

Tsallis entropy に関する作用素不等式と トレース不等式

Operator inequalities and trace inequalities derived from Tsallis entropies

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1 Trace inequalities of Tsallis entropy

We define q -logarithm function as follows;

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q}, \quad (x \geq 0, q \geq 0, q \neq 1).$$

Then we have the following properties;

- (1) $\lim_{q \rightarrow 1} \ln_q x = \log x$. (in uniformly)
- (2) $\ln_q xy = \ln_q x + \ln_q y + (1 - q) \ln_q x \ln_q y$.
- (3) $\ln_q x$: concave in x for $q \geq 0$.

Definition 1 (Tsallis entropy) For density operator ρ on a finite dimensional Hilbert space $\mathcal{H}$, Tsallis entropy $S_q(\rho)$ is defined by

$$S_q(\rho) = \frac{\text{Tr}[\rho^q - \rho]}{1 - q}, \quad (q \geq 0, q \neq 1).$$

Proposition 1 *We have the following properties;*

- (1) $\lim_{q \rightarrow 1} S_q(\rho) = -\text{Tr}[\rho \log \rho]$.
- (2) $S_q(\rho_1 \otimes \rho_2) = S_q(\rho_1) + S_q(\rho_2) + (1 - q)S_q(\rho_1)S_q(\rho_2)$.

Lemma 1 $S_q(\rho) \leq \ln_q d$, ($d = \dim \mathcal{H}$).

Proof. Since $\ln_q x$ is concave, we have

$$D_q(A|B) = -\sum_{j=1}^d a_j \ln_q \frac{b_j}{a_j} \geq -\ln_q \left(\sum_{j=1}^d a_j \frac{b_j}{a_j} \right) = 0.$$

We put $A = \{a_j\}$, $B = \{u_j\}$, $u_j = \frac{1}{d}$ ($1 \leq j \leq d$). Then

$$D_q(A|B) = -d^{q-1}(S_q(A) - \ln_q d) \geq 0.$$

Thus $S_q(A) \leq \ln_q d$.

q.e.d.

Lemma 2 *If f is a concave function satisfying $f(0) = f(1) = 0$, then*

$$|f(t+s) - f(t)| \leq \max\{f(s), f(1-s)\},$$

where $s \in [0, 1/2]$, $t \in [0, 1]$, $0 \leq s+t \leq 1$.

Proof. We put

$$r(t) = f(s) - f(t+s) + f(t).$$

Then

$$r'(t) = -f'(t+s) + f'(t).$$

Since f' is a monotone decreasing function, $r'(t) \geq 0$. Thus we have $r(t) \geq 0$ by $r(0) = 0$. Therefore $f(t+s) - f(t) \leq f(s)$. We also put

$$\ell(t) = f(t+s) - f(t) + f(1-s).$$

Then

$$\ell'(t) = f'(t+s) - f'(t).$$

Since f' is a monotone decreasing function, $\ell'(t) \leq 0$. Thus we have $\ell(t) \geq 0$ by $\ell(1-s) = 0$. Therefore $-f(1-s) \leq f(t+s) - f(t)$. Thus we have the result. q.e.d.

Lemma 3 If $|u - v| \leq 1/2$, then $|\eta_q(u) - \eta_q(v)| \leq \eta_q(|u - v|)$,
 where $\eta_q(x) = \frac{x^q - x}{1 - q}$, $q \in [0, 2]$, $u, v \in [0, 1]$.

Proof. Since η_q is a concave function with $\eta_q(0) = \eta_q(1)$, we have

$$|\eta_q(t + s) - \eta_q(t)| \leq \max\{\eta_q(s), \eta_q(1 - s)\}$$

for $s \in [0, \frac{1}{2}]$ and $t \in [0, 1]$ satisfying $0 \leq t + s \leq 1$ by Lemma 2.
 Since $\eta_q(x)$ is a monotone increasing function on $[0, q^{1/(1-q)}]$ and $q^{1/(1-q)} \leq \frac{1}{2}$
 for $q \in (0, 2]$,

$$\max\{\eta_q(s), \eta_q(1 - s)\} = \eta_q(s).$$

Thus we have the result by letting $u = t + s$ and $v = t$. q.e.d.

Lemma 4 Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be eigenvalues of Hermitian matrix A and
 $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ be eigenvalues of Hermitian matrix B . Then we have the
 following;

$$\text{Tr}[|A - B|] \geq \sum_{i=1}^n |\lambda_i - \mu_i|.$$

Theorem 1 (Generalized Fannes's inequality) For two density operators
 ρ_1, ρ_2 on $\mathcal{H}$ and $q \in [0, 2]$, if $\|\rho_1 - \rho_2\|_1 \leq q^{1/(1-q)}$, then

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \|\rho_1 - \rho_2\|_1^q \ln_q d + \eta_q(\|\rho_1 - \rho_2\|_1),$$

where $d = \dim \mathcal{H}$ and $\|A\|_1 = \text{Tr}[|A|]$.

Proof. Let $\lambda_1^{(i)} \geq \dots \geq \lambda_n^{(i)}$ be eigenvalues of ρ_i .

We set

$$\epsilon = \sum_{j=1}^d \epsilon_j, \quad \epsilon_j = |\lambda_j^{(1)} - \lambda_j^{(2)}|.$$

From Lemma 2,

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \sum_{j=1}^d |\eta_q(\lambda_j^{(1)}) - \eta_q(\lambda_j^{(2)})| \leq \sum_{j=1}^d \eta_q(\epsilon_j).$$

By $\ln_q(xy) = \ln_q x + x^{1-q} \ln_q y$ and Lemma 1, we have

$$\begin{aligned}
\sum_{j=1}^d \eta_q(\epsilon_j) &= -\sum_{j=1}^d \epsilon_j^q \ln_q \epsilon_j = \epsilon \left\{ -\sum_{j=1}^d \frac{\epsilon_j^q}{\epsilon} \ln_q \left(\frac{\epsilon_j}{\epsilon} \right) \right\} \\
&= \epsilon \left\{ -\sum_{j=1}^d \frac{\epsilon_j^q}{\epsilon} \ln_q \frac{\epsilon_j}{\epsilon} - \sum_{j=1}^d \frac{\epsilon_j^q}{\epsilon} \left(\frac{\epsilon_j}{\epsilon} \right)^{1-q} \ln_q \epsilon \right\} \\
&= \epsilon_q \sum_{j=1}^d \eta_q \left(\frac{\epsilon_j}{\epsilon} \right) + \eta_q(\epsilon) \leq \epsilon^q \ln_q d + \eta_q(\epsilon).
\end{aligned}$$

Therefore we have

$$|S_q(\rho_1) - S_q(\rho_2)| \leq \epsilon^q \ln_q d + \eta_q(\epsilon).$$

From Lemma 3, we have $\|\rho_1 - \rho_2\|_1 \geq \epsilon$. And $\eta_q(x)$ is monotone increase on $x \in [0, q^{1/(1-q)}]$. In addition, x^q is monotone increase on $x \in [0, 2]$. Thus we have theorem.

q.e.d.

Since $\lim_{q \rightarrow 1} q^{1/(1-q)} = 1/e$, we have

Corollary 1 (Fannes's inequality) *For two density operators ρ_1, ρ_2 on $\mathcal{H}$, if $\|\rho_1 - \rho_2\|_1 \leq 1/e$, then*

$$|S_1(\rho_1) - S_1(\rho_2)| \leq \|\rho_1 - \rho_2\|_1 \log d + \eta_1(\|\rho_1 - \rho_2\|_1),$$

where $S_1(\rho) = -\text{Tr}[\rho \log \rho]$, $\eta_1(x) = -x \log x$.

2 Operator inequalities of Tsallis relative operator entropy

We change the notation ($\lambda = 1 - q$). That is, for $\lambda \in (0, 1]$,

$$\ln_\lambda x = \frac{x^\lambda - 1}{\lambda}.$$

Definition 2 (Tsallis relative operator entropy) *For $A > 0, B > 0, \lambda \in (0, 1]$, Tsallis relative operator entropy $T_\lambda(A|B)$ is defined by*

$$T_\lambda(A|B) = A^{1/2} \ln_\lambda(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

Proposition 2 *we have the following properties;*

- (1) $\lim_{\lambda \rightarrow 0} T_\lambda(A|B) = S(A|B) = A^{1/2} \log(A^{-1/2} B A^{-1/2}) A^{1/2}$.
- (2) $T_\lambda(\alpha A | \alpha B) = \alpha T_\lambda(A|B)$, $\alpha \in \mathbb{R}^+$.
- (3) *If $B \leq C$, then $T_\lambda(A|B) \leq T_\lambda(A|C)$.*
- (4) $T_\lambda(A_1 + A_2 | B_1 + B_2) \geq T_\lambda(A_1 | B_1) + T_\lambda(A_2 | B_2)$.
- (5) $T_\lambda(\alpha A_1 + \beta A_2 | \alpha B_1 + \beta B_2) \geq \alpha T_\lambda(A_1 | B_1) + \beta T_\lambda(A_2 | B_2)$.
- (6) $T_\lambda(U A U^* | U B U^*) = U T_\lambda(A|B) U^*$.
- (7) $\Phi(T_\lambda(A|B)) \leq T_\lambda(\Phi(A) | \Phi(B))$, *where U is an unital positive linear map.*

Remark 1 *Same properties are shown for a more general case by Fujii et.al.*

Solodarity $A \sharp B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$ *for operator monotone f .*

Since

$$\frac{x^{-\lambda} - 1}{-\lambda} \leq \log x \leq \frac{x^\lambda - 1}{\lambda}$$

for $x > 0, \lambda > 0$, we have the following.

Proposition 3 *For $A > 0, B > 0, \lambda \in (0, 1]$, we have the following;*

$$T_{-\lambda}(A|B) \leq S(A|B) \leq T_\lambda(A|B).$$

Since

$$1 - \frac{1}{x} \leq \ln_\lambda x \leq x - 1$$

for $x > 0, 0 < \lambda \leq 1$, we have the following.

Proposition 4 *For $A > 0, B > 0, \lambda \in (0, 1]$, we have the following;*

$$A - AB^{-1}A \leq T_\lambda(A|B) \leq B - A.$$

Since

$$x^\lambda \left(1 - \frac{1}{\alpha x}\right) + \ln_\lambda \frac{1}{\alpha} \leq \ln_\lambda x \leq \frac{x}{\alpha} - 1 - x^\lambda \ln_\lambda \frac{1}{\alpha}$$

for $\alpha > 0, x > 0, 0 < \lambda \leq 1$, we have the following.

Theorem 2 For $A > 0, B > 0, \alpha > 0, \lambda \in (0, 1]$, we have the following;

$$A\sharp_{\lambda}B - \frac{1}{\alpha}A\sharp_{\lambda-1}B + (\ln_{\lambda}\frac{1}{\alpha})A \leq T_{\lambda}(A|B) \leq \frac{1}{\alpha}B - A - (\ln_{\lambda}\frac{1}{\alpha})A\sharp_{\lambda}B,$$

where $A\sharp_{\lambda}B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}$.

We have the following by taking $\lambda \rightarrow 0, \alpha = 1$, respectively;

Corollary 2 For $A > 0, B > 0, \alpha > 0$, we have the following;

$$(1 - \log \alpha)A - \frac{1}{\alpha}AB^{-1}A \leq S(A|B) \leq (\log \alpha - 1)A + \frac{1}{\alpha}B.$$

For $A > 0, B > 0$, we have the following;

$$A - AB^{-1}A \leq S(A|B) \leq B - A.$$

Lemma 5 For $X > 0, Y > 0, a \in \mathbb{R}$, we have

$$(X \otimes Y)^a = X^a \otimes Y^a.$$

Theorem 3 For $A_1, A_2, B_1, B_2 > 0, \lambda \in (0, 1]$, we have the following;

$$T_{\lambda}(A_1 \otimes A_2 | B_1 \otimes B_2) = T_{\lambda}(A_1 | B_1) \otimes A_2 + A_1 \otimes T_{\lambda}(A_2 | B_2) + \lambda T_{\lambda}(A_1 | B_1) \otimes T_{\lambda}(A_2 | B_2).$$

Proof. From Lemma 5, we have for $X > 0, Y > 0, \lambda \in (0, 1]$,

$$\begin{aligned} \ln_{\lambda}(X \otimes Y) &= (\ln_{\lambda} X) \otimes I + I \otimes (\ln_{\lambda} Y) \\ &\quad + \lambda (\ln_{\lambda} X) \otimes (\ln_{\lambda} Y). \end{aligned}$$

By putting $X = A_1^{-1/2}B_1A_1^{-1/2}, Y = A_2^{-1/2}B_2A_2^{-1/2}$ and by multiplying $A_1^{1/2} \otimes A_2^{1/2}$ from both sides. we have the theorem. q.e.d.

Corollary 3 For $A_1, A_2, B_1, B_2 > 0$, we have

$$S(A_1 \otimes A_2 | B_1 \otimes B_2) = S(A_1 | B_1) \otimes A_2 + A_1 \otimes S(A_2 | B_2).$$

Since we put $B_1 = B_2 = I, A_i = \rho_i$, we have the following;

Corollary 4 (pseudo additivity) *For ρ_1, ρ_2 , we have*

$$S_\lambda(\rho_1 \otimes \rho_2) = S_\lambda(\rho_1) + S_\lambda(\rho_2) + \lambda S_\lambda(\rho_1) S_\lambda(\rho_2).$$

Corollary 5 *From Theorem 3 we have the following inequalities;*

(1) *For $\lambda \in (0, 1]$ and $0 < A_i \leq B_i$ ($i = 1, 2$), we have*

$$(a) \quad T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2).$$

$$(b) \quad T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_2 | B_2).$$

(2) *For $\lambda \in (0, 1]$ and $0 < B_i \leq A_i$ ($i = 1, 2$), we have*

$$(c) \quad T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \leq \lambda T_\lambda(A_1 | B_1) \otimes T_\lambda(A_2 | B_2).$$

$$(d) \quad T_\lambda(A_1 \otimes A_2 | B_1 \otimes B_2) \geq T_\lambda(A_1 | B_1) \otimes A_2 + A_1 \otimes T_\lambda(A_1 | B_2).$$

3 Trace inequalities of Tsallis relative entropy

Definition 3 (Tsallis relative entropy) *For density operators ρ, σ , Tsallis relative entropy is defined by*

$$D_\lambda(\rho | \sigma) = \frac{\text{Tr}[\rho - \rho^{1-\lambda} \sigma^\lambda]}{\lambda}, \quad \lambda \in (0, 1].$$

Theorem 4 $D_\lambda(\rho | \sigma) \leq -\text{Tr}[T_\lambda(\rho | \sigma)]$.

Proof. We remark that

$$A \sharp_\alpha B = A^{1/2} (A^{-1/2} B A^{-1/2})^\alpha A^{1/2}$$

is α power mean. By Theorem 3.4 in Hiai-Petz [3], we have

$$\text{Tr}[e^{A \sharp_\alpha B}] \leq \text{Tr}[e^{(1-\alpha)A + \alpha B}].$$

for any $\alpha \in [0, 1]$. We put $A = \log \rho, B = \log \sigma$.

$$\text{Tr}[\rho \sharp_\alpha \sigma] \leq \text{Tr}[e^{\log \rho^{1-\alpha} + \log \sigma^\alpha}].$$

We apply Golden-Thompson inequality

$$\text{Tr}[e^{A+B}] \leq \text{Tr}[e^A e^B]$$

for any Hermitian operators A, B . Then we have

$$\text{Tr}[e^{\log \rho^{1-\alpha} + \log \sigma^\alpha}] \leq \text{Tr}[e^{\log \rho^{1-\alpha}} e^{\log \sigma^\alpha}] = \text{Tr}[\rho^{1-\alpha} \sigma^\alpha].$$

Thus we have

$$\text{Tr}[\rho^{1/2}(\rho^{-1/2} \sigma \rho^{-1/2})^\alpha \rho^{1/2}] \leq \text{Tr}[\rho^{1-\alpha} \sigma^\alpha].$$

q.e.d.

Corollary 6 (Hiai-Petz) $\text{Tr}[\rho(\log \rho - \log \sigma)] \leq -\text{Tr}[\rho \log(\rho^{-1/2} \sigma \rho^{-1/2})]$.

Definition 4 (Tsallis relative entropy) For positive operators A, B and $0 < \lambda \leq 1$, we define

$$D_\lambda(A\|B) = \frac{\text{Tr}[A - A^{1-\lambda} B^\lambda]}{\lambda}.$$

Theorem 5 (Generalized Bogoliubov inequality) For positive operators A, B and $0 < \lambda \leq 1$, we have the following;

$$D_\lambda(A\|B) \geq \frac{\text{Tr}[A] - (\text{Tr}[A])^{1-\lambda} (\text{Tr}[B])^\lambda}{\lambda}.$$

Proof. It follows by the application of the Holder's inequality:

$$|\text{Tr}[XY]| \leq \text{Tr}[|X|^s]^{1/s} \text{Tr}[|Y|^t]^{1/t}$$

for $1 < s, t < \infty$, $1/s + 1/t = 1$.

q.e.d.

Corollary 7 (Peierls-Bogoliubov inequality) For positive operators A, B , we have the following;

$$\text{Tr}[A(\log A - \log B)] \geq \text{Tr}[A](\log \text{Tr}[A] - \log \text{Tr}[B]).$$

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