

Coefficient conditions for certain classes concerning starlike functions of complex order

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Abstract

For functions $f(z)$ which are starlike of complex order b ($b \neq 0$) in the open unit disk $\mathbb{U}$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_0 = 0, a_1 = 1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $\mathbb{U}$. If $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$), then we say that $p(z)$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$).

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As usual, in the present investigation, we write

$$\mathcal{S}^* \equiv \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} \equiv \mathcal{K}(0).$$

Classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [5].

Next, a function $f(z) \in \mathcal{A}$ is called λ -spiral like of order α in $\mathbb{U}$ if and only if

$$\operatorname{Re} \left[e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right] > 0 \quad (z \in \mathbb{U})$$

for some real λ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$) and α ($0 \leq \alpha < 1$). We denote this class by $\mathcal{SP}(\lambda, \alpha)$.

Moreover, for some non-zero complex number b , we consider the subclasses $\mathcal{S}_b^*$ and $\mathcal{K}_b$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_b^* = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] > 0 \quad (b \neq 0; z \in \mathbb{U}) \right\}$$

and

$$\mathcal{K}_b = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} \right) \right] > 0 \quad (b \neq 0; z \in \mathbb{U}) \right\}.$$

If a function $f(z)$ belongs to the class $\mathcal{S}_b^*$ or $\mathcal{K}_b$, we say that $f(z)$ is starlike or convex of complex order b ($b \neq 0$), respectively. In [3], Nasr and Aouf introduced the class $\mathcal{S}_b^*$.

Then, we can see that

$$\mathcal{S}_{1-\alpha}^* = \mathcal{S}^*(\alpha), \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha) \quad \text{and} \quad \mathcal{S}_{(1-\alpha)e^{-i\lambda} \cos \lambda}^* = \mathcal{SP}(\lambda, \alpha).$$

Example 1.1

$$f(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^n (j+2(b-1))}{(n-1)!} z^n \in \mathcal{S}_b^* \quad (b \neq 0)$$

and

$$f(z) = \begin{cases} \frac{1 - (1-z)^{1-2b}}{1-2b} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^n (j+2(b-1))}{n!} z^n \in \mathcal{K}_b & \left(b \neq \frac{1}{2} \right) \\ \log \left(\frac{1}{1-z} \right) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n \in \mathcal{K}_{\frac{1}{2}} = \mathcal{K} \left(\frac{1}{2} \right). \end{cases}$$

We apply the following lemma to obtain our results.

Lemma 1.2 A function $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$) if and only if

$$p(z) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})$$

for all $|x| = 1$.

Then, by using Lemma 1.2, various conditions for starlike functions are studied. The following results are enumerated as the some examples.

Lemma 1.3 A function $f(z) \in \mathcal{A}$ is in $\mathcal{S}^*(\alpha)$ if and only if

$$(1.3) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \quad (z \in \mathbb{U}; |x| = 1)$$

where

$$A_n = \frac{n+1-2\alpha+(n-1)x}{2-2\alpha} a_n.$$

Silverman, Silvia, and Telage [6] have given

Remark 1.4 The relation (1.3) of Lemma 1.3 is equivalent to

$$\frac{1}{z} \left(f(z) * \frac{z + \frac{x+2\alpha-1}{2-2\alpha} z^2}{(1-z)^2} \right) \neq 0 \quad (z \in \mathbb{U}, |x| = 1)$$

where $*$ means the convolution or Hadamard product of two functions.

Furthermore, letting $\alpha = 0$ in Lemma 1.3, Nezhmetdinov and Ponnusamy [4] have given the sufficient conditions for coefficients of $f(z)$ to be in the class $\mathcal{S}^*$.

Hayami, Owa and Sirivastava [2] have shown the following results.

Theorem 1.5 If $f(z) \in \mathcal{A}$ satisfies the following condition

$$\begin{aligned} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j+1-2\alpha)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha) \end{aligned}$$

for some α ($0 \leq \alpha < 1$), $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Theorem 1.6 If $f(z) \in \mathcal{A}$ satisfies the following condition

$$\begin{aligned} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k j(j+1-2\alpha)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k j(j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha) \end{aligned}$$

for some α ($0 \leq \alpha < 1$), $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{K}(\alpha)$.

Theorem 1.7 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j - \alpha + (1 - \alpha)e^{-2i\lambda})(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1 - \alpha) \cos \lambda$$

for some α ($0 \leq \alpha < 1$), λ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$), $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

2 Main results

Main result for starlike of complex order b is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1+2b)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2|b|$$

for some $b \in \mathbb{C}$ ($b \neq 0$), $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{S}_b^*$.

Proof. Let us define the function $p(z)$ by $p(z) = 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right)$ for $f(z) \in \mathcal{A}$.

Applying Lemma 1.2, $f(z) \in \mathcal{S}_b^*$ if and only if

$$(2.1) \quad p(z) = 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})$$

for all $|x| = 1$.

Then, we need not consider Lemma 1.2 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1} \quad (|x| = 1).$$

Hence, the relation (2.1) is equivalent to

$$(2.2) \quad 2bz + \sum_{n=2}^{\infty} \left\{ (n-1+2b) + x(n-1) \right\} n^j a_n z^n \neq 0.$$

Dividing the both sides of (2.2) by $2bz$ ($z \neq 0$), we obtain that

$$1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0$$

where

$$B_n = \frac{(n-1+2b) + x(n-1)}{2b} n^j a_n \quad (n \geq 2).$$

Therefore, it is sufficient that we prove

$$\left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right) (1-z)^{\beta}(1+z)^{\gamma} = 1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^n \left\{ \sum_{j=1}^k B_j (-1)^{k-j} \binom{\gamma}{k-j} \right\} \binom{\delta}{n-k} \right] z^{n-1} \neq 0$$

where $\beta, \gamma \in \mathbb{R}$ and $B_1 = 1$. Thus, if $f(z)$ satisfies

$$\begin{aligned} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1+2b)(-1)^{k-j} \binom{\gamma}{k-j} a_j \right\} \binom{\delta}{n-k} \right| \right. \\ \left. + |x| \cdot \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1)(-1)^{k-j} \binom{\gamma}{k-j} a_j \right\} \binom{\delta}{n-k} \right| \right] \leq 2|b| \end{aligned}$$

then $f(z) \in \mathcal{S}_b^*$. The proof of Theorem 2.1 is completed. $\square$

We next derive the coefficient condition for functions $f(z)$ to be in the class $\mathcal{K}_b$.

Theorem 2.2 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\begin{aligned} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k j(j-1+2b)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k j(j-1)(-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2|b| \end{aligned}$$

for some $b \in \mathbb{C}$ ($b \neq 0$), $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{K}_b$.

Proof. Since $zf'(z) \in \mathcal{S}_b^*$ if and only if $f(z) \in \mathcal{K}_b$ and since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

replacing a_j in Theorem 2.1 by ja_j , we easily prove Theorem 2.2. $\square$

Putting $\beta = \gamma = 0$ in Theorem 2.1 and Theorem 2.2, we have

Corollary 2.3 *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} \left\{ |n-1+2b| + (n-1) \right\} |a_n| \leq 2|b|$$

for some $b \in \mathbb{C}$ ($b \neq 0$), then $f(z) \in \mathcal{S}_b^*$.

Corollary 2.4 *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n \left\{ |n - 1 + 2b| + (n - 1) \right\} |a_n| \leq 2|b|$$

for some $b \in \mathbb{C}$ ($b \neq 0$), then $f(z) \in \mathcal{K}_b$.

Finally, taking $b = 1 - \alpha$ in Theorem 2.1 and Theorem 2.2, or $b = (1 - \alpha)e^{-i\lambda} \cos \lambda$ in Theorem 2.1, we arrive Theorem 1.5, Theorem 1.6 and Theorem 1.7.

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