

Stationary patterns for a cooperative model with nonlinear diffusion

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1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

$$(P) \begin{cases} u_t = \Delta \left[\left(1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) & \text{in } \Omega \times (0, T), \\ v_t = \Delta v + v(-b + du - v) & \text{in } \Omega \times (0, T), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot), \quad v(\cdot, 0) = v_0(\cdot) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$; a, b, c, d and μ are all positive constants; α is a non-negative constant; u_0 and v_0 are given non-negative functions which are not identically zero. System (P) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions u and v represent population densities of two cooperative species, respectively; a and $-b$ denote the intrinsic growth rates of the respective species; c and d denote interaction coefficients. When $\alpha = 0$, (P) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of (P), the nonlinear diffusion term $\alpha \Delta \{u/(\mu + v)\}$ describes a situation where species u tends to leave low-density areas of species v . This situation is natural because relations between u and v are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including

density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

$$(SP) \begin{cases} \Delta \left[\left(1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) = 0 & \text{in } \Omega, \\ \Delta v + v(-b + du - v) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

$$\frac{a}{b} > \frac{1}{d} > c. \quad (1.1)$$

From now on, we will always assume (1.1). It is well known that, if $\alpha = 0$, then every solution of (P) converges to a unique positive constant steady-state

$$(u^*, v^*) := \left(\frac{a - bc}{1 - cd}, \frac{ad - b}{1 - cd} \right)$$

uniformly as $t \rightarrow \infty$; see [6]. This implies the following proposition.

Proposition 1.1. *Let $\alpha = 0$. Then (u^*, v^*) is a unique positive solution of (SP).*

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on α to show the emergence of stationary patterns for (SP).

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ denote eigenvalues of $-\Delta$ with the homogeneous Neumann boundary condition on $\partial\Omega$ and let m_i denote the algebraic multiplicity of λ_i . Then we have the following theorem.

Theorem 1.1. *Suppose that $\{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1})$ for some $l \geq 1$ and that $\sum_{i=1}^l m_i$ is odd. Then there exists a positive constant $\alpha^* = \alpha^*(a, b, c, d, \mu)$ such that (SP) has at least one positive non-constant solution for each $\alpha > \alpha^*$.*

We are also interested in the limiting patterns of (SP) as $\alpha \rightarrow \infty$. Under the restriction $N \leq 3$, we obtain the following limiting system as $\alpha \rightarrow \infty$.

Theorem 1.2. Suppose $N \leq 3$ and $b > \mu$. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^\infty$ be any sequence such that $\lim_{i \rightarrow \infty} \alpha_i = \infty$ and positive functions (u_i, v_i) satisfy (SP) with $\alpha = \alpha_i$. Then, by passing to a subsequence if necessary, it holds that

$$\lim_{i \rightarrow \infty} (u_i, v_i) = (\tau(\mu + \bar{v}), \bar{v}) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}),$$

where τ is a positive constant satisfying $1 < d\tau < b/\mu$, $\bar{v}$ is a positive function in Ω and $(\tau, \bar{v})$ satisfies

$$\begin{cases} \Delta \bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\} dx = 0. \end{cases} \quad (1.2)$$

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large α . We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\bar{\Omega})$ are defined by

$$\|\psi\|_p := \left(\int_{\Omega} |\psi(x)|^p dx \right)^{1/p} \quad \text{and} \quad \|\psi\|_{\infty} := \max_{x \in \bar{\Omega}} |\psi(x)|,$$

respectively.

2 Stability of the constant solution (u^*, v^*)

In this section, we will analyze the linearized stability of the constant stationary solution (u^*, v^*) for (P).

The linearized eigenvalue problem of (P) at (u^*, v^*) is given by

$$\begin{cases} -\left(1 + \frac{\alpha}{\mu + v^*}\right) \Delta h + \frac{\alpha u^*}{(\mu + v^*)^2} \Delta k + u^* h - c u^* k = \eta h & \text{in } \Omega, \\ -\Delta k - d v^* h + v^* k = \eta k & \text{in } \Omega, \\ \frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

We know that (u^*, v^*) is linearly stable when $\alpha = 0$. Using the expansions of h and k in terms of eigenfunctions of $-\Delta$, one can see that η is an eigenvalue of (2.1) if and only if

$$\det \begin{pmatrix} -\eta + \left(1 + \frac{\alpha}{\mu + v^*}\right) \lambda_i + u^* & -\frac{\alpha u^*}{(\mu + v^*)^2} \lambda_i - c u^* \\ -d v^* & -\eta + \lambda_i + v^* \end{pmatrix} = 0$$

for some $i \geq 0$. In particular, $\eta = 0$ is an eigenvalue of (2.1) if and only if

$$\frac{\lambda_i}{(\mu + v^*)^2} \{(\mu + v^*)(\lambda_i + v^*) - du^*v^*\} \alpha + (\lambda_i + u^*)(\lambda_i + v^*) - cdu^*v^* = 0$$

for some $i \geq 0$. Note that $(\lambda_i + u^*)(\lambda_i + v^*) - cdu^*v^* > 0$ for all $i \geq 0$ because of (1.1). Thus it is easy to see that the linearized stability of (u^*, v^*) changes as α increases in (P) if and only if

$$\begin{aligned} (\mu + v^*)(\lambda_1 + v^*) - du^*v^* &= (\mu + v^*)\lambda_1 + v^*(\mu + v^* - du^*) \\ &= (\mu + v^*)\lambda_1 + v^*(\mu - b) \\ &< 0. \end{aligned}$$

Therefore, $b > \mu$ is necessary for the linearized stability of (u^*, v^*) to change (and so we do not discuss the case $b \leq \mu$, especially, $-b \geq 0$). This means that the difference in the intrinsic growth rates between two species u and v contributes to creating stationary patterns in (SP).

3 Proof of Theorem 1.1

3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function U by

$$U = \left(1 + \frac{\alpha}{\mu + v}\right) u. \quad (3.1)$$

Clearly, there exists a one-to-one correspondence between $(u, v) > 0$ and $(U, v) > 0$. As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:

$$(EP) \begin{cases} \Delta U + \frac{\mu + v}{\mu + v + \alpha} U \left(a - \frac{\mu + v}{\mu + v + \alpha} U + cv \right) = 0 & \text{in } \Omega, \\ \Delta v + v \left(-b + d \frac{\mu + v}{\mu + v + \alpha} U - v \right) = 0 & \text{in } \Omega, \\ \frac{\partial U}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).

3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

Lemma 3.1. *Suppose that $g \in C(\bar{\Omega} \times \mathbb{R})$.*

(i) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \quad \text{on } \partial\Omega,$$

and $w(x_0) = \max_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.

(ii) *If $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \quad \text{on } \partial\Omega,$$

and $w(x_0) = \min_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Now we can derive the following a priori estimates.

Lemma 3.2. *Let ζ be any fixed positive number. Then there exist two positive constants $C_*(\zeta) = C_*(\zeta, a, b, c, d, \mu) < C^*(\zeta) = C^*(\zeta, a, b, c, d, \mu)$ such that, if $\alpha \leq \zeta$, then any positive solution (U, v) of (EP) satisfies*

$$a \leq U(x) \leq C^*(\zeta) \quad \text{and} \quad C_*(\zeta) \leq v(x) \leq C^*(\zeta) \quad \text{for all } x \in \bar{\Omega}.$$

Proof. Let $U(x_0) = \max_{\bar{\Omega}} U$ and $v(y_0) = \max_{\bar{\Omega}} v$ with $x_0, y_0 \in \bar{\Omega}$. Applying Lemma 3.1 to (EP), we have

$$\max_{\bar{\Omega}} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)} (a + cv(x_0))$$

and

$$\max_{\bar{\Omega}} v \leq -b + d \frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha} U(y_0) \leq -b + d \max_{\bar{\Omega}} U. \quad (3.2)$$

Thus

$$\begin{aligned} \max_{\bar{\Omega}} U &\leq a + cv(x_0) + \zeta \frac{a + cv(x_0)}{\mu + v(x_0)} \\ &\leq a + c(-b + d \max_{\bar{\Omega}} U) + \zeta \max \left\{ \frac{a}{\mu}, c \right\}. \end{aligned}$$

Therefore, we see

$$\max_{\bar{\Omega}} U \leq \frac{a - bc + \zeta \max\{a/\mu, c\}}{1 - cd}. \quad (3.3)$$

It follows from (3.2) and (3.3) that

$$\max_{\bar{\Omega}} v \leq -b + \frac{d(a - bc + \zeta \max\{a/\mu, c\})}{1 - cd} = \frac{ad - b + \zeta d \max\{a/\mu, c\}}{1 - cd}. \quad (3.4)$$

Hence we have obtained the desired upper bound of (U, v) .

Let $U(z_0) = \min_{\bar{\Omega}} U$ with some $z_0 \in \bar{\Omega}$. Using Lemma 3.1 to the first equation of (EP), we get

$$\min_{\bar{\Omega}} U \geq \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)}(a + cv(z_0)) \geq a. \quad (3.5)$$

Thus we have obtained the desired lower bound of U .

Finally, we derive a lower bound of v by contradiction. Suppose that there exist a certain positive constant ζ_0 and a sequence $\{(U_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ such that $\alpha_i \leq \zeta_0$ for all $i \in \mathbb{N}$, $\lim_{i \rightarrow \infty} \alpha_i = \alpha_{\infty}$ for some non-negative constant α_{∞} ,

$$\lim_{i \rightarrow \infty} \min_{\bar{\Omega}} v_i = 0 \quad (3.6)$$

and positive functions (U_i, v_i) satisfy

$$\begin{cases} \Delta U_i + \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left(a - \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i \right) = 0 & \text{in } \Omega, \\ \Delta v_i + v_i \left(-b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = 0 & \text{in } \Omega, \\ \frac{\partial U_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that

$$\|v_i\|_{W^{2,p}(\Omega)} \leq C(\zeta_0)$$

with some positive constant $C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)$ independent of i . If $p > N$, then Sobolev's embedding theorem implies $\{v_i\}_{i=1}^{\infty}$ is compact in $C^1(\bar{\Omega})$. Consequently, there exists a subsequence, which is still denoted by $\{v_i\}_{i=1}^{\infty}$, such that

$$\lim_{i \rightarrow \infty} v_i = v_{\infty} \quad \text{in } C^1(\bar{\Omega}) \quad (3.8)$$

with some non-negative function $v_{\infty} \in C^1(\bar{\Omega})$. Similarly, there exists a non-negative function $U_{\infty} \in C^1(\bar{\Omega})$ such that

$$\lim_{i \rightarrow \infty} U_i = U_{\infty} \quad \text{in } C^1(\bar{\Omega}). \quad (3.9)$$

Therefore, v_{∞} satisfies

$$\Delta v_{\infty} + v_{\infty} \left(-b + d \frac{\mu + v_{\infty}}{\mu + v_{\infty} + \alpha_{\infty}} U_{\infty} - v_{\infty} \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial v_{\infty}}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. By standard elliptic regularity theory we have $v_\infty \in C^2(\bar{\Omega})$ and thus v_∞ is a classical solution of the above equation. Then it follows from (3.6), (3.8) and the strong maximum principle that $v_\infty \equiv 0$ in $\bar{\Omega}$. We can easily see from the above argument that U_∞ satisfies

$$\Delta U_\infty + \frac{\mu}{\mu + \alpha_\infty} U_\infty \left(a - \frac{\mu}{\mu + \alpha_\infty} U_\infty \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial U_\infty}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either $U_\infty \equiv a(\mu + \alpha_\infty)/\mu$ or $U_\infty \equiv 0$ in $\bar{\Omega}$. Combining (3.5) and (3.9), we can conclude $U_\infty \equiv a(\mu + \alpha_\infty)/\mu$ in $\bar{\Omega}$. Hence

$$\lim_{i \rightarrow \infty} \left(-b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = ad - b > 0 \quad \text{uniformly in } \Omega$$

by (1.1) and this means

$$v_i \left(-b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) > 0 \quad \text{in } \Omega$$

for sufficiently large $i \in \mathbb{N}$ because $v_i > 0$ in Ω . On the other hand, from the second equation of (3.7), we have

$$\int_{\Omega} v_i \left(-b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) dx = - \int_{\Omega} \Delta v_i dx = - \int_{\partial\Omega} \frac{\partial v_i}{\partial n} d\sigma = 0$$

for all $i \in \mathbb{N}$. This is a contradiction; thus our proof is complete. $\square$

3.3 Completion of the proof of Theorem 1.1

Set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For each $\alpha \geq 0$, define an operator F_α by

$$F_\alpha \begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} (-\Delta + I)^{-1} \left[U + \frac{\mu+v}{\mu+v+\alpha} U \left(a - \frac{\mu+v}{\mu+v+\alpha} U + cv \right) \right] \\ (-\Delta + I)^{-1} \left[v + v \left(-b + d \frac{\mu+v}{\mu+v+\alpha} U - v \right) \right] \end{pmatrix},$$

where I is the identity map from $C(\bar{\Omega})$ into itself, and $(-\Delta + I)^{-1}$ is the inverse operator of $-\Delta + I$ subject to the homogeneous Neumann boundary condition on $\partial\Omega$. It is easy to see that $F_\alpha : X \rightarrow X$ is well-defined, and that by elliptic regularity theory and Sobolev's embedding theorem, F_α is a continuous and compact operator for each $\alpha \geq 0$. From these observations, one can define the Leray-Schauder degree of $I - F_\alpha$ at 0 in a suitable open set. Furthermore, (U, v) is a positive solution of $(I - F_\alpha)(U, v) = 0$ if and only if (U, v) is a positive solution of (EP).

In view of (3.1), we set

$$U_\alpha^* = \left(1 + \frac{\alpha}{\mu + v^*} \right) u^*.$$

Hence (U_α^*, v^*) is a zero point of $I - F_\alpha$. Then we can calculate the index of $I - F_0$ at (u^*, v^*) and the index of $I - F_\alpha$ at (U_α^*, v^*) for sufficiently large α , which are denoted by $\text{index}(I - F_0, (u^*, v^*))$ and $\text{index}(I - F_\alpha, (U_\alpha^*, v^*))$, respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

Lemma 3.3. *It holds that $\text{index}(I - F_0, (u^*, v^*)) = 1$.*

Lemma 3.4. *Suppose that $\{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1})$ for some $l \geq 1$. Then there exists a positive constant $\alpha^* = \alpha^*(a, b, c, d, \mu)$ such that, if $\alpha > \alpha^*$, then*

$$\text{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^l m_i},$$

where m_i denotes the algebraic multiplicity of λ_i defined in Section 1.

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent (EP) as $(\text{EP})_\alpha$ to indicate the dependence on α .

Proof of Theorem 1.1. Fix any $\alpha > \alpha^*$, where α^* is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants $C_*(\alpha) = C_*(\alpha, a, b, c, d, \mu) < C^*(\alpha) = C^*(\alpha, a, b, c, d, \mu)$ such that

$$a \leq U(x) \leq C^*(\alpha) \quad \text{and} \quad C_*(\alpha) \leq v(x) \leq C^*(\alpha) \quad \text{for all } x \in \bar{\Omega}$$

for any positive solution (U, v) of $(\text{EP})_\nu$ with any $\nu \in [0, \alpha]$. We define

$$S = \left\{ (U, v) \in X \mid \frac{a}{2} \leq U \leq 2C^*(\alpha), \quad \frac{C_*(\alpha)}{2} \leq v \leq 2C^*(\alpha) \text{ in } \bar{\Omega} \right\};$$

so that $I - F_\nu$ has no zero point on the boundary of S for any $\nu \in [0, \alpha]$. Note that $I - F_0$ has a unique zero point (u^*, v^*) in S . On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

$$\deg(I - F_\alpha, S, 0) = \deg(I - F_0, S, 0) = \text{index}(I - F_0, (u^*, v^*)) = 1. \quad (3.10)$$

Suppose that $(\text{EP})_\alpha$ has no positive non-constant solution, i.e. $I - F_\alpha$ has a unique zero point (U_α^*, v^*) in S . Then from the assumption $\sum_{i=1}^l m_i$ being odd and Lemma 3.4, it follows that

$$\deg(I - F_\alpha, S, 0) = \text{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^l m_i} = -1,$$

which contradicts (3.10). Thus we complete the proof. $\square$

4 Proof of Theorem 1.2

We first state some a priori estimates independent of α .

Lemma 4.1. *Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of α such that any positive solution (u, v) of (SP) satisfies*

$$\|u\|_\infty \leq C_0 \quad \text{and} \quad \|v\|_\infty \leq C_0.$$

Lemma 4.1 can be proved by combining the L^2 -estimates for positive solutions of (SP) (independent of α and N) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^\infty$ be any sequence such that $\lim_{i \rightarrow \infty} \alpha_i = \infty$ and positive functions (u_i, v_i) satisfy (SP) with $\alpha = \alpha_i$. Set

$$\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i} \right) u_i.$$

Note that positive functions (ψ_i, v_i) satisfy

$$\begin{cases} \Delta \psi_i + \frac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 & \text{in } \Omega, \\ \Delta v_i + v_i(-b + du_i - v_i) = 0 & \text{in } \Omega, \\ \frac{\partial \psi_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and that $\{\psi_i\}_{i=1}^\infty$ is bounded independently of i by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^\infty$, such that

$$\lim_{i \rightarrow \infty} \psi_i = \tau \quad \text{in } C^1(\bar{\Omega})$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$\lim_{i \rightarrow \infty} v_i = \bar{v} \quad \text{in } C^1(\bar{\Omega}) \tag{4.1}$$

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$\lim_{i \rightarrow \infty} u_i = \lim_{i \rightarrow \infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in } C^1(\bar{\Omega}). \tag{4.2}$$

We will show that τ is a positive constant. Observe that τ satisfies

$$\Delta\tau = 0 \quad \text{in } \Omega, \quad \frac{\partial\tau}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. A standard elliptic regularity theory ensures $\tau \in C^2(\bar{\Omega})$; so that τ must be a non-negative constant. Let $v_i(x_i) = \max_{\bar{\Omega}} v_i$ with some $x_i \in \bar{\Omega}$. It follows from Lemma 3.1 that

$$u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d} (> 0)$$

for all $i \in \mathbb{N}$. This fact, together with (4.2), yields $\tau > 0$.

We next prove $(\tau, \bar{v})$ satisfies (1.2). Note that $\bar{v}$ satisfies

$$\Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 \quad \text{in } \Omega, \quad \frac{\partial\bar{v}}{\partial n} = 0 \quad \text{on } \partial\Omega \quad (4.3)$$

in a weak sense. In the standard manner, one can see that $\bar{v} \in C^2(\bar{\Omega})$ and $\bar{v}$ is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either $\bar{v} \equiv 0$ or $\bar{v} > 0$ in Ω . We show $\bar{v} > 0$ in Ω by contradiction. Suppose that $\bar{v} \equiv 0$ in Ω . Then it follows from (4.1) and (4.2) that

$$\lim_{i \rightarrow \infty} a - u_i + cv_i = a - \tau\mu \quad \text{and} \quad \lim_{i \rightarrow \infty} -b + du_i - v_i = -b + d\tau\mu$$

uniformly in Ω . On the other hand,

$$\int_{\Omega} u_i(a - u_i + cv_i)dx = \int_{\Omega} v_i(-b + du_i - v_i)dx = 0 \quad (4.4)$$

for all $i \in \mathbb{N}$. Consequently, $a - \tau\mu = -b + d\tau\mu = 0$ because of $u_i > 0$ and $v_i > 0$ in Ω and thus $ad - b = 0$. This contradicts (1.1). Therefore $\bar{v} > 0$ in Ω .

By (4.1), (4.2) and (4.4), it is clear that

$$\int_{\Omega} (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\}dx = \int_{\Omega} (\mu + \bar{v})\{a - \tau(\mu + \bar{v}) + c\bar{v}\}dx = 0.$$

Hence it only remains to show $1 < d\tau < b/\mu$. By the assumption of Theorem 1.2,

$$-b + d\tau\mu < -\mu + d\tau\mu = \mu(d\tau - 1).$$

It thus follows from Lemma 3.1 and (4.3) that if $d\tau - 1 \leq 0$, then $\max_{\bar{\Omega}} \bar{v} \leq 0$ and this contradicts $\bar{v} > 0$ in Ω . Therefore, $d\tau > 1$. Using Lemma 3.1 and $\bar{v} > 0$ in Ω again, we obtain $d\tau < b/\mu$. Hence we complete the proof. $\square$

5 Remarks about the limiting system (1.2)

We easily see that $(\tau, \bar{v}) = (u^*/(\mu + v^*), v^*)$ is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction $N \leq 3$:

$$\begin{cases} \Delta \bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

Set

$$w = \frac{d\tau - 1}{b - d\tau\mu} \bar{v},$$

where $1 < d\tau < b/\mu$. Then (5.1) is rewritten in the following equivalent form:

$$\begin{cases} \frac{1}{b - d\tau\mu} \Delta w - w + w^2 = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

We note that, if $(0 <) b - d\tau\mu \ll 1$, then (5.2) has no positive non-constant solution (see [4]). Therefore, $b \gg 1$ is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

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