

On the Asymptotic Behaviors of the Generalized
Spherical Functions on Semisimple Lie Groups

Masaaki Eguchi, Hiroshima University.

(広島大 総合科 江口 正 晃)

1. INTRODUCTION. This is an abstract note of [6]. Though the main result of this note is correct for more general Lie groups, called of class H , than semisimple Lie groups, for simplicity we restrict ourselves to semisimple case. So we now assume that G is a connected semisimple Lie group of the non-compact type with finite center. Let $G=KAN$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ the corresponding Iwasawa decompositions of G and its Lie algebra $\mathfrak{g}$.

The Eisenstein integrals, that is the matrix elements of representations of principal series for G , play an essential role in harmonic analysis on G . Therefore it is very important to know their asymptotic behaviors. In fact, the leading terms of the asymptotic expansions of them are the Harish-Chandra C -functions and, as is well known, they relate closely with the Plancherel measure on G [8,9,10,11]. Moreover, we need to know their behaviors of higher order to carry out further analysis on G . In this note we focus our attention on the Harish-Chandra expansions of Eisenstein integrals and their coefficients. For the zonal spherical function $\phi_\nu(x) = \int_K e^{(\nu-\rho)(H(xk))} dk$ ($x \in G$), when $x = h$ varying in the positive Weyl chamber A^+ of A , $\phi_\nu(h)$ is expanded into an infinite series by Harish-Chandra [7] as follows:

$$\phi_v(h) = e^{-\rho(\log h)} \sum_{s \in W} c(sv) \Phi(sv:h),$$

$$\Phi(v:h) = \sum_{\lambda \in L} \Gamma_{\lambda}(v-\rho) e^{(v-\rho)(\log h)} \quad (h \in A^+).$$

Here $c(\cdot)$ is the Harish-Chandra c -function and Γ_{λ} ($\lambda \in L$) are the coefficients. In his paper [14] Gangolli gave a remarkable estimate for these coefficients. The purpose of this note is to give the Gangolli estimate for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals.

2. PRELIMINARIES. Let M be the centralizer of A in K . Denote by F_R and F_C the real dual space of $\mathfrak{a}$ and its complexification, respectively. Write $F = (-1)^{1/2} F_R$. Let $\tau = (\tau_1, \tau_2)$ be a double unitary representation of K on a finite dimensional Hilbert space V . Put

$$V_M = \{v \in V; \tau_1(m)v = v\tau_2(m) \text{ for any } m \in M\}.$$

Then the following integral is called the Eisenstein integral or the generalized spherical function:

$$(1) \quad E(v:v;x) = \int_K \tau_1(\kappa(xk)) v \tau_2(k^{-1}) e^{(v-\rho)(H(xk))} dk.$$

Let ω be the Casimir operator. Then E satisfies the following differential equation:

$$(2) \quad E(v:v;x;\omega) = \{\langle v, v \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_m)\} E(v:v;x).$$

Here ω_m denotes the Casimir operator on M .

Let θ be the Cartan involution of $\mathfrak{g}$ with respect to k . Let $\mathfrak{s}$ be the subspace of all $X \in \mathfrak{g}$ such that $\theta(X) = -X$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{h}$ and put $\mathfrak{h}_k = \mathfrak{h} \cap k$. Let Σ denote the set of all roots of $(\mathfrak{g}, \mathfrak{h})$ and $\Sigma_0 = \{\alpha_1, \dots, \alpha_{\ell}\}$ the set of all simple roots in Σ . We consider the lexicographic order in F_R defined by $\alpha_1, \dots, \alpha_{\ell}$ and fix a com-

patible order in the dual space of $\mathfrak{h}^* = \mathfrak{a} + (-1)^{1/2} \mathfrak{h}_k$. Let Δ_+ denote the set of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ such that $\tilde{\alpha} = \alpha|_{\mathfrak{a}} \neq 0$. For each $\alpha \in \Delta_+$ define the element $Q_{\alpha} \in \mathfrak{a}$ so that $\tilde{\alpha}(H) = \langle Q_{\alpha}, H \rangle$ for all $H \in \mathfrak{a}$. For each $\alpha \in \Delta_+$, choose the root vectors $X_{\pm\alpha} \in \mathfrak{g}_C^{\pm\alpha}$ so that $B(X_{\alpha}, X_{-\alpha}) = 1$, B denoting the Killing form, and write them as $X_{\pm\alpha} = Y_{\pm\alpha} + Z_{\pm\alpha}$ ($Y_{\pm\alpha} \in \mathfrak{k}_C$, $Z_{\pm\alpha} \in \mathfrak{p}_C$).

The following lemma gives the radial part of the Casimir operator.

Lemma 1. Denote the radial part of ω (resp. ω_m) by $\mathcal{R}(\omega)$ (resp. $\mathcal{R}(\omega_m)$).

Then we have

$$\begin{aligned}
 (3) \quad \mathcal{R}(\omega) &= \mathcal{R}(\omega_m) + \sum_{i=1}^2 H_i^2 + \sum_{\alpha \in \Delta_+} \coth(\alpha) Q_{\alpha} \\
 &\quad - 2 \sum_{\alpha \in \Delta_+} (\text{sh}(\alpha))^{-2} \{1 \otimes 1 \otimes Y_{\alpha} Y_{-\alpha} + Y_{\alpha} Y_{-\alpha} \otimes 1 \otimes 1\} \\
 &\quad + 4 \sum_{\alpha \in \Delta_+} (\text{sh}(\alpha))^{-1} \coth(\alpha) (Y_{\alpha} \otimes 1 \otimes Y_{-\alpha}).
 \end{aligned}$$

3. THE HARISH-CHANDRA EXPANSION AND THE MAIN THEOREM. Put

$$L = \{\lambda = n_1 \alpha_1 + \dots + n_{\ell} \alpha_{\ell}, \quad n_i \in \mathbb{Z}_+, i=1, \dots, \ell\}.$$

If $\lambda = n_1 \alpha_1 + \dots + n_{\ell} \alpha_{\ell} \in L$, $m(\lambda) = n_1 + \dots + n_{\ell}$ is called its level. Let γ be the endomorphism of $\text{Hom}_C(V_M, V_M)$ defined by

$$\gamma(T) = [\tau_2(\omega_m), T], \quad T \in \text{Hom}_C(V_M, V_M).$$

Let $\gamma_1, \dots, \gamma_t$ be the set of all distinct eigenvalues of γ with multiplicities $m_1, \dots, m_t$, respectively. It is known that they are all real. We assume that $\gamma_1 < \dots < \gamma_t$. We review the definition of Γ_{λ} ($\lambda \in L$). Let $\Gamma_0 \equiv 1$. For $\lambda \neq 0$, let Γ_{λ} be the function on F_C with values in $\text{Hom}_C(V_M, V_M)$ given by the following recursion formula:

$$\begin{aligned}
(4) \quad \{2\lambda - \langle \lambda, \lambda - 2\rho \rangle\} \Gamma_{\lambda} - \gamma(\Gamma_{\lambda}) &= 2 \sum_{\alpha \in \Delta_+} \sum_{n \geq 1} \{\tilde{\alpha} - \langle \tilde{\alpha}, \lambda - 2n\tilde{\alpha} \rangle\} \Gamma_{\lambda - 2n\tilde{\alpha}} \\
&+ 8 \sum_{\alpha \in \Delta_+} \sum_{n \geq 1} (2n-1) \{\tau_1(Y_{\alpha}) \tau_2(Y_{-\alpha})\} \Gamma_{\lambda - (2n-1)\tilde{\alpha}} \\
&- 8 \sum_{\alpha \in \Delta_+} \sum_{n \geq 1} n \{\tau_1(Y_{\alpha} Y_{-\alpha}) + \tau_2(Y_{\alpha} Y_{-\alpha})\} \Gamma_{\lambda - 2n\tilde{\alpha}},
\end{aligned}$$

where $\Gamma_{\lambda} \equiv 0$ if $\lambda \notin L$.

Denote by L' the set of $\lambda \neq 0$ in L . For each i ($1 \leq i \leq t$) and $\lambda \in L'$ put

$$\sigma_{\lambda, i} = \{v \in F_C; 2\langle \lambda, v \rangle = \langle \lambda, \lambda \rangle + \gamma_i\}.$$

Let T denote the complement of the set $\bigcup_{\lambda \in L'} \bigcup_i \sigma_{\lambda, i}$ in F_C . Let T' be the set of all $v \in F'_C$ such that $wv \in T$ for all $w \in W$, W denoting the Weyl group of (G, A) .

Theorem 2. (Harish-Chandra).

(i) For a fixed $v \in T$,

$$h \mapsto \Phi(v; h) = \sum_{\lambda \in L} \Gamma_{\lambda} (v - \rho) h^{v - \lambda} \quad \text{is analytic on } A^+.$$

(ii) $\Phi(v; h; e^0 \cdot \mathcal{R}(\omega) \cdot e^{-\rho}) = (\langle v, v \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_m)) \Phi(v; h).$

(iii) $h^0 E(v; v; h) = \sum_{w \in W} \Phi(wv; h) C(w; v) v, \quad v \in T',$

where $C(w; v)$ ($w \in W$) are certain meromorphic functions on F_C with values in $\text{Hom}_C(V_M, V_M)$.

Fix $a > 0$ and put

$$R(a) = \{\xi + \eta \in L'; \xi \in F, \eta \in F_R, -\eta + a\rho \in Cl(F_R^+)\}$$

We want to know the behavior of Γ_λ in the cone $R(a)$ and consider the following finite set (may be empty):

$$L'_1(a) = \{\lambda \in L' : -\langle \lambda, \lambda \rangle + 2a\langle \lambda, \rho \rangle - \gamma_1 \geq 0\}.$$

which is the set of $\lambda \in L'$ such that the determinant of the coefficient of Γ_λ in (4) takes value 0 in $R(a)$. Put

$$p_\lambda(v) \equiv 1 \quad \text{if } \lambda \notin L'_1(a);$$

$$p_\lambda(v) = \prod_{\substack{1 \leq i \leq t \\ d(a:\lambda) \leq -\gamma_i}} (2\langle \lambda, v \rangle - \langle \lambda, \lambda \rangle - \gamma_i)^{m_i},$$

$$d'(\lambda) = \sum_{1 \leq i \leq t, d(a:\lambda) \leq -\gamma_i} m_i$$

for $\lambda \in L'_1(a)$, where $d(a:\lambda) = \langle \lambda, \lambda \rangle - 2a\langle \lambda, \rho \rangle$. If $\lambda, \lambda' \in L$ and $\lambda - \lambda' \in L$ then we denote it by $\lambda \gg \lambda'$. We also put

$$P(v) = \prod_{\lambda \in L'_1(a)} p_\lambda(v), \quad d = \sum_{\lambda \in L'_1(a)} d'(\lambda) < +\infty;$$

$$P_\lambda(v) = \prod_{\substack{\lambda' \in L' \\ \lambda' \ll \lambda}} p_{\lambda'}(v), \quad d(\lambda) = \sum_{\substack{\lambda' \in L' \\ \lambda' \ll \lambda}} d'(\lambda')$$

for $\lambda \in L'_1(a)$. Then as is easily seen, all singularities of Γ_λ in the domain $R(a)$ concentrate on the polynomial $P(\lambda)$. The following result is the main theorem.

Theorem 3. There exist constants $D, d_1 > 0$, depending only on τ , which satisfy

$$\|P_\lambda(v) \Gamma_\lambda(v-\rho)\| \leq D(1 + |v| + m(\lambda))^{2d_{m(\lambda)}} d_1$$

uniformly in $\lambda \in L', v \in R(a)$.

4. A SKETCH OF THE PROOF OF THE THEOREM. We give in this section a sketch of the proof. We need the following lemma.

Lemma 4. Put $H = \log h$ ($h \in A^+$) and $\Delta(h) = h^{2\rho} \prod_{\alpha \in \Delta_+} (1 - h^{-2\alpha})$ ($h \in A$). Then

we have

$$\begin{aligned}
 (5) \quad & \Delta(h)^{1/2} \circ \mathcal{R}(\omega) \circ \Delta(h)^{-1/2} = \mathcal{R}(\omega_m) + \sum_{i=1}^{\ell} H_i^2 - \langle \rho, \rho \rangle \\
 & + \sum_{\alpha \in \Delta_+} \langle \tilde{\alpha}, \tilde{\alpha} \rangle \sum_{j \geq 1} j e^{-2j\alpha(H)} - \sum_{\substack{\alpha, \beta \in \Delta_+ \\ \alpha \neq \beta}} \langle \tilde{\alpha}, \tilde{\alpha} \rangle \sum_{\substack{j \geq 1 \\ k \geq 0}} e^{-2j\alpha(H) - 2k\beta(H)} \\
 & - 8 \sum_{\alpha \in \Delta_+} \sum_{j \geq 1} j e^{-2j\alpha(H)} (1 \otimes 1 \otimes Y_{\alpha} Y_{-\alpha} + Y_{\alpha} Y_{-\alpha} \otimes 1 \otimes 1) \\
 & + 8 \sum_{\alpha \in \Delta_+} \sum_{j \geq 0} (2j+1) e^{-(2j+1)\alpha(H)} (Y_{\alpha} \otimes 1 \otimes Y_{-\alpha}).
 \end{aligned}$$

The most important thing is that $\Delta(h)^{1/2} \circ \mathcal{R}(\omega) \circ \Delta(h)^{-1/2}$ in the lemma is an operator of the Sturm-Liouville type. If we consider the function Ψ given by the following in stead of Φ itself:

$$\Psi(v:h) = \Delta(h)^{1/2} h^{-\rho} \Phi(v:h) \quad h \in A^+,$$

then Ψ satisfies the following differential equation:

$$(6) \quad (\Delta(h)^{1/2} \circ \mathcal{R}(\omega) \circ \Delta(h)^{-1/2})_T = (\langle v, v \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_m)) \Psi.$$

Expand Ψ into the series

$$\Psi(v:h) = h^v \sum_{\lambda \in L} a_{\lambda}(v) h^{-\lambda} \quad h \in A^+.$$

Then, using Lemma 4, we obtain the recursion formula for $a_{\lambda}(v)$:

$$(7) \quad [2\langle \lambda, v \rangle - \langle \lambda, \lambda \rangle] a_{\lambda}(v) - \gamma(a_{\lambda}(v))$$

$$\begin{aligned}
&= \sum_{\alpha \in \Delta_+} [\langle \tilde{\alpha}, \tilde{\alpha} \rangle - 8F_\alpha] \sum_{j \geq 1} j a_{\lambda-2j\tilde{\alpha}}(v) - \sum_{\substack{\alpha, \beta \in \Delta_+ \\ \alpha \neq \beta}} \langle \tilde{\alpha}, \tilde{\beta} \rangle \sum_{\substack{j \geq 1 \\ k \geq 0}} a_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(v) \\
&\quad + 8 \sum_{\alpha \in \Delta_+} G_\alpha \sum_{j \geq 1} (2j-1) a_{\lambda-(2j-1)\tilde{\alpha}}(v),
\end{aligned}$$

where F_α and G_α are defined by

$$F_\alpha = \tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha}); \quad G_\alpha = \tau_1(Y) \circ \tau_2(Y_{-\alpha}).$$

We pay attention to the fact that all singularities of a_λ in the domain $R(a)$ are concentrated upon P_λ and put

$$Q_\lambda(v) = P_\lambda(v)(1 + |v| + |\lambda|)^{-2d(\lambda)} \quad \text{and} \quad q_\lambda(v) = p_\lambda(v)(1 + |v| + |\lambda|)^{-2d'(\lambda)}.$$

Moreover, we define $b_\lambda(v)$ for all $\lambda \in L$ by

$$\begin{aligned}
b_0(v) &\equiv 1 & \text{if } \lambda = 0; \\
b_\lambda(v) &= Q_\lambda(v)a_\lambda(v) & \text{if } \lambda \in L'.
\end{aligned}$$

Then we obtain the following recursion formula for $b_\lambda(v)$. We put $\gamma(\lambda:v) = (2\langle \lambda, v \rangle - \langle \lambda, \lambda \rangle)I - \gamma$.

$$\begin{aligned}
(8) \quad \gamma(\lambda:v)b_\lambda(v) &= \sum_{\alpha \in \Delta_+} [\langle \tilde{\alpha}, \tilde{\alpha} \rangle - 8F_\alpha] q_\lambda(v) \sum_{j \geq 1} Q_{\lambda,j}^1(v) b_{\lambda-2j\tilde{\alpha}}(v) \\
&\quad - \sum_{\substack{\alpha, \beta \in \Delta_+ \\ \alpha \neq \beta}} \langle \tilde{\alpha}, \tilde{\beta} \rangle q_\lambda(v) \sum_{\substack{j \geq 1 \\ k \geq 0}} Q_{\lambda,j,k}(v) b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(v) \\
&\quad + 8 \sum_{\alpha \in \Delta_+} G_\alpha q_\lambda(v) \sum_{j \geq 1} Q_{\lambda,j}^2(v) b_{\lambda-(2j-1)\tilde{\alpha}}(v).
\end{aligned}$$

Where the polynomials $Q_{\lambda,j}^1$, $Q_{\lambda,j,k}$ and $Q_{\lambda,j}^2$ are given by the relation

$$Q_{\lambda,j}^1(v) Q_{\lambda-2j\tilde{\alpha}}(v) = Q_{\lambda,j,k}(v) Q_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(v) = Q_{\lambda,j}^2(v) Q_{\lambda-(2j-1)\tilde{\alpha}}(v) = Q_\lambda(v) q_\lambda(v)^{-1}$$

An argument parallel to [14] leads to our assertion. For more detail, see [6].

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