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Greatest Common Divisor of the Dimensions of Irreducible Representations of the General Linear Group

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Abstract

Let $V_{\lambda,\mathbf{GL}_k}$ be the irreducible polynomial representation of the general linear group $\mathbf{GL}_k(\mathbb{C})$ corresponding to a partition λ. The main result is that the greatest common divisor of $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$, where λ runs through the partitions of size n, is equal to $k/\gcd(n,k)$. By using this result and the Schur-Weyl duality, we show that the class function $\sigma \mapsto k^{l(\mathrm{type}(\sigma))-1}$ is the character of some representation of $\mathfrak{S}_n$ over $\mathbb{C}$ if and only if k is relatively prime to n. We also present an analogous result for other irreducible Coxeter groups, generalizing Sommers' result on Weyl groups.

1 Introduction and main results

The motivation for considering a class function of the form $\sigma \mapsto k^{l(\mathrm{type}(\sigma))-1}$ arises from parking functions. In the course of studying this class function, we find a simple fact about the divisibility of $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$. In this section, we present the main results of this paper after introducing parking spaces.

We denote by $\mathfrak{S}_n$ the symmetric group of order n. Recall that every permutation σ in $\mathfrak{S}_n$ is decomposed as the product of disjoint cycles. We arrange the lengths of the cycles in nonincreasing order and get a partition of n, denoted by $\mathrm{type}(\sigma)$, which is called the *cycle type* of σ. Remark that the length of $\mathrm{type}(\sigma)$, denoted by $l(\mathrm{type}(\sigma))$, is the number of cycles that appear in the cyclic decomposition of σ.

There are some representations of $\mathfrak{S}_n$ whose character value depends only on the length of the cycle type. Parking spaces are examples of such representations, and are also interesting objects in combinatorics.

Definition 1.1. *A function $f : \{1,\dots,n\} \to \{1,\dots,n\}$ is called a parking function (of length n) if*

$$\#f^{-1}(\{1,\dots,i\}) \geq i$$

for all $i \in \{1,\dots,n\}$. We also denote by PF_n the set of all parking functions of length n.

The name of this function originates in combinatorics. Imagine that n cars want to park at the n parking spots, labelled 1, 2, ..., n on a straight line. The i-th car wants to park at the $f(i)$-th spot, and the cars will park according to their preferences successively. If the parking spot is already occupied, the car parks at the next available spot. In this situation, all the n cars can park in the n parking spots if and only if f is a parking function.

Example 1.2. The symbol ⓘ stands for the i-th car.

1. $(f(1), f(2), f(3), f(4)) = (1, 1, 4, 3)$ is a parking function.

④	③	②	①	⟶	①	②	④	③
3	4	1	1		1	2	3	4

2. $(f(1), f(2), f(3), f(4)) = (3, 3, 1, 4)$ is not a parking function, because the car ④ cannot park.

④	③	②	①	⟶	③		①	②	④
4	1	3	3		1	2	3	4	×

If $f \in \mathrm{PF}_n$ and $\sigma \in \mathfrak{S}_n$, then $f \circ \sigma^{-1} \in \mathrm{PF}_n$, so $\mathfrak{S}_n$ acts on PF_n (on the left). In fact, the same action can be constructed in another way. In the following, we use the notation $\mathbb{Z}_k$ instead of $\mathbb{Z}/k\mathbb{Z}$ for a positive integer k, and denote by $\mathbb{Z}_k^n$ the direct product $\overbrace{\mathbb{Z}_k \times \cdots \times \mathbb{Z}_k}^{n}$. We often identify $\mathbb{Z}_k$ with the set $\{0, 1, \ldots, k-1\}$. We consider the left action of $\mathfrak{S}_n$ on the abelian group $\mathbb{Z}_{n+1}^n$ by permuting the coordinates. This action stabilizes the subgroup $\langle (1^n) \rangle$ generated by $(1^n) := (1, \ldots, 1) \in \mathbb{Z}_{n+1}^n$, so it induces the action on the quotient group.

Proposition 1.3 ([2, Proposition 2.6.1]). *The permutation action of $\mathfrak{S}_n$ on PF_n is isomorphic to the action on $\mathbb{Z}_{n+1}^n / \langle (1^n) \rangle$.*

The permutation representation on the $\mathbb{C}$-vector space $\mathbb{C}[\mathrm{PF}_n]$ with basis PF_n is called the *parking space*. The character of the representation $\mathbb{C}[\mathrm{PF}_n]$ is known to be calculated as follows.

Proposition 1.4. *The character of the $\mathfrak{S}_n$-representation $\mathbb{C}[\mathrm{PF}_n]$ maps $\sigma \in \mathfrak{S}_n$ to $(n+1)^{l(\mathrm{type}(\sigma))-1}$.*

At this point, a simple question arises for a class function $\varphi_k^{(n)}$ on $\mathfrak{S}_n$ defined as follows.

Definition 1.5. *For a positive integer k, define a class function $\varphi_k^{(n)} : \mathfrak{S}_n \to \mathbb{C}$ by*

$$\varphi_k^{(n)}(\sigma) := k^{l(\mathrm{type}(\sigma))-1} \quad (\sigma \in \mathfrak{S}_n).$$

Question. *What is the condition on a positive integer k for the class function $\varphi_k^{(n)}$ to be the character of some representation of $\mathfrak{S}_n$ over $\mathbb{C}$?*

By generalizing parking spaces, we can give a partial answer to this question: if k is relatively prime to n, we can construct a desired representation.

Proposition 1.6. *Let k and n be relatively prime positive integers. Then the character of the permutation representation $\mathbb{C}[\mathbb{Z}_k^n/\langle(1^n)\rangle]$ is given by $\varphi_k^{(n)}$.*

This provides a sufficient condition on k so that $\varphi_k^{(n)}$ is the character of some representation of $\mathfrak{S}_n$. Actually we obtain the following theorem.

Theorem 1.7. *Let k be a positive integer. The class function $\varphi_k^{(n)}$ on $\mathfrak{S}_n$ is the character of some representation of $\mathfrak{S}_n$ over $\mathbb{C}$ if and only if k is relatively prime to n.*

To prove this theorem, we use the Schur-Weyl duality and consider the dimensions of the polynomial irreducible representations of the general linear group $\mathbf{GL}_k(\mathbb{C})$. We denote by $V_{\lambda,\mathbf{GL}_k}$ the irreducible representation of $\mathbf{GL}_k(\mathbb{C})$ corresponding to a partition λ. The key ingredient to prove Theorem 1.7 is the following theorem on the divisibility of $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$.

Theorem 1.8. *Let n and k be positive integers. Denote by $\gcd\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\}$ the greatest common divisor of the set $\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\}$. Then we have*

$$\gcd\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\} = \frac{k}{\gcd(n,k)}.$$

This is the main result of this paper, and Theorem 1.7 is derived from Theorem 1.8. The statement of Theorem 1.8 can be divided into the following two propositions.

Proposition 1.9. $\gcd\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\}$ *divides* $k/\gcd(n,k)$.

Proposition 1.10. *For any partition $\lambda \vdash n$, $k/\gcd(n,k)$ divides $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$.*

In this paper, we prove Theorem 1.7 by using Theorem 1.8 in Section 3, after assembling some well-known results on the combinatorial representation theory in Section 2. The proof of Theorem 1.8 is outlined in Sections 4–6. In Section 4, we prove Proposition 1.9 by using some properties of symmetric polynomials and Kummer's theorem, which states how many times a binomial coefficient is divisible

by a given prime number. Proposition 1.10 can be proved in two different ways. One proof is done by constructing a representation of $\mathfrak{S}_n$, and is given in Section 5. Another one uses a q-analogue of Theorem 1.8, which is presented in Section 6. Finally, in Section 7, we generalize Theorem 1.7 to other finite irreducible Coxeter groups, which is a result obtained after the workshop. This generalization can be viewed as an extension of Sommers' result in [6].

Throughout this paper, we denote by $\mathbb{N}$ the set of all nonnegative integers. We abbreviate $\mathbf{GL}_k(\mathbb{C})$ as $\mathbf{GL}_k$, and assume that all representations are finite dimensional linear representations over $\mathbb{C}$ unless otherwise specified.

2 Preliminaries

We summarize some results of the combinatorial representation theory of symmetric groups and general linear groups. We recall how to parameterize the irreducible representations of $\mathfrak{S}_n$ and $\mathbf{GL}_k$, and the connection with symmetric polynomials. The results arranged in this section can be found in [1, Chapters 6, 7, 11] or [5, Chapter I and its Appendix A].

The irreducible representations of $\mathfrak{S}_n$ are known to be parameterized by the partitions of n.

Definition 2.1. *Let $\lambda \vdash n$, and let T be any filling of the Young diagram of shape λ with numbers 1, ..., n. Define two subgroups of $\mathfrak{S}_n$ as*

$$R := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ stabilizes each row of } T \text{ as sets}\},$$
$$C := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ stabilizes each column of } T \text{ as sets}\},$$

and define two elements in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$ over $\mathbb{C}$ as

$$a := \sum_{\sigma \in R} \sigma, \qquad b := \sum_{\tau \in C} (\operatorname{sgn} \tau)\tau.$$

Define $c := ab$ and let $S^\lambda := (\mathbb{C}[\mathfrak{S}_n])c$ be the left ideal generated by c.

Proposition 2.2 ([1, Section 6.2], [5, Section I.7]).

1. *The $\mathfrak{S}_n$-representation S^λ is irreducible and does not depend on the choice of a filling T.*
2. *If $\lambda \neq \mu$, then S^λ is not equivalent to S^μ.*
3. *The S^λ $(\lambda \vdash n)$ form a complete set of representatives of the isomorphism classes of the irreducible representations of $\mathfrak{S}_n$.*

We denote by χ^λ the character of $\mathfrak{S}_n$-representation S^λ for $\lambda \vdash n$.

The irreducible $\mathbb{C}[\mathfrak{S}_n]$-modules S^λ can be used to construct irreducible polynomial representations of $\mathbf{GL}_k$. Here, the *polynomial representation* of $\mathbf{GL}_k$ means the one whose matrix elements are polynomials in the coordinate functions $x_{ij} : \mathbf{GL}_k \to \mathbb{C}\ ;\ A = (a_{ij}) \mapsto a_{ij}$. We also notice that $\mathfrak{S}_n$ acts on $(\mathbb{C}^k)^{\otimes n}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \tag{2.1}$$

for $\sigma \in \mathfrak{S}_n$, $v_i \in \mathbb{C}^k$.

Definition 2.3. *For a partition λ of n, we define*

$$V_{\lambda,\mathbf{GL}_k} := \mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\big(S^\lambda, (\mathbb{C}^k)^{\otimes n}\big),$$

on which $A \in \mathbf{GL}_k$ acts by composing the linear map $A^{\otimes n}$ on the left.

Proposition 2.4 ([1, Section 7.2], [5, Chapter I Appendix A.8]).

1. *If $l(\lambda) \leq k$, then $V_{\lambda,\mathbf{GL}_k}$ is an irreducible $\mathbf{GL}_k$-representation; otherwise $V_{\lambda,\mathbf{GL}_k} = 0$.*
2. *If $\lambda \neq \mu$ $(l(\lambda), l(\mu) \leq k)$, then $V_{\lambda,\mathbf{GL}_k}$ is not equivalent to $V_{\mu,\mathbf{GL}_k}$.*
3. *The $V_{\lambda,\mathbf{GL}_k}$ $(l(\lambda) \leq k)$ form a complete set of representatives of the isomorphism classes of the irreducible polynomial representations of $\mathbf{GL}_k$.*

There is an interesting connection, called the Schur-Weyl duality, between the representations of $\mathfrak{S}_n$ and $\mathbf{GL}_k$, which plays an essential role in this paper.

Theorem 2.5 ([1, Section 7.2], [5, Chapter I Appendix A.5]). *As left $\mathfrak{S}_n \times \mathbf{GL}_k$-modules, we have*

$$(\mathbb{C}^k)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda \boxtimes V_{\lambda,\mathbf{GL}_k},$$

where $S^\lambda \boxtimes V_{\lambda,\mathbf{GL}_k}$ is the outer tensor product of the $\mathfrak{S}_n$-module S^λ and the $\mathbf{GL}_k$-module $V_{\lambda,\mathbf{GL}_k}$.

Finally, we define some basic symmetric polynomials, which have interesting relations with the representation theory of $\mathfrak{S}_n$ and $\mathbf{GL}_k$.

Definition 2.6. *We define two kinds of symmetric polynomials $e_m(X_1, \ldots, X_k)$ and $h_m(X_1, \ldots, X_k)$ as*

$$e_m(X_1, \ldots, X_k) := \sum_{1 \leq i_1 < \cdots < i_m \leq k} X_{i_1} \cdots X_{i_m},$$

$$h_m(X_1, \ldots, X_k) := \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq k} X_{i_1} \cdots X_{i_m}.$$

The polynomials $e_m(X_1, \ldots, X_k)$ *and* $h_m(X_1, \ldots, X_k)$ *are called the elementary symmetric polynomial and the complete symmetric polynomial respectively.*

Note that $e_0(X_1, \ldots, X_k) = h_0(X_1, \ldots, X_k) = 1$, and $e_m(X_1, \ldots, X_k) = 0$ when $m > k$. For a partition λ, we define

$$e_\lambda(X_1, \ldots, X_k) := \prod_{i=1}^{\infty} e_{\lambda_i}(X_1, \ldots, X_k),$$

$$h_\lambda(X_1, \ldots, X_k) := \prod_{i=1}^{\infty} h_{\lambda_i}(X_1, \ldots, X_k).$$

Definition 2.7. *For a partition* λ *of length at most* k*, we define*

$$s_\lambda(X_1, \ldots, X_k) := \frac{\det\left(X_i^{\lambda_j+k-j}\right)_{1\le i,j\le k}}{\det\left(X_i^{k-j}\right)_{1\le i,j\le k}}.$$

For a partition λ *of length longer than* k*, we define* $s_\lambda(X_1, \ldots, X_k) := 0$*. We call* $s_\lambda(X_1, \ldots, X_k)$ *the Schur polynomial.*

Schur polynomials are indispensable to combinatorial representation theory. One of the most important property of them is the following relation with representations of general linear groups.

Proposition 2.8 ([1, Section 11.2], [5, Chapter I Appendix A.8])**.** *The character value of the* $\mathbf{GL}_k$*-representation* $V_{\lambda,\mathbf{GL}_k}$ *at the diagonal matrix* $\mathrm{diag}(d_1, \ldots, d_k) \in \mathbf{GL}_k$ *is equal to the Schur polynomial* $s_\lambda(d_1, \ldots, d_k)$*.*

In general, the dimension of a representation of a group can be calculated by evaluating the character of the representation at the identity element of the group. Since the identity element of $\mathbf{GL}_k$ is the identity matrix, we have the following corollary.

Corollary 2.9. *For an arbitrary partition* λ*,* $\dim_{\mathbf{C}} V_{\lambda,\mathbf{GL}_k}$ *is equal to* $s_\lambda(1^k) := s_\lambda(\overbrace{1, \ldots, 1}^{k})$.

3 From Theorem 1.8 to Theorem 1.7

In this section, we derive Theorem 1.7 from Theorem 1.8. The Schur-Weyl duality plays an essential role in this argument, and relates the class functions $\varphi_k^{(n)}$ with the dimensions $\dim_{\mathbf{C}} V_{\lambda,\mathbf{GL}_k}$.

Proposition 3.1. *The class function $\varphi_k^{(n)}$ can be expressed as*

$$\varphi_k^{(n)} = \sum_{\lambda \vdash n} \frac{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}}{k} \chi^{\lambda}. \tag{3.1}$$

Proof. Let $\theta_k^{(n)}$ be the character of the representation $(\mathbb{C}^k)^{\otimes n}$ of $\mathfrak{S}_n$ defined as (2.1). We see that $\theta_k^{(n)}$ is calculated as

$$\theta_k^{(n)}(\sigma) = k^{l(\mathrm{type}(\sigma))} \quad (\sigma \in \mathfrak{S}_n) \tag{3.2}$$

by considering the decomposition of σ into disjoint cycles. On the other hand, the function $\theta_k^{(n)}$ is shown to be expressed as

$$\theta_k^{(n)} = \sum_{\lambda \vdash n} (\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}) \chi^{\lambda} \tag{3.3}$$

by viewing the isomorphism in Theorem 2.5 as $\mathbb{C}[\mathfrak{S}_n]$-isomorphism. Combining (3.2) and (3.3), we have

$$k^{l(\mathrm{type}(\sigma))} = \sum_{\lambda \vdash n} (\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}) \chi^{\lambda}(\sigma) \quad (\sigma \in \mathfrak{S}_n). \tag{3.4}$$

Dividing the both sides by k, we get (3.1). □

Since any representation of $\mathfrak{S}_n$ over $\mathbb{C}$ is completely reducible, the class function $\varphi_k^{(n)}$ is the character of some representation of $\mathfrak{S}_n$ if and only if $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$ is divisible by k for all $\lambda \vdash n$. This observation and Proposition 3.1 show that Theorem 1.7 is derived from Theorem 1.8.

4 Proof of Proposition 1.9

We now prove that $\gcd\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\}$ divides $k/\gcd(n,k)$ (Proposition 1.9) by examining the divisibility of binomial coefficients.

Recall that Schur polynomials and elementary symmetric polynomials of homogeneous degree n generate the same $\mathbb{Z}$-module of symmetric polynomials of homogeneous degree n:

$$\sum_{\lambda \vdash n} \mathbb{Z} s_{\lambda}(X_1, \ldots, X_k) = \sum_{\lambda \vdash n} \mathbb{Z} e_{\lambda}(X_1, \ldots, X_k). \tag{4.1}$$

Substituting $X_1 = \cdots = X_k = 1$ in both sides, and using Corollary 2.9 and the fact

$$e_m(1^k) = \binom{k}{m},$$

we have

$$\langle \dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n \rangle_{\mathbb{Z}} = \left\langle \prod_{i=1}^{\infty} \binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n \right\rangle_{\mathbb{Z}},$$

where $\langle a_i \mid i \in I \rangle_R$ denotes the ideal of a commutative ring R generated by the subset $\{a_i \mid i \in I\} \subset R$. (Notice that if $m > k$, the binomial coefficient $\binom{k}{m}$ is defined to be 0.) Thus we see that

$$\gcd\{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \mid \lambda \vdash n\} = \gcd\left\{ \prod_{i=1}^{\infty} \binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n \right\}.$$

Therefore, in order to prove Proposition 1.9, it suffices to prove

$$\gcd\left\{ \prod_{i=1}^{\infty} \binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n \right\} \;\middle|\; \frac{k}{\gcd(n,k)}.$$

(For elements a, b in a commutative ring R, $a \mid b$ means that b is divisible by a.) It becomes easier to prove this if we decompose $\gcd\left\{ \prod \binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n \right\}$ into prime factors and reduce this argument to each prime powers.

Definition 4.1. *Let p a prime number, r be a positive integer, and $L \in \mathbb{N}$. We write $p^L \parallel r$ when $p^L \mid r$ and $p^{L+1} \nmid r$. We also write such L as $\varepsilon_p(r)$.*

Proposition 1.9 follows from the following proposition.

Proposition 4.2. *If $L \in \mathbb{N}$ and a prime number p satisfy*

$$p^L \;\Big\|\; \gcd\left\{ \prod_{i=1}^{\infty} \binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n \right\},$$

then we have

$$p^L \;\Big|\; \frac{k}{\gcd(n,k)}.$$

For the proof of Proposition 4.2, we need to know the divisibility of binomial coefficients by prime numbers. The next theorem is known as Kummer's theorem.

Theorem 4.3 ([4]). *For a prime number p and positive integers m, r with $m \geq r$, $\varepsilon_p\left(\binom{m}{r}\right)$ is equal to the number of borrows required when subtracting r from m in the base p representation.*

Example 4.4. Take $\binom{18}{5}$ and examine how many times it is divisible by 2. Since $\binom{18}{5} = 2^3 \cdot 3^2 \cdot 7 \cdot 17$, we have $\varepsilon_2\left(\binom{18}{5}\right) = 3$. On the other hand, the representations of 18 and 5 in base 2 are $(10010)_2$, $(101)_2$ respectively, and we need 3 borrows when subtracting 5 from 18 in base 2.

$$\begin{array}{cccccc} & \not{1}^{0} & \not{0}^{1} & 0 & \not{1}^{0} & 0 \\ - & & & 1 & 0 & 1 \\ \hline & & 1 & 1 & 0 & 1 \end{array}$$

Proof of Proposition 4.2. Let p be a prime number and put

$$L := \varepsilon_p\left(\gcd\left\{\prod_{i=1}^{\infty}\binom{k}{\lambda_i} \;\middle|\; \lambda \vdash n\right\}\right), \quad K := \varepsilon_p(k).$$

We choose s, $r \in \mathbb{N}$ $(0 \leq r < p^K)$ so that $n = sp^K + r$, and define

$$\mu := (\underbrace{p^K, \ldots, p^K}_{s}, r) \vdash n.$$

Then we have

$$p^L \;\Bigg|\; \prod_{i=1}^{s+1}\binom{k}{\mu_i} \tag{4.2}$$

by the assumption. Since $p^K \parallel k$, k is expressed in base p as

$$k = a_K p^K + a_{K+1}p^{K+1} + \cdots \quad (a_K \neq 0).$$

From this expression, we see that no borrow is required when subtracting p^K from k. It follows from Kummer's theorem that p does not divide $\binom{k}{p^K}$. Hence we have

$$\varepsilon_p\left(\prod_{i=1}^{s+1}\binom{k}{\mu_i}\right) = \varepsilon_p\left(\binom{k}{p^K}^s \cdot \binom{k}{r}\right) = \varepsilon_p\left(\binom{k}{r}\right). \tag{4.3}$$

We may assume $r > 0$. (If $r = 0$, then (4.2) and (4.3) imply $L = 0$, so p^L divides $k/\gcd(n,k)$ as desired.) Let $R := \varepsilon_p(r)$. Since $r < p^K$, we have $R < K$ and the base p representation of r should be

$$r = b_R p^R + b_{R+1}p^{R+1} + \cdots + b_{K-1}p^{K-1} \quad (b_R \neq 0).$$

If we subtract r from k in base p, the number of borrows required is precisely $K - R$:

$$\begin{array}{ccccccccccc} & \cdots & a_{K+1} & \not{a_K}^{a_K-1} & \not{0}^{p-1} & \cdots & \not{0}^{p-1} & 0 & 0 & \cdots & 0 \\ - & & & & b_{K-1} & \cdots & b_{R+1} & b_R & 0 & \cdots & 0 \\ \hline \end{array}$$

Therefore, Kummer's theorem implies that $\varepsilon_p\left(\binom{k}{r}\right) = K - R$. By (4.2) and (4.3), we see that $L \leq K - R$. On the other hand, we have $\varepsilon_p(n) = R$ because $n = sp^K + r$ $(r < p^K)$. Hence it follows that $\varepsilon_p\bigl(k/\gcd(n,k)\bigr) = K - R \geq L$, which implies that

$$p^L \,\Big|\, \frac{k}{\gcd(n,k)}$$

and the proof is done. □

5 Proof of Proposition 1.10

We prove that $k/\gcd(n,k)$ divides $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$ for any partition $\lambda \vdash n$ (Proposition 1.10) by constructing a representation of $\mathfrak{S}_n$ whose character is given by $\gcd(n,k) \cdot \varphi_k^{(n)}$. The result of this section is due to Professor Soichi Okada.

First, we obtain

$$\gcd(n,k)k^{l(\mathrm{type}(\sigma))-1} = \sum_{\lambda \vdash n} \frac{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}}{k/\gcd(n,k)} \chi^{\lambda}(\sigma) \quad (\sigma \in \mathfrak{S}_n) \tag{5.1}$$

by dividing both sides of the equation (3.4) by $k/\gcd(n,k)$. We define a class function $\xi_k^{(n)} : \mathfrak{S}_n \to \mathbb{C}$ by $\xi_k^{(n)}(\sigma) := \gcd(n,k)k^{l(\mathrm{type}(\sigma))-1}$ for $\sigma \in \mathfrak{S}_n$.

Proposition 5.1. *There exists a representation of $\mathfrak{S}_n$ which has the character $\xi_k^{(n)}$.*

Since

$$\xi_k^{(n)} = \sum_{\lambda \vdash n} \frac{\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}}{k/\gcd(n,k)} \chi^{\lambda}$$

by (5.1), Proposition 5.1 implies that all the coefficients of χ^{λ} must be nonnegative integers. That is,

$$\frac{k}{\gcd(n,k)} \,\Big|\, \dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$$

for all $\lambda \vdash n$.

We prove Proposition 5.1 by constructing an explicit representation of $\mathfrak{S}_n$ which has the character $\xi_k^{(n)}$.

Proof of Proposition 5.1. Define

$$X_0 := \{(x_1, \ldots, x_n) \in \mathbb{Z}_k^n \mid x_1 + \cdots + x_n = 0,\ 1,\ \ldots,\ \gcd(n,k) - 1\},$$

and consider the permutation representation of $\mathfrak{S}_n$ corresponding to X_0. We denote by $\tilde{\xi}_k^{(n)}$ the character of this representation. Our goal is to prove $\xi_k^{(n)} = \tilde{\xi}_k^{(n)}$.

For a positive integer $i \in \{1, 2, \ldots, k/\gcd(n,k) - 1\}$, we define

$$X_i := \{(x_1, \ldots, x_n) \in \mathbb{Z}_k^n \mid x_1 + \cdots + x_n = i \cdot \gcd(n,k), \ldots, (i+1) \cdot \gcd(n,k) - 1\}.$$

(This notation is compatible with X_0.) Then we have the permutation representation $\mathbb{C}[X_i]$ of $\mathfrak{S}_n$. We choose $a, b \in \mathbb{Z}$ such that $an + bk = \gcd(n,k)$. Then

$$\begin{array}{ccc} X_0 & \to & X_i \\ (x_1, \ldots, x_n) & \mapsto & (x_1 + ai, \ldots, x_n + ai) \end{array}$$

is proved to be an $\mathfrak{S}_n$-equivariant bijection for all $i \in \{1, 2, \ldots, k/\gcd(n,k) - 1\}$. Since $\mathbb{Z}_k^n$ is decomposed as a disjoint union

$$\mathbb{Z}_k^n = \coprod_{i=0}^{\frac{k}{\gcd(n,k)}-1} X_i,$$

we have

$$\mathbb{C}[\mathbb{Z}_k^n] = \bigoplus_{i=0}^{\frac{k}{\gcd(n,k)}-1} \mathbb{C}[X_i] \cong (\mathbb{C}[X_0])^{\oplus k/\gcd(n,k)}$$

as $\mathfrak{S}_n$-modules. Since $\mathbb{C}[\mathbb{Z}_k^n]$ and $(\mathbb{C}^k)^{\otimes n}$ are isomorphic as $\mathfrak{S}_n$-modules, the character of the $\mathfrak{S}_n$-representation $\mathbb{C}[\mathbb{Z}_k^n]$ is given by $\theta_k^{(n)}$ in Section 3. Then the equation (3.2) implies

$$k^{l(\mathrm{type}(\sigma))} = \frac{k}{\gcd(n,k)} \tilde{\xi}_k^{(n)}(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$

Hence

$$\tilde{\xi}_k^{(n)}(\sigma) = \gcd(n,k) \cdot k^{l(\mathrm{type}(\sigma))-1} = \xi_k^{(n)}(\sigma) \quad (\sigma \in \mathfrak{S}_n)$$

as desired. □

6 q-Analogue of Theorem 1.8

Proposition 1.10 is also proved by considering a q-analogue of Theorem 1.8. The advantage of this method is that one can reduce the proof to determining the common roots of the principal specialization of Schur polynomials.

The *q-integer* of $r \in \mathbb{N}$ is defined to be

$$[r]_q := \frac{1 - q^r}{1 - q} = 1 + q + \cdots + q^{r-1}.$$

Similarly, we define a q-factorial and a q-binomial coefficient as

$$[r]_q! := [r]_q \cdot [r-1]_q \cdot \cdots \cdot [2]_q \cdot [1]_q,$$
$$\begin{bmatrix} r \\ s \end{bmatrix}_q := \frac{[r]_q!}{[r-s]_q! \cdot [s]_q!}.$$

Now, we consider a q-analogue of $\dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k}$. Recall that the dimension of $V_{\lambda,\mathbf{GL}_k}$ is given by $s_\lambda(1^k)$ (Corollary 2.9). The q-analogue of $s_\lambda(1^k)$ should be $s_\lambda(1, q, \ldots, q^{k-1})$, which is called the *principal specialization* of the Schur polynomial $s_\lambda(X_1, X_2, \ldots, X_k)$. From these observations, the following theorem is a natural q-analogue of Theorem 1.8:

Theorem 6.1. *We have*

$$\gcd\{s_\lambda(1, q, \ldots, q^{k-1}) \mid \lambda \vdash n\} = \frac{[k]_q}{[\gcd(n,k)]_q}. \tag{6.1}$$

Here, the greatest common divisor on the left hand side is taken in the ring $\mathbb{Q}[q]$, and required to be monic.

Theorem 6.1 is a generalization of [2, Proposition 2.5.1], and the proof below is almost the same as Haiman's proof in [2]. Here, we just give an overview of the proof of Theorem 6.1.

Sketch of the proof of Theorem 6.1. We prove the equation (6.1) by comparing the roots of both sides. Analogous to the equation (4.1), we have

$$\sum_{\lambda \vdash n} \mathbb{Z} s_\lambda(X_1, \ldots, X_k) = \sum_{\lambda \vdash n} \mathbb{Z} h_\lambda(X_1, \ldots, X_k).$$

Substituting $X_i = q^{i-1}$, we have

$$\sum_{\lambda \vdash n} \mathbb{Z} s_\lambda(1, q, \ldots, q^{k-1}) = \sum_{\lambda \vdash n} \mathbb{Z} h_\lambda(1, q, \ldots, q^{k-1}),$$

which implies

$$\gcd\{s_\lambda(1, q, \ldots, q^{k-1}) \mid \lambda \vdash n\} = \gcd\{h_\lambda(1, q, \ldots, q^{k-1}) \mid \lambda \vdash n\}.$$

Next we examine the right hand side of (6.1). For a positive integer d, we denote by $\Phi_d(q)$ the d-th cyclotomic polynomial:

$$\Phi_d(q) := \prod_{\zeta} (q - \zeta),$$

where the product is taken over all primitive d-th roots ζ of 1. The following is a well-known property of cyclotomic polynomials:

$$\prod_{d|m} \Phi_d(q) = q^m - 1.$$

Thus, the right hand side of (6.1) is calculated as

$$\begin{aligned}\frac{[k]_q}{[\gcd(n,k)]_q} &= \frac{1-q^k}{1-q^{\gcd(n,k)}} = \frac{\prod_{d|k} \Phi_d(q)}{\prod_{d|\gcd(n,k)} \Phi_d(q)} \\ &= \prod_{\substack{d|k \\ d\nmid n}} \Phi_d(q) = \prod_{\substack{d|k \\ d\nmid n}} \prod_{\zeta} (q-\zeta).\end{aligned}$$

From these observations, it suffices to prove the following claim.

Claim.

1. *We have*

$$\begin{aligned}&\{z \in \mathbb{C} \mid z \text{ is a common root of } h_\lambda(1,q,\dots,q^{k-1}) \ (\lambda \vdash n)\} \\ &\qquad = \coprod_{\substack{d|k \\ d\nmid n}} \{z \in \mathbb{C} \mid z \text{ is a primitive } d\text{-th root of } 1\}.\end{aligned}$$

2. *If z is a common root of $h_\lambda(1,q,\dots,q^{k-1})$ $(\lambda \vdash n)$, then z is a simple root of $h_\mu(1,q,\dots,q^{k-1})$ for some $\mu \vdash n$.*

Claim is not trivial, but can be proved without much difficulty by using the following lemma.

Lemma 6.2. *Let d be a positive integer dividing k, and $r \in \mathbb{N}$.*

1. *If $d \mid r$, then no primitive d-th root of 1 is a root of $h_r(1,q,\dots,q^{k-1})$.*
2. *If $d \nmid r$, then any primitive d-th root of 1 is a simple root of $h_r(1,q,\dots,q^{k-1})$.*

This lemma is easily proved by the formula on the principal specialization of the complete symmetric polynomial:

$$h_r(1,q,\dots,q^{k-1}) = \begin{bmatrix} k+r-1 \\ r \end{bmatrix}_q = \frac{(1-q^k)(1-q^{k+1})\cdots(1-q^{k+r-1})}{(1-q)(1-q^2)\cdots(1-q^r)},$$

which can be found in [5, Section I.2 Example 2] for example. □

From Theorem 6.1, we see that for all $\lambda \vdash n$,

$$\frac{[k]_q}{[\gcd(n,k)]_q} \;\Big|\; s_\lambda(1, q, \ldots, q^{k-1}) \quad \text{in } \mathbb{Z}[q].$$

(Note that $[k]_q/[\gcd(n,k)]_q$ is a monic polynomial with integer coefficients.) By taking the limit $q \to 1$, we have

$$\frac{k}{\gcd(n,k)} \;\Big|\; \dim_{\mathbb{C}} V_{\lambda,\mathbf{GL}_k} \quad \text{in } \mathbb{Z}$$

for all $\lambda \vdash n$. Thus this gives another proof of Proposition 1.10.

Finally, we remark that Theorem 6.1 is not a generalization of Theorem 1.8 in a strict sense. This is because taking the limit and taking the gcd are not commutative operations in general.

Example 6.3. Consider $(q^2+1)(q+1)^2$ and $(q+1)^3$. Then we have

$$\lim_{q\to 1} \gcd\{(q^2+1)(q+1)^2, (q+1)^3\} = \lim_{q\to 1}(q+1)^2 = 4,$$
$$\gcd\{\lim_{q\to 1}(q^2+1)(q+1)^2, \lim_{q\to 1}(q+1)^3\} = \gcd(8,8) = 8.$$

Another choice of q-analogue, which recovers Theorem 1.8 as $q \to 1$, should be the following. However, we have not been able to prove this conjecture.

Conjecture. *In the ring $\mathbb{Z}[q]$, we have*

$$\langle s_\lambda(1, q, \ldots, q^{k-1}) \mid \lambda \vdash n \rangle_{\mathbb{Z}[q]} = \left\langle \frac{1-q^k}{1-q^{\gcd(n,k)}} \right\rangle_{\mathbb{Z}[q]}.$$

7 Generalization to Coxeter Groups

We reformulate Theorem 1.7 in terms of Coxeter groups. (In this section, we follow the terminology and notations in [3].) Some of the results in this section are obtained in a collaboration with Professor Soichi Okada.

Notice that the symmetric group $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}. The geometric representation (over $\mathbb{R}$) of type A_{n-1} is given by the subspace of $\mathbb{R}^n$ defined by

$$V := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}.$$

Remark that $l(\mathrm{type}(\sigma)) - 1$ is equal to the $\mathbb{R}$-dimension of the fixed-point subspace

$$\mathrm{Fix}_V(\sigma) := \{x \in V \mid \sigma x = x\}.$$

This interpretation of Theorem 1.7 suggests a general formulation of the question mentioned in Section 1.

Question. *Let W be a finite Coxeter group, and V be the geometric representation of W. What is the condition on a positive integer k for the class function $\varphi_k^W : W \to \mathbb{C}\ ;\ w \mapsto k^{\dim_{\mathbb{R}} \mathrm{Fix}_V(w)}$ to be the character of some representation of W over $\mathbb{C}$?*

By Theorem 1.7, the answer to this question in type A_{n-1} is that "k is relatively prime to n." More generally, we can give the answer to this question when W is irreducible. (However, non-irreducible cases are not yet settled.) Here we just present the result at this point.

Theorem 7.1. *For a finite irreducible Coxeter group W, the class function φ_k^W is the character of some representation of W over $\mathbb{C}$ if and only if the following condition is satisfied.*

type	condition on k
A_{n-1}	k is relatively prime to n
B_n, D_n	$2 \nmid k$
$I_2(m)$ (m is even)	$k = 1$ or "$k \geq m-1$ and $k^2 \equiv 1 \bmod 2m$"
$I_2(m)$ (m is odd)	$k = 1$ or "$k \geq m-1$ and $k^2 \equiv 1 \bmod m$"
E_6, E_7, F_4	$2 \nmid k$ and $3 \nmid k$
E_8	$2 \nmid k$ and $3 \nmid k$ and $5 \nmid k$
H_3	$k \equiv 1, 5, 9 \bmod 10$
H_4	$k \equiv 1, 11, 19, 29 \bmod 30$

The result above can be seen as a generalization of Sommers' result in [6]. He constructs a representation of a Weyl group W over $\mathbb{C}$ whose character is given by φ_k^W, when k is "very good" in the sense of [6]. For a Weyl group W, we see that k is "very good" if and only if k satisfies the condition given in Theorem 7.1. In other words, our result shows that Sommers' representations exhaust all the representations of Weyl groups with characters of the form φ_k^W. Theorem 7.1 is also new in that the noncrystallographic types H and I are examined.

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