

INFINITE DIMENSIONAL LIE ALGEBRAS, VERTEX ALGEBRAS AND W-ALGEBRAS

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1. INTRODUCTION

1.1. One of the distinguished features of infinite dimensional Lie algebras is the modular invariance of the characters of certain representations. There are two celebrated examples for this phenomena: One is the integrable highest weight representations of an affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a simple Lie algebra $\mathfrak{g}$ at a fixed level [KP], and the other is the minimal series representations [FFu] of the Virasoro algebra Vir with a fixed central charge.

However there is a relevant difference in these two examples: The Virasoro algebra is a single Lie algebra, while affine Lie algebras constitute a family of Lie algebras. Therefore it is natural to consider a generalization of the Virasoro algebra.

The *W-algebras* can be regarded as such a generalization of the Virasoro algebra. Some people say that this is the reason why they are called the “W-algebras” (because the letter “W” comes right after “V” alphabetically). The first example of a W-algebra was discovered by Zamalodchikov [Za] in his study of classification of conformal field theory (see [BS] and reference therein.).

1.2. In general, there is the W-algebra $\mathcal{W}(\mathfrak{g})$ associated with any simple Lie algebra $\mathfrak{g}$ ([FF2]). The simplest W-algebra is the W-algebra $\mathcal{W}(\mathfrak{sl}_2)$ associated with $\mathfrak{sl}_2$. This is nothing but the Virasoro algebra (or more precisely, the corresponding vertex algebra). The Virasoro algebra Vir is the Lie algebra with the following generators and the relations:

$$\begin{aligned} \text{generators: } & L_n \ (n \in \mathbb{Z}), \ c \\ \text{relations: } & [L_n, c] = 0 \\ & [L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m+n,0}c. \end{aligned}$$

The next simplest W-algebra is the one associated with $\mathfrak{sl}_3$; $\mathcal{W}(\mathfrak{sl}_3)$ is defined by the following generators and relations:

$$\begin{aligned} \text{generators: } & \mathbf{c}, L_n \ (n \in \mathbb{Z}), W_n \ (n \in \mathbb{Z}), \\ \text{relations: } & [\mathbf{c}, \mathcal{W}(\mathfrak{sl}_3)] = 0, \\ & [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c}, \\ & [L_m, W_n] = (2m-n)W_{m+n}, \\ & [W_m, W_n] \\ & = (m-n)\left\{\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right\}L_{m+n} \\ & + \frac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n} + \frac{\mathbf{c}}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}, \end{aligned}$$

where

$$(1) \quad \Lambda_n = \sum_{k \leq 0} L_k L_{n-k} + \sum_{k \geq 0} L_{n-k} L_k - \frac{3}{10}(n+2)(n+3)L_n.$$

In the above formula, the pole at $\mathbf{c} = -22/5$ can be removed if we multiply W_n by $22 + 5\mathbf{c}$, and therefore it is inessential. More serious is the existence of the infinite sum of the quadratic term of the form $L_{n-k}L_k$. This means that the above *does not define* a Lie algebra in the usual sense. In general, W-algebras are no more Lie algebras and one should understand them as *vertex algebras* (see [K2, FB, BD] for the definition of vertex algebras).

1.3. As we have seen in the above, $\mathcal{W}(\mathfrak{g})$ has a complicated algebraic structure except for the case that $\mathfrak{g} = \mathfrak{sl}_2$. In fact, even the defining relations of the generators are not known for a general $\mathcal{W}(\mathfrak{g})$! Thus, instead of defining it by generators and relations, W-algebras are usually defined by a cohomological method. This method is called the *quantized Drinfeld-Sokolov reduction*, or simply the *quantum reduction*, and was discovered by Feigin and Frenkel [FF2]. This is a powerful method, in the sense that it not only gives a uniform definition of $\mathcal{W}(\mathfrak{g})$, but also defines a functor from a suitable category (the category $\mathcal{O}$) of $\hat{\mathfrak{g}}$ -modules to the category of $\mathcal{W}(\mathfrak{g})$ -modules. Frenkel, Kac and Wakimoto [FKW] conjectured that one can obtain a family of modular invariant representations of $\mathcal{W}(\mathfrak{g})$ from the modular invariant representations (admissible representations) of $\hat{\mathfrak{g}}$ via this functor. If this is true then one can surely say that $\mathcal{W}(\mathfrak{g})$ is a generalization of *Vir*, for it inherits our favorite property of the Virasoro algebra.

1.4. The propose of this note to describe the representation theory of $\mathcal{W}(\mathfrak{g})$ via quantum reduction. In particular, we explain how the conjecture of Frenkel, Kac and Wakimoto follows from our general results.

2. FINITE DIMENSIONAL CASE

2.1. Recall that $\hat{\mathfrak{g}}$ is an affinization (or a *chiralization*) of the finite dimensional Lie algebra $\mathfrak{g}$. In this sense, the Virasoro algebra *Vir* is a chiralization of its zero mode, “ CL_0 ”. And because L_0 corresponds to the Casimir operator (via the Sugawara construction), one can think of $Vir = \mathcal{W}(\mathfrak{sl}_2)$ as a chiralization of the center $\mathcal{Z}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$. This is true in general:

$\mathcal{W}(\mathfrak{g})$ is a chiralization of the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$.

2.2. Kostant's Theorem. Let e be a principal nilpotent element of $\mathfrak{g}$. For instance, if $\mathfrak{g} = \mathfrak{sl}_n$, then e has the form

$$e = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By the Jacobson-Morozov theorem there exists a corresponding $\mathfrak{sl}_2$ -triple $\{e, h_0, f\}$:

$$[h_0, e] = 2e, \quad [h_0, f] = -2f, \quad [e, f] = h_0.$$

Then we have the eigenspace decomposition of $\mathfrak{g}$ with respect to the adjoint action of $\rho^\vee := h_0/2$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g}; [\rho^\vee, x] = jx\}.$$

Because e is principal, this gives a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_+ = \sum_{j>0} \mathfrak{g}_j, \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}_- = \sum_{j<0} \mathfrak{g}_j.$$

Let $\Delta_+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots, $\Delta_- = -\Delta_+$, $\Delta = \Delta_+ \sqcup \Delta_-$.

Define $p \in \mathfrak{n}_-^*$ by

$$p(x) = (x, e).$$

Here $(\cdot, \cdot)$ is the normalized invariant inner product of $\mathfrak{g}$. Then $p([\mathfrak{n}_-, \mathfrak{n}_-]) = 0$ and p defines a character of $\mathfrak{n}_-$.

Let $\mathcal{Cl}$ be the Clifford algebra associated with the space $\mathfrak{n}_- \oplus \mathfrak{n}_-^*$ and the natural bilinear form on it. Then $\mathcal{Cl}$ has the following generators and relations:

generators: $\psi_\alpha, \psi_\alpha^* \quad (\alpha \in \Delta_-),$

relations: $\{\psi_\alpha, \psi_\beta^*\} = \delta_{\alpha, \beta}, \quad \{\psi_\alpha, \psi_\beta\} = \{\psi_\alpha^*, \psi_\beta^*\} = 0.$

We shall regard

$$U(\mathfrak{g}) \otimes \mathcal{Cl}$$

as a superalgebra with even generators $\mathfrak{g} \ni x = x \otimes 1$ and odd generators $\psi_\alpha = 1 \otimes \psi_\alpha, \psi_\alpha^* = 1 \otimes \psi_\alpha^*$.

Define an odd element $Q^{\text{st}} \in U(\mathfrak{g}) \otimes \mathcal{Cl}$ by

$$Q^{\text{st}} = \sum_{\alpha \in \Delta_-} x_\alpha \psi_\alpha^* - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_-} c_{\alpha, \beta}^\gamma \psi_\alpha^* \psi_\beta^* \psi_\gamma.$$

Here x_α is a (fixed) root vector of root α and $c_{\alpha, \beta}^\gamma$ is the structure constant. Then by direct calculation one can check that $[Q^{\text{st}}, Q^{\text{st}}] = 0$, or equivalently,

$$(Q^{\text{st}})^2 = 0.$$

We remark that the "st" suffix stands for "standard", because Q^{st} is the differential of the standard Lie algebra cohomology or homology.

Set

$$Q := Q^{\text{st}} + p,$$

where p is considered as an element of $Cl \subset U(\mathfrak{g}) \otimes Cl$:

$$p = \sum_{\alpha \in \Delta_-} p(x_\alpha) \psi_\alpha^*.$$

Lemma 1. $[p, p] = [Q^{\text{st}}, p] = 0$. Therefore $[Q, Q] = 0$, or equivalently $Q^2 = 0$.

By Lemma 1 it follows that

$$(\text{ad } Q)^2 = 0$$

on $U(\mathfrak{g}) \otimes Cl$. Hence we can consider $(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$ as a homology complex by setting

$$\begin{aligned} \deg u &= 0 \quad (u \in U(\mathfrak{g})), \\ \deg \psi_\alpha &= 1, \quad \deg \psi_\alpha^* = -1 \quad (\alpha \in \Delta_-). \end{aligned}$$

Then the corresponding homology

$$H_\bullet(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) = \bigoplus_{i \in \mathbb{Z}} H_i(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$$

inherits the graded superalgebra structure from $U(\mathfrak{g}) \otimes Cl$.

Theorem 1 (Kostant [Ko], Kostant-Sternberg [KS], cf. [A3, Theorem 2.3.2]).

- (i) $H_{i \neq 0}(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) = 0$.
- (ii) The map

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \rightarrow & H_0(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) \\ z & \mapsto & z \otimes 1 \end{array}$$

is an isomorphism of $\mathbb{C}$ -algebras.

2.3. Reduction Functor. Let $\Lambda(\mathfrak{n}_-)$ be the Grassmann algebra of $\mathfrak{n}_-$. Then $\Lambda(\mathfrak{n}_-)$ is naturally a module over Cl . Thus, for a $\mathfrak{g}$ -module M ,

$$C(M) := M \otimes \Lambda(\mathfrak{n}_-)$$

is naturally a module over $U(\mathfrak{g}) \otimes Cl$. Thus, $(C(M), Q)$ again has the structure of homology complex. Let

$$H_\bullet(M) := H_\bullet(C(M), Q).$$

By definition $(C(M), Q)$ is identical to the Chevalley complex for calculating the Lie algebra homology $H_\bullet(\mathfrak{n}_-, M \otimes \mathbb{C}_p)$. Hence

$$(2) \quad H_\bullet(M) = H_\bullet(\mathfrak{n}_-, M \otimes \mathbb{C}_p).$$

On the other hand, the $U(\mathfrak{g}) \otimes Cl$ -module structure of $C(M)$ induces a $\mathcal{Z}(\mathfrak{g})$ -module structure on $H_i(M)$, because $\mathcal{Z}(\mathfrak{g}) = H_0(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$. Therefore we have obtained the following functor:

$$(3) \quad \begin{array}{ccc} H_i(?) : \mathfrak{g}\text{-Mod} & \rightarrow & \mathcal{Z}(\mathfrak{g})\text{-Mod} \\ M & \mapsto & H_i(M). \end{array}$$

Let $\mathcal{O}$ be the BGG category [BGG] of $\mathfrak{g}$. Let $M(\lambda) \in \mathcal{O}$ the Verma module of highest weight λ , $L(\lambda) \in \mathcal{O}$ the unique irreducible quotient of $M(\lambda)$. Then it is known that the following are equivalent:

- (i) The Gelfand-Kirillov dimension $\text{Dim } L(\lambda)$ of $L(\lambda)$ is maximal, i.e. $\text{Dim}(L(\lambda)) = \dim \mathfrak{n}_-$.
- (ii) $L(\lambda) = M(\lambda)$,

(iii) λ is anti-dominant, i.e. $\lambda(\alpha^\vee) \notin \mathbb{N}$ for all $\alpha \in \Delta_+$.

The following assertion was essentially proved by Kostant [Ko] (cf. [A3, Section 2])

Theorem 2.

- (i) $H_{i \neq 0}(M) = 0$ for all $M \in \mathcal{O}$.
- (ii) $H_0(L(\lambda)) = \begin{cases} \mathbb{C}_{\gamma_\lambda} & \text{if } \dim L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \dim L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$

Here $\mathbb{C}_{\gamma_\lambda} = \mathcal{Z}(\mathfrak{g}) / \text{Ker } \gamma_\lambda$ and $\gamma_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is the central character defined as the evaluation at $M(\lambda)$.

By Theorem 2 (i), the functor $H_0(?)$ is exact. Moreover, by Theorem 2 (ii), one can obtain each simple $\mathcal{Z}(\mathfrak{g})$ -module as the image of the functor $H_0(?)$.

Remark 1. More is known for the functor $H_0(?)$. According to Soergel [S] and Backelin [Ba], it holds that

$$\text{Hom}_{\mathcal{O}}(M, P) \cong \text{Hom}_{\mathcal{Z}(\mathfrak{g})}(H_0(M), H_0(P))$$

provided that P is projective in $\mathcal{O}$ (cf. [A3, Section 2]).

3. CHIRALIZATION OF THE CENTER

3.1. We now “chiralize” the construction of the previous section to define affine W-algebras. To this end we “chiralize” the every data used for the cohomological realization of $\mathcal{Z}(\mathfrak{g})$ in Theorem 2. Thus

- $\mathfrak{g}$ is replaced by the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$, where K is the central element and D is the degree operator;
- $\mathfrak{n}_-$ is replaced by its loop algebra $L\mathfrak{n}_- = \mathfrak{n}_- \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}}$;
- $\mathcal{C}l$ is replaced by the Clifford algebra $\widehat{\mathcal{C}l}$ associated with $L\mathfrak{n}_- \oplus (L\mathfrak{n}_-)^*$ and its natural symmetric bilinear form, where $(L\mathfrak{n}_-)^*$ is the graded dual of $L\mathfrak{n}_-$. This algebra may be defined by the following generators and relations:

$$\begin{aligned} \text{generators: } & \psi_\alpha(n), \psi_\alpha^*(n) \quad (\alpha \in \Delta_-, n \in \mathbb{Z}), \\ \text{relations: } & \{\psi_\alpha(m), \psi_\beta^*(n)\} = \delta_{\alpha, \beta} \delta_{m+n, 0}, \\ & \{\psi_\alpha(m), \psi_\beta(n)\} = \{\psi_\alpha^*(m), \psi_\beta^*(n)\} = 0; \end{aligned}$$

- $Q = Q^{\text{st}} + p$ is replaced by $\widehat{Q} = \widehat{Q}^{\text{st}} + \widehat{p}$, where

$$\begin{aligned} \widehat{Q}^{\text{st}} &= \sum_{\substack{\alpha \in \Delta_- \\ k \in \mathbb{Z}}} x_\alpha(-k) \psi_\alpha^*(k) - \frac{1}{2} \sum_{\substack{\alpha, \beta, \gamma \in \Delta_- \\ k+l+m=0}} c_{\alpha, \beta}^\gamma \psi_\alpha^*(k) \psi_\beta^*(l) \psi_\gamma(m), \\ \widehat{p} &= \sum_{\alpha \in \Delta_-} p(x_\alpha) \psi_\alpha^*(0), \end{aligned}$$

$$\text{where } x(k) = x \otimes t^k \in \widehat{\mathfrak{g}}.$$

By analogy with Theorem 1, we want to define the affine W-algebra $\mathcal{W}(\mathfrak{g})$ as

$$“\mathcal{W}(\mathfrak{g}) = H_0(U(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}, \text{ad } \widehat{Q})”.$$

But this does not make sense, for the appearance of the infinite sum in the formula of $\widehat{Q}^{\text{st}}$. Thus we need to make a suitable completion of $U(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$. We also specialize

the value of the central element $K \in \widehat{\mathfrak{g}}$ at a given complex number $k \in \mathbb{C}$. So let $U_k(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(K - k \text{id})$. The algebra $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$ is naturally graded:

$$U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} = \bigoplus_{d \in \mathbb{Z}} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_d,$$

where the grading is taken from the relation

$$(4) \quad \deg x(n) = \deg \psi_\alpha(n) = \deg \psi_\alpha^*(n) = n, \deg 1 = 0.$$

Give $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$ the linear topology defined by the decreasing sequence where

$$\mathcal{I}_N = \bigoplus_{d \in \mathbb{Z}} (\mathcal{I}_N)_d, \quad (\mathcal{I}_N)_d = \sum_{j \geq N} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_{d-j} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_j.$$

Let $\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}$ be the corresponding completion:

$$\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}} = \varprojlim_N (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} / \mathcal{I}_N).$$

Then Q is a well-defined element of the topological algebra $\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}$, and one can define

$$(5) \quad H_\bullet(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) := \varprojlim_N H_\bullet(U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} / \mathcal{I}_N, \text{ad } \widehat{Q}).$$

But

$$(6) \quad \mathcal{W}_k(\mathfrak{g}) = H_0(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) \quad (k \in \mathbb{C}).$$

is still not a correct definition of W-algebra, because what is defined by (6) is a topological algebra in the usual sense, but an affine W-algebra should be defined as a vertex algebra. So what we actually mean by (6) is the following statement:

Theorem 3 ([A3, Theorem 3.11.1]). *There is an isomorphism*

$$\mathcal{U}(\mathcal{W}_k(\mathfrak{g})) \cong H_0(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}),$$

where $\mathcal{U}(V) = \bigoplus_{d \in \mathbb{Z}} \mathcal{U}(V)_d$ is the universal enveloping algebra of a vertex algebra V (in the sense of Frenkel and Zhu [FZ]).

Remark 2. The vanishing $H_{i \neq 0}(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) = 0$ also holds.

We will not define the W-algebra $\mathcal{W}_k(\mathfrak{g})$ itself in this note. Instead, we take (6) as its definition because a $\mathcal{W}_k(\mathfrak{g})$ -module M is by definition a $\mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))$ -module (such that $\dim \mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))_n \cdot v < \infty$ for all $v \in V$ and $n \geq 0$). But it should be remarked that Theorem 3 follows from the corresponding statement for the vertex algebra $\mathcal{W}_k(\mathfrak{g})$ itself. This was proved for generic k by Feigin and Frenkel [FF2], for a general k and $\mathfrak{g} = \mathfrak{sl}_n$ by de Bore and Tjin [dBT2] and for a general k and a general $\mathfrak{g}$ by Frenkel [FB].

Remark 3. The W-algebra $\mathcal{W}_k(\mathfrak{g})$ considered here is not a simple vertex algebra in general.

Remark 4. If $k \neq -h^\vee$, then $\mathcal{W}_k(\mathfrak{g})$ has the structure of the vertex operator algebra and has the central charge

$$c(k) = \text{rank } \mathfrak{g} - 12(\kappa |\rho^\vee|^2 - \langle \rho, \rho^\vee \rangle + |\rho|^2 / \kappa), \quad (\kappa = k + h^\vee).$$

Remark 5. It is known that $\mathcal{W}_{-k^\vee}(\mathfrak{g})$ is commutative. This is one of the results of Feigin-Frenkel [FF2].

To give a more precise relationship between $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{W}_k(\mathfrak{g})$, let us introduce the notion of *Zhu algebra* $\text{Zh}(V)$ of a (graded) vertex algebra V .

$$\text{Zh}(V) := \mathcal{U}(V)_0 / \overline{\sum_{p>0} \mathcal{U}(V)_{-p} \mathcal{U}(V)_p},$$

where $\overline{}$ denotes the closure. By definition the following assertion is clear.

Theorem 4 (Zhu [Zhu]). *There is a one-to-one correspondence between irreducible V -modules and irreducible $\text{Zh}(V)$ -modules.*

For example, consider the universal affine vertex algebra $V_k(\mathfrak{g})$ associated with $\mathfrak{g}$ at level k . Then $\mathcal{U}(V_k(\mathfrak{g})) = \widehat{U_k(\widehat{\mathfrak{g}})}$ and we have $\text{Zh}(V_k(\mathfrak{g})) = U(\mathfrak{g})$. This reflects the fact that $\widehat{\mathfrak{g}}$ (or more precisely $V_k(\mathfrak{g})$) is a chiralization of $\mathfrak{g}$. Since $\mathcal{W}_k(\mathfrak{g})$ is a chiralization of $\mathcal{Z}(\mathfrak{g})$, it is natural to expect the following assertion:

Theorem 5 ([A3, Theorem 3.13.2]). *The Zhu algebra $\text{Zh}(\mathcal{W}_k(\mathfrak{g}))$ of $\mathcal{W}_k(\mathfrak{g})$ is naturally isomorphic to $\mathcal{Z}(\mathfrak{g})$.*

By Theorems 4, 5, irreducible $\mathcal{W}_k(\mathfrak{g})$ -modules are parameterized by the central characters of $\mathcal{Z}(\mathfrak{g})$. Let $L(\gamma)$ denote the irreducible $\mathcal{W}_k(\mathfrak{g})$ -module corresponding to the central character γ . Then $L(\gamma)$ is the quotient of the Verma module $M(\gamma)$ with highest weight γ , which has the PBW type basis.

3.2. As in the finite dimensional case we functionally obtain the $\mathcal{W}_k(\mathfrak{g})$ -modules in the following way: Let $\Lambda^{\frac{\infty}{2}}(Ln_-)$ be the irreducible representation of $\widehat{\mathcal{Cl}}$ generated by the vector $\mathbf{1}$ satisfying the following relations:

$$\psi_\alpha(n)\mathbf{1} = \psi_\alpha^*(n+1)\mathbf{1} = 0 \quad (\alpha \in \Delta_-, n \geq 0).$$

Denote by $\widehat{\mathcal{O}}_k$ the BGG category of $\widehat{\mathfrak{g}}$ at level k . Then

$$\widehat{C}(M) := M \otimes \Lambda^{\frac{\infty}{2}}(Ln_-)$$

with $M \in \widehat{\mathcal{O}}_k$ is naturally a module over $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{Cl}}$, and its action extends to the smooth action of $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{Cl}}$. In particular the action of $\widehat{Q}$ is well-defined on $\widehat{C}(M)$. Thus the homology

$$(7) \quad \widehat{H}_\bullet(M) := H_\bullet(\widehat{C}(M), \widehat{Q})$$

well-defined and is naturally a module over $\mathcal{W}_k(\mathfrak{g})$. Note that $\widehat{H}_i(M)$ is naturally graded (cf. (4)):

$$(8) \quad \widehat{H}_i(M) = \bigoplus_{d \in \mathbb{C}} \widehat{H}_i(M)_d.$$

If k is not critical then (8) is essentially the L_0 -eigenspace decomposition. Set $\text{ch } \widehat{H}_i(M) = \sum_{d \in \mathbb{C}} q^d \dim H_i(M)_d$ whenever it is well-defined.

Remark 6. By definition we have $\widehat{H}_\bullet(M) = H_{\frac{\infty}{2}+\bullet}(Ln_-, M \otimes \mathbb{C}_{\widehat{p}})$, the Feigin's semi-infinite Ln_- -homology with the coefficient in $M \otimes \mathbb{C}_{\widehat{p}}$ ([Fe]).

Let $\widehat{M}(\widehat{\lambda})$ be the Verma module of $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda}$, $\widehat{L}(\widehat{\lambda})$ the unique simple quotient of $\widehat{M}(\widehat{\lambda})$.

Theorem 6 ([A3]). *For any $k \in \mathbb{C}$ we have the following.*

- (i) $\widehat{H}_{i \neq 0}(M) = 0$ for all objects M of $\widehat{\mathcal{O}}_k$.
- (ii) Let $\widehat{\lambda}$ be a weight of $\widehat{\mathfrak{g}}$ at level k , λ the classical part of $\widehat{\lambda}$ (i.e. the restriction of $\widehat{\lambda}$ to $\mathfrak{h}$). Then

$$\widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \begin{cases} L(\gamma_\lambda) & \text{if } \dim L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \dim L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$$

By Theorem 6 it follows that the functor

$$\widehat{H}_0(?) : \mathcal{O}_k \rightarrow \mathcal{W}_k(\mathfrak{g})\text{-Mod}$$

is exact for any $k \in \mathbb{C}$.

Write the formal character $\text{ch } \widehat{L}(\widehat{\lambda})$ of $L(\lambda)$ as

$$\text{ch } \widehat{L}(\widehat{\lambda}) = \sum_{\widehat{\mu}} m_{\widehat{\lambda}, \widehat{\mu}} \text{ch } \widehat{M}(\widehat{\mu}), \quad (m_{\widehat{\lambda}, \widehat{\mu}} \in \mathbb{Z}).$$

Then the following assertion follows from Theorem 6:

Theorem 7 ([A3]). $\text{ch } \widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \sum_{\widehat{\mu}} m_{\widehat{\lambda}, \widehat{\mu}} q^{\widehat{\mu}(\mathbf{D})} \prod_{i \geq 0} (1 - q^{-i})^{-\text{rank } \mathfrak{g}}$.

Recall that the integer $m_{\lambda, \mu}$ is known by Kashiwara-Tanisaki-Cassian [KT1, KT2, KT3, Ca] provided that $k \neq -h^\vee$. Therefore by Theorems 6 and 7 we have obtained the character formula of *all* the irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g})$ for any $k \in \mathbb{C} \setminus \{-h^\vee\}$.

Remark 7. It may be worth emphasizing that Theorems 6 and 7 remain valid even at the critical level $k = -h^\vee$, and the result for this case in particular implies the Kac-Kazhdan conjecture [KK], which was proved by Hayashi [Ha] and others [GW, FF1, Ku] by computational methods (see [A4] for details).

3.3. Frenkel-Kac-Wakimoto Conjecture. Note that our functor $\widehat{H}_0(?)$ kills integrable representations of $\widehat{\mathfrak{g}}$. However there are a wider class of modular invariant representations of $\widehat{\mathfrak{g}}$; they are called Kac-Wakimoto *admissible representations* [KW1, KW2].

The simple module $\widehat{L}(\widehat{\lambda})$ is called admissible if $\widehat{\lambda}$ is an admissible weight. An admissible weight is a weight $\widehat{\lambda}$ that satisfies the following:

- (i) $\widehat{\lambda}$ is regular dominant;
- (ii) the $\mathbb{Q}$ -span of $\widehat{\Delta}(\widehat{\lambda})^\vee := \{\alpha \in \widehat{\Delta}_+^{\vee, \text{re}}; \widehat{\lambda}(\alpha) \in \mathbb{Z}\} = \text{the } \mathbb{Q}\text{-span of } \widehat{\Delta}_+^{\vee, \text{re}}.$

The condition (i) implies that the corresponding Kazhdan-Lusztig polynomial is trivial. Therefore $\widehat{L}(\widehat{\lambda})$ has the Weyl-Kac type character formula:

$$\text{ch } \widehat{L}(\widehat{\lambda}) = \sum_{w \in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} \text{ch } \widehat{M}(w \circ \widehat{\lambda}),$$

where $\widehat{W}(\widehat{\lambda})$ is the integral Weyl group of $\widehat{\mathfrak{g}}$, generated by the reflections r_α with $\alpha^\vee \in \widehat{\Delta}(\widehat{\lambda})^\vee$. The condition (ii) implies that $\widehat{W}(\widehat{\lambda})$ is an infinite Coxeter group, and $\text{ch } \widehat{L}(\widehat{\lambda})$ is written in terms of some theta functions ([KW1, KW2]).

If the classical part λ of an admissible weight $\widehat{\lambda}$ is anti-dominant, then $\widehat{\lambda}$ is called a *non-degenerate* admissible weight. Let $Pr_k^{\text{non-deg}}$ be the set of non-degenerate admissible weight at level k .

And as explained in Introduction, the conjecture of Frenkel, Kac and Wakimoto [FKW] follows from Theorems 6 and 7:

Corollary 1 (Frenkel-Kac-Wakimoto Conjecture [FKW]). *Let $\widehat{\lambda}$ be an non-degenerated admissible weight of $\widehat{\mathfrak{g}}$, λ the classical part of $\widehat{\lambda}$. Then*

$$\text{ch } \mathbf{L}(\gamma_\lambda) = \sum_{w \in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} q^{(w \circ \widehat{\lambda})(D)} \prod_{i \geq 1} (1 - q^{-i})^{-\text{rank } \mathfrak{g}}.$$

As explained in [FKW], from Corollary 1 it follows that the (modified) characters of

$$\{\mathbf{L}(\gamma_\lambda); \lambda \text{ is the classical part of } \widehat{\lambda} \in Pr_k^{\text{non-deg}}\}$$

are modular invariant, i.e. the linear space spanned by their (modified) characters are invariant under the natural action of $SL_2(\mathbb{Z})$. In the case that $\mathfrak{g} = \mathfrak{sl}_2$, they are exactly the minimal series representations of *Vir*.

4. GENERALIZATION TO OTHER NILPOTENT ORBITS

4.1. In the above construction we started with the principal nilpotent element of $\mathfrak{g}$. However the above construction can be generalized to cases of other nilpotent elements:

Let e be a nilpotent element which corresponds to a nice parabolic subalgebra ([BW]) of $\mathfrak{g}$. Then it is straightforward to generalize the previous construction to e (cf. [dBT1, dBT2, KRW]). As a result, instead of $\mathcal{Z}(\mathfrak{g})$, we obtain the *finite W-algebra* $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ [dBT1] associated with $(\mathfrak{g}, e)$, which is the endomorphism ring of the *generalized Gelfand-Graev representation* ([Ka], cf. [Pr, GG, BG]). The corresponding affine W-algebra $\mathcal{W}_k(\mathfrak{g}, e)$ has $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ as its Zhu algebra (cf. [DK]).

We have the similar result as Theorems 6 and 7 for this case ([A5]); The difficulty is that the representation theory of $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ is not known very much in general, except for the type *A* cases; Recently Brundan and Kleshchev [BK] established important results on the representation theory of finite W-algebras for these cases. Thanks to their result, for the type *A* cases one obtains the character formula for each irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g}, e)$ (see [A5] for details).

If e does not corresponds to a nice parabolic subalgebra then the construction of W-algebras becomes more involving. The most general construction was made by Kac, Roan and Wakimoto [KRW], which applies to the Lie superalgebra case also. One of the remarkable discoveries of Kac, Roan and Wakimoto [KRW] is that almost all the superconformal algebras (such as $N = 2, 3, 4$ superconformal algebra) appears as a W-algebra associated with some Lie superalgebra $\mathfrak{g}$ and its *minimal nilpotent element*. As principal nilpotent element cases, their representation theory (such as characters of irreducible representations) can be completely described through the reduction functor (see [A2] for details).

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