

Periodic Orbits of some kinds of Periodic Systems

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ABSTRACT

We apply a modified version of the method of Sinai and others by means of computer to the existence of a closed orbit which appears in the periodic system of special type.

## 1. INTRODUCTION

Ja. Sinai studied the ordinary differential equations having the form  $\dot{x}_i = f_i(x_1, \dots, x_d)$ ,  $i=1, \dots, d$ , where  $f_i$  are polynomials of degree not more than two. He applied his criterion to the Lorentz model and showed rigorously the existence of a closed orbit.

In this paper we consider a time dependent system

$$\dot{x}_i = f_i(t, X), \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad i=1,2$$

where  $f_i(t, X)$  are some special types described below. We use Sinai's criterion and estimate each constant with the use of a computer.

## 2. A CRITERION FOR THE EXISTENCE OF A CLOSED ORBIT

We consider a differential equation in  $R^2$

$$\dot{X} = F(t, X)$$

where  $F(t, X) = \begin{pmatrix} f_1(t, X) \\ f_2(t, X) \end{pmatrix}$  is defined as follows:

$$f_1(t, X) = f_1(X)$$

$$f_2(t, X) = f_2(X) + f(t).$$

$f_i(x)$  ( $i=1,2$ ) are polynomials of degree not more than two and  $f(t)$  is defined as follows:

$$f(t) = bf_a(t), \quad f_a(t) = \begin{cases} 1 - \frac{2}{a}t & \text{for } 0 \leq t \leq a, \\ -3 + \frac{2}{a}t & \text{for } a \leq t \leq 2a \end{cases}$$

where  $a, b$  are some real constants. We put

$$\bar{F}(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \end{pmatrix}.$$

Then  $F(t, X)$  can be rewritten as follows:

$$F(t, X) = \bar{F}(X) + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

This system is periodic in  $t$  of period  $T = 2a$ . Let  $S_t X$  be the solution of (2.1) with initial condition  $X$ . Let  $P$  be the Poincaré map induced by the solution  $S_t$ . We put

$$Y = X - X^0, \quad Q(Y) = P(X) - X^0, \quad X \in \mathbb{R}^2$$

and write  $Q$  in the form

$$Q(Y) = Y^0 + LY + K(Y).$$

Here  $Y^0 = P(X^0) - X^0$ ,  $L$  is the matrix of linear part of  $Q$  at the point  $Y=0$  and  $K$  is a nonlinear term. The mapping  $K$  satisfies the following condition; there exist positive constants  $\rho_0$ ,  $K_0$  such that  $|K(Y^1) - K(Y^2)| \leq K_0 \rho |Y^1 - Y^2|$  for an arbitrary  $\rho \leq \rho_0$  and arbitrary  $Y^1, Y^2$ ,  $\|Y^1\| \leq \rho$ ,  $\|Y^2\| \leq \rho$ . This inequality expresses the quadratic character of  $K$ . We put

$$\varepsilon = |P(X^0) - X^0|.$$

Criterion. (J. Sinai[1]) For some  $\bar{\rho}_0 < \rho_0$   $(L-E)^{-1} (\varepsilon/\bar{\rho}_0 + K_0 \bar{\rho}_0) \leq 1$ .

It is easily proved under this criterion, in the  $\bar{\rho}_0$  neighbourhood of  $X^0$  there exists one and only one fixed point of  $Q$ . Now, we estimate  $L$ ,  $\varepsilon$ ,  $K_0$  with the use of a computer.

### 3. PSEUDO ORBITS

We introduce the pseudo orbits which play the fundamental role in our

problem. We define  $(X, Y) = x_1 y_1 + x_2 y_2$  where  $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ . We write  $f_i(X)$  in the form

$$f_i(X) = (\ell_i, X) + (B^i X, X)$$

where  $(\ell_i, X)$  is linear form and  $(B^i X, X)$  is quadratic form. In vector notation we denote  $\bar{F}(X) = (\ell, X) + (\bar{B}X, X)$  where  $(\ell, X) = \begin{pmatrix} (\ell_1, X) \\ (\ell_2, X) \end{pmatrix}$  and  $(\bar{B}X, X) = \begin{pmatrix} (B^1 X, X) \\ (B^2 X, X) \end{pmatrix}$ . We fix a constant  $\Delta$  called the step and we write

$$f(t) = At + B$$

Suppose a pseudo orbit  $X_i$  is given, we define a pseudo orbit  $X_{i+1}$  in the following. Put  $t_i = i\Delta$ . We take the zeroth approximation  $X_0(t) \equiv X_i$ ; then the first approximation is given by

$$X_1(t) = X_i + \int_{t_i}^t F(s, X_0(s)) ds.$$

Consider the second approximation

$$X_2(t_{i+1}) = X_i + \int_{t_i}^{t_{i+1}} F(s, X_1(s)) ds.$$

We define the function  $R$  as follows

$$R(x_i) \equiv X_i + \Delta[(\ell, X_i) + (\bar{B}X_i, X_i) + \begin{pmatrix} 0 \\ At_i + B \end{pmatrix}] + \Delta^2 M,$$

$$M = \frac{1}{2}(\ell, \bar{F}(X_i)) + (\bar{B}X_i, \bar{F}(X_i)) + \frac{1}{2}(At_i + B) \begin{pmatrix} \ell_{12} \\ \ell_{22} \end{pmatrix} + (At_i + B) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ A/2 \end{pmatrix},$$

where

$$X_i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \ell_j = \begin{pmatrix} \ell_{j1} \\ \ell_{j2} \end{pmatrix}, \quad B^j = (B_{mn}^j), \quad \alpha_j = B_{12}^j x_1 + B_{22}^j x_2, \quad j = 1, 2.$$

$R$  is the main part of the second approximation. This implies that we ignore the following terms of degree  $V$  than or equal to three with respect to  $\Delta$ , i.e.

$$\Delta^3 \left[ \frac{1}{3} (\overline{B} \overline{F}(X_i), F(X_i) + \frac{A}{6} \begin{pmatrix} \ell_{12} \\ \ell_{22} \end{pmatrix} + \frac{2}{3} (At_i + B) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \frac{1}{3} (At_i + B)^2 \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \right],$$

$$\Delta^4 \left[ \frac{A}{4} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \frac{A}{4} At_i + B \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \right], \text{ and}$$

$$\Delta^5 \left[ \frac{A^2}{20} \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \right]$$

where  $\beta_j = B_{12}^j f_1(X_i) + B_{22}^j f_2(X_i)$ ,  $j = 1, 2$ .

One may think that we abandon too many terms. But we assert that we are able to ignore them taking  $\Delta$  so small. Now we define  $X_{i+1}$  using  $R(X_i)$

$$|R(X_i) - X_{i+1}| \leq \alpha$$

where  $\alpha$  is the error arising from roundoff errors. The value  $\alpha$  depends on the precision of the computer.

#### 4. PRELIMINARIES

We give some notation. Let  $X^0$  be fixed.  $\gamma = \{X \in R^2; X = S_t X^0, 0 \leq t \leq T\}$ .  $F'(t, X)$  is the matrix  $\left( \frac{\partial f_i(t, X)}{\partial x_j} \right)$  which is equal to  $\left( \frac{\partial f_i(X)}{\partial x_j} \right)$ . We consider the variational equation corresponding to  $\gamma$ , namely  $dZ/dt = F'(t, S_t X^0)Z$  where the right-handside is equal to  $F'(S_t X^0)Z$ . We denote by  $L(t_1, t_2)$  the fundamental matrix of solution of this system on the interval  $t_1 \leq t \leq t_2$ .

We put

$$C_1 = \sup_{0 \leq t_1 \leq t_2 \leq T} \|L(t_1, t_2)\|.$$

Because  $f_i(X)$  are polynomials of degree not more than two, we can find a constant  $C_2$  for which

$$\sum_{i,j,k} \left| \frac{\partial^2 f_i(X)}{\partial x_j \partial x_k} y_j z_k \right| \leq C_2 |Y| |Z|, \quad x \in W_\rho(\gamma)$$

where  $W_\rho(\gamma)$  is the  $\rho$ -neighbourhood of  $\gamma$ . Further, let

$$C_3 = \sup_{x \in W_\rho(\gamma)} \|F'(t, X)\|, \quad C_4 = \sup_{x \in W_\rho(\gamma)} |F(t, X)|.$$

We put also  $\delta_1 X(t) \equiv S_t X - S_t X^0 - L(0, t)(X - X^0)$ . Then the next theorem can be proved analogously to the proof in [1].

Theorem A. (Ja. Sinai [1]) Let  $|X - X^0| \leq \rho$  where  $\rho$  satisfies the inequality  $\rho \leq 1/2TC_1^2C_2$ . Then for  $B_1(t) = 2C_1^3C_2t$

$$|\delta_1 X(t)| \leq B_1(t) |X - X^0|^2 \quad 0 \leq t \leq T.$$

## 5. ESTIMATION OF $\varepsilon$

Let  $\Delta$  be the step and  $n\Delta = T$ , where  $T$  is the period of (2.1).

We consider the pseudo orbits  $\{X_i\}$  as before. Then we have

$$\varepsilon = |S_T X^0 - X^0| \leq |S_T X^0 - X_n| + |X_n - X^0|.$$

Here  $|X_n - X^0|$  is found from the result of calculation. Therefore we have only to estimate  $|S_T X^0 - X_n|$ . Let us put  $Z_i \equiv S_{i\Delta} X^0 - X_i$ . We define recurrence equations  $V_{i+1}$  such that

$$Z_{i+1} = L(t_i, t_{i+1})Z_i + V_{i+1}.$$

Then using the property of fundamental matrix,  $Z_{i+1}$  can be rewritten as follows:

$$Z_{i+1} = \sum_{j=0}^{i+1} L(t_j, t_{i+1})V_j.$$

Therefore it is sufficient to estimate  $V_j$ .  $V_{j+1}$  can be rewritten as follows:

$$V_{j+1} = X_{j+1} - S_{\Delta} X_j + \delta_1 Z_j$$

where  $\delta_1 Z_j = S_{\Delta} X_j - S_{\Delta}(S_j \Delta X^0) - L(t_j, t_{j+1}) Z_j$ . First, we estimate  $X_{j+1} - S_{\Delta} X_j$ . Let us assume the following inequalities with respect to  $U$  and  $\Delta$

$$\left| \frac{A}{6} \begin{pmatrix} \ell_{12} \\ \ell_{22} \end{pmatrix} + \frac{2}{3}(AT+B) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \frac{1}{3}(AT+B)^2 \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \right| \leq U$$

$$\left| \frac{A}{4} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \Delta + \frac{A}{4}(AT+B) \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \Delta + \frac{A^2}{20} \begin{pmatrix} B_{22}^1 \\ B_{22}^2 \end{pmatrix} \Delta^2 \right| \leq 1$$

$$\exp(C_3 \sqrt{2} \Delta) - [1 + C_3 \sqrt{2} \Delta + \frac{1}{2} (C_3 \sqrt{2} \Delta)^2] \leq \frac{1}{5} (C_3 \sqrt{2})^3 \Delta^3.$$

Then the following inequalities hold

$$\begin{aligned} |X_{j+1} - S X_j| &\leq |X_{j+1} - R X_j| + |R X_j - S_{\Delta} X_j| \\ &\leq \alpha + \exp(C_3 \sqrt{2} \Delta) - [1 + C_3 \sqrt{2} \Delta + \frac{1}{2} (C_3 \sqrt{2} \Delta)^2] + (\frac{1}{3} C_2 C_4^2 + U + 1) \Delta^3 \\ &\leq \alpha + [\frac{1}{5} (C_3 \sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1] \Delta^3. \end{aligned}$$

On the other hand, under the conditions of Theorem A we have

$$|\delta_1 Z_j| \leq B_1(\Delta) |Z_j|^2 \leq B_1(\Delta) C_1^2 \left[ \sum_{i=0}^j |v_i| \right]^2.$$

Now we make the inductive hypothesis

$$|v_j| \leq \bar{A} \Delta^2 \quad 0 \leq j \leq k,$$

and we find a condition on  $\bar{A}$  under which the inequality is also valid for  $j = k+1$ .

$$\begin{aligned}
|v_{k+1}| &\leq |\delta_1 z_k| + |x_{k+1} - s_\Delta x_k| \\
&\leq B_1(\Delta) C_1^2 (k+1) \bar{A}^2 \Delta^4 + \Delta^2 \left[ \frac{\alpha}{\Delta^2} + \Delta \left\{ \frac{1}{5} (C_3 \sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1 \right\} \right] \\
&\leq \Delta^2 [B_1(\Delta) C_1^2 T^2 \bar{A}^2 + \frac{\alpha}{\Delta^2} + \Delta \left\{ \frac{1}{5} (C_3 \sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1 \right\}].
\end{aligned}$$

Let  $\bar{A}$  be the least root of the quadratic equation

$$\bar{A} - B_1(\Delta) C_1^2 T^2 \bar{A}^2 = \frac{\alpha}{\Delta^2} + \Delta \left\{ \frac{1}{5} (C_3 \sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1 \right\}.$$

Then we have

$$|v_{k+1}| \leq \bar{A} \Delta^2.$$

Lastly we can estimate  $|z_i|$  as follows:

$$|z_i| \leq \left| \sum_{j=0}^n L(j\Delta, n\Delta) v_j \right| \leq C_1 \sum_{j=0}^n |v_j| \leq C_1 n \bar{A} \Delta^2.$$

## 6. ESTIMATION OF L

In our case we have  $L = L(0, T)$ .

We define the matrix  $\bar{L}(0, i\Delta)$  using the pseudo orbits  $\{x_i\}$  as follows:

$$L(0, i\Delta) = [E + \Delta \bar{F}(x_{i-1})] \bar{L}(0, (i-1)\Delta) + \delta L_i$$

$$\|\delta L_i\| \leq \beta$$

where  $\delta L_i$  is the error arising from roundoff errors. Then  $\bar{L}(0, i\Delta)$  is the approximate value of the matrix  $L(0, i\Delta)$ . In order to estimate the error, we have

$$L(0, (i+1)\Delta) - \bar{L}(0, (i+1)\Delta) = [E + \Delta \bar{F}'(x_i)] [L(0, i\Delta) - \bar{L}(0, i\Delta)] + \delta_1 L_{i+1}$$

where  $\delta_1 L_{i+1} = [L(i\Delta, (i+1)\Delta) - (E + \Delta \bar{F}'(x_i)) L(0, i\Delta) - \delta L_{i+1}]$ . Now we



can write

$$L(0, (i+1)\Delta) - \bar{L}(0, (i+1)\Delta) = \sum_{j=0}^i \prod_{j=k}^i [E + \Delta \bar{F}'(X_1)] \delta_1 L_k.$$

Let us take  $C_1$  so that (see Ja. Sinai [1])

$$\left\| \prod_{j=k}^i (E + \Delta \bar{F}'(X_1)) \right\| \leq C_1.$$

Then we have

$$\|L(0, T) - \bar{L}(0, T)\| \leq C_1 \sum_{k=0}^n \|\delta_1 L_k\|.$$

Therefore it is sufficient to estimate  $\|\delta_1 L_k\|$  for our purpose. The following inequalities hold.

$$\begin{aligned} \|\delta_1 L_{i+1}\| &\leq \| [L(i\Delta, (i+1)\Delta) - (E + \Delta \bar{F}'(S_{i\Delta} X^0))] L(0, i\Delta) \| \\ &\quad + \Delta \| (\bar{F}'(S_{i\Delta} X^0) - \bar{F}'(X_1)) L(0, i\Delta) \| + \|\delta L_{i+1}\| \\ \|\bar{F}'(S_{i\Delta} X^0) - \bar{F}'(X_1)\| &\leq C_2 |X_1 - S_{i\Delta} X^0| \leq C_1 C_2 i \bar{A} \Delta^2 \end{aligned}$$

Let us take  $\Delta$  so that

$$\exp(C_3 \sqrt{2} \Delta) - (1 + C_3 \sqrt{2} \Delta) \leq 2C_3^2 \Delta^2.$$

Then we have

$$\|\delta_1 L_{i+1}\| \leq \Delta^2 (2C_1 C_3^2 + C_1 C_2 C_4 + C_1^3 C_2 i \bar{A} + \frac{\beta}{\Delta^2})$$

Finally, we get the following

$$\|L(0, T) - \bar{L}(0, T)\| \leq C_1 T (2C_1 C_3^2 \Delta + C_1 C_2 C_4 + C_1^3 C_2 T \bar{A} + \frac{\beta}{\Delta}).$$

This completes the estimation of  $L$ .

7. ESTIMATION OF  $K_0$ 

In order to evaluate the constant  $K_0$ , we need two lemmas.

Lemma B. (S. De Gregorio and others [2]) If  $|X - X_0| \leq \rho_0$ ,  $\rho_0 = 1/C_1^2 C_2 T$ , then for  $0 \leq t \leq T$ ,

$$|s_t X - s_t X^0| \leq 2C_1 |X - X^0|.$$

Lemma C. ([2]) If  $|X^1 - X^0| \leq \rho_0$ ,  $|X^2 - X^0| \leq \rho_0$ , then for  $0 \leq t \leq T$

$$|s_t X^1 - s_t X^2| \leq 8C_1 |X^1 - X^2|.$$

The proof is analogous to the proof in [2]. Now we are able to estimate  $K_0$ .

Proposition D. If  $\rho \leq \rho_0$  and  $|X^1 - X^0| \leq \rho$ ,  $|X^2 - X^0| \leq \rho$ , then

$$|K(Y^1) - K(Y^2)| \leq K_0 \rho |Y^1 - Y^2|$$

where  $Y^1 = X^1 - X^0$ ,  $Y^2 = X^2 - X^0$  and

$$K_0 = 16TC_1^3 C_2.$$

Proof. By the definition of  $Q(Y)$  we have

$$s_T X^i = PX^0 - X^0 + LY^i + K(Y^i), \quad i = 1, 2.$$

From this we have

$$|K(Y^1) - K(Y^2)| = s_T X^1 - s_T X^2 - L(Y^1 - Y^2).$$

We obtain the equality

$$s_T X^i - s_T X^0 = L(0, T)Y^i + \frac{1}{2} \int_0^T ds L(s, T)h^i(s), \quad i = 1, 2$$

where  $h^i(s) = (\bar{F}''(s_s X^1 - s_s X^0), (s_s X^1 - s_s X^0))$  and hence

$$s_T X^1 - s_T X^2 = L(0, T)(Y^1 - Y^2) + \frac{1}{2} \int_0^T ds L(s, T)(h^1(s) - h^2(s)).$$

Since  $L = L(0, T)$ , we have

$$K(Y^1) - K(Y^2) = \frac{1}{2} \int_0^T ds L(s, T)(h^1(s) - h^2(s)).$$

From this we have

$$|K(Y^1) - K(Y^2)| \leq \frac{1}{2} TC_1 C_2 \sup_{0 \leq s \leq T} |s_s X^1 - s_s X^2| \{ \sup_{0 \leq s \leq T} |s_s X^1 - s_s X^0| + \sup_{0 \leq s \leq T} |s_s X^2 - s_s X^0| \}.$$

By the above two lemmas we have

$$|K(Y^1) - K(Y^2)| \leq 16 TC_1^3 C_2 \rho |Y^1 - Y^2|.$$

This completes the estimation of  $K_0$ .

## REFERENCES

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