib_j I)$, $1 \leq j \leq q$, and $y_0 \in \mathbf{Y}$.

- 2) *Any τ -periodic solution $u(t)$ of Equation (1) is written as*

$$u(t) = u_0(t) + \sum_{j=1}^p e^{ib_j t} u_j,$$

with some vectors $u_j \in \mathcal{N}(A - ib_j I)$, $1 \leq j \leq p$.

Proof Since $u_0(t)$ is the τ -periodic solution of Equation (1), $u_0(0) \in \text{SP}_{\mathbf{X}} \subset \mathbf{Y} \oplus \mathbf{N}_0$. Let $x(t)$ be a bounded solution of Equation (1) on $\mathbb{R}_+$. Then it follows from Theorem 3.2 that $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$. Therefore $x(0) - u_0(0) \in \mathbf{Y} \oplus \mathbf{N}_0$; it is expressed as

$$x(0) - u_0(0) = \sum_{j=1}^q x_j + y_0,$$

where $x_j \in \mathcal{N}(ib_j I - A)$ and $y_0 \in \mathbf{Y}$. Since $x(t) - u_0(t)$ is a solution of the homogeneous equation, we have

$$\begin{aligned} x(t) - u_0(t) &= U(t)[x(0) - u_0(0)] \\ &= U(t) \sum_{j=1}^p x_j + U(t) \sum_{j=p+1}^q x_j + U_{\mathbf{Y}}(t)y_0 \\ &= \sum_{j=1}^p e^{ib_j t} x_j + \sum_{j=p+1}^q e^{ib_j t} x_j + U_{\mathbf{Y}}(t)y_0, \end{aligned}$$

as required. The remainder is obvious. Therefore the proof of the theorem is completed. $\square$

Corollary 3.5 *The following statements are valid.*

- 1) *There is a τ -periodic solution to Equation (1) ; $\dim SP = p$.*
- 2) *There is an asymptotically τ -periodic solution to Equation (1).*
- 3) *If $p < q$, there is an asymptotically quasi-periodic solution to Equation (1).*
- 4) *If $p = q$, every bounded solution is an asymptotically τ -periodic solution to Equation (1).*
- 5) *If $p = q = r$, and if $R(\lambda, A) := (\lambda I - A)^{-1}$ has a pole of order 1 at $\lambda = \lambda_j$, $1 \leq j \leq p$, then all τ -periodic solutions of Equation (1) are stable.*

Proof Statements 1,2,3,4) are trivial. Assume that the conditions in 5) hold. Then we have $X = Y \oplus N_0$. Hence there is a positive constant H such that $\|U(t)x\| \leq H\|x\|$ for $t \geq 0, x \in X$. Let $u_0(t)$ be a τ -periodic solution of Equation (1). Then for every solution $u(t)$, $u(t) - u_0(t)$ is a solution of the homogeneous equation. Hence $u(t) - u_0(t) = U(t)(u(0) - u_0(0))$, which implies $\|u(t) - u_0(t)\| \leq H\|u(0) - u_0(0)\|$ for $t \geq 0$. Therefore the assertion 5) is valid. $\square$

In a subsequent paper, we will consider the case where

$$\limsup_{n \rightarrow \infty} \|S_n(U(\tau))b_f\| = \infty.$$

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