

Refinement of Strichartz estimates for Airy equation and application

Dedicated to Professor Yoshio Tsutsumi on his 60th birthday

By

Satoshi MASAKI* and Jun-ichi SEGATA**

Abstract

In this paper we review our recent results [19, 20] on the refinement of the Strichartz estimate for the Airy equation. As an application of this estimate, we construct a minimal non-scattering solution to the mass sub-critical generalized Korteweg-de Vries equation.

§ 1. Introduction

We consider the space-time estimates for the solution $e^{-t\partial_x^3} f$ to the Airy equation

$$(1.1) \quad \begin{cases} \partial_t u + \partial_x^3 u = 0 & t, x \in \mathbb{R}, \\ u(0, x) = f(x) & x \in \mathbb{R}, \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a given data. As with the Schrödinger equation, the Strichartz estimate for (1.1) is well-known (see [10] for instance). Grünrock [7] generalized the Strichartz estimate for (1.1) to the hat-Lebesgue space. More precisely, he obtained the following space-time estimates.

Theorem 1.1 (Stein-Tomas type estimate [7]). *Let $4/3 < p \leq \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$(1.2) \quad \left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}} \leq C \|f\|_{\dot{L}^p}$$

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*Department of systems innovation, Graduate school of Engineering Science, Osaka University, Toyonaka Osaka, 560-8531, Japan.

e-mail: masaki@sigmath.es.osaka-u.ac.jp

**Mathematical Institute, Tohoku University, 6-3, Aoba, Aramaki, Aoba-ku, Sendai 980-8578, Japan.

e-mail: segata@m.tohoku.ac.jp

holds for any $f \in \hat{L}^p$, where the space $\hat{L}^p$ is defined for $1 \leq p \leq \infty$ by

$$\hat{L}^p = \hat{L}^p(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) \mid \|f\|_{\hat{L}^p} = \|\hat{f}\|_{L^{p'}} < \infty\},$$

where $\hat{f}$ stands for Fourier transform of f with respect to space variable and p' denotes the Hölder conjugate of p .

The space $\hat{L}^p$ has appeared in study of Fourier restriction theorem [6, 25]. Recently, well-posedness of the nonlinear dispersive equations (nonlinear Schrödinger and generalized KdV equations,...) are studied by Grünrock [7] and Hyakuna-Tsutsumi [8] in the framework of $\hat{L}^p$ space. Notice that the Hausdorff-Young inequality yields

$$\begin{aligned} L^p &\hookrightarrow \hat{L}^p && \text{if } 1 \leq p \leq 2, \\ \hat{L}^p &\hookrightarrow L^p && \text{if } 2 \leq p \leq \infty. \end{aligned}$$

The key point to prove the Stein-Tomas type estimate (1.2) is to reduce the *linear* form into a *bi-linear* form. For the reader's convenience, we give the proof of the estimate (1.2) in Appendix A.

We consider an improvement of the Stein-Tomas type estimate (1.2). To state our main result more precisely, we now introduce generalized Morrey space and generalized hat-Morrey space.

Definition 1.2. For $j, k \in \mathbb{Z}$, let $\tau_k^j = [k2^{-j}, (k+1)2^{-j})$.

(i) For $1 \leq q \leq p \leq \infty$ and $1 \leq r \leq \infty$, we define a generalized Morrey norm $\|\cdot\|_{M_{q,r}^p}$ by

$$\|f\|_{M_{q,r}^p} = \left\| |\tau_k^j|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^j)} \right\|_{\ell_{j,k}^r},$$

where, the case $p = q$ and $r < \infty$ is excluded.

(ii) For $1 \leq p \leq q \leq \infty$ and $1 \leq r \leq \infty$, we introduce a generalized hat-Morrey norm by

$$\|f\|_{\hat{M}_{q,r}^p} := \|\hat{f}\|_{M_{q',r}^{p'}} = \left\| |\tau_k^j|^{\frac{1}{q} - \frac{1}{p}} \|\hat{f}\|_{L^{q'}(\tau_k^j)} \right\|_{\ell_{j,k}^r}.$$

Banach spaces $M_{q,r}^p$ and $\hat{M}_{q,r}^p$ are defined as sets of tempered distributions of which above norms are finite, respectively.

The space $\hat{M}_{q,r}^p$ naturally appears in the context of the estimate for the maximal function [3, 22, 23] and the refinement of Storchartz's estimate [4, 21, 12, 26, 1, 24]. Notice that the inclusion relation $\hat{L}^p \hookrightarrow \hat{M}_{q,r}^p$ holds for $1 \leq q' < p' < r \leq \infty$, see [20, Proposition A.1].

The first main theorem is as follows.

Theorem 1.3 (Refinement of Stein-Tomas type estimate [20]). *Let $4/3 \leq p < \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$(1.3) \quad \left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}} \leq C \|f\|_{\hat{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}^p},$$

holds for any $f \in \hat{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}^p$.

Since the embedding $\hat{L}^p \hookrightarrow \hat{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}^p$ holds for $p > 4/3$, the inequality (1.3) is an improvement of Stein-Tomas estimate (1.2). This kind of refinement is known for the Schrödinger equation [4, 21, 26, 1] and the Airy equation with $p = 2$ [12, 24]. Although the inequality (1.3) holds for $p = 4/3$, the function space $\hat{M}_{2,4}^{4/3}$ is inferior in quality.

We give an outline of the proof of Theorem 1.3. As with the proof of the Stein-Tomas type estimate (1.2), the key point to prove the inequality (1.3) is to reduce the *linear* form into a *bi-linear* form;

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}}^2 = \left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta \right\|_{L_{t,x}^{\frac{3p}{2}}}.$$

Change of variables and Hausdorff-Young's inequality yield

$$(1.4) \quad \left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}}^2 \leq C \left\{ \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \right\}^{\frac{1}{(\frac{3p}{2})'}},$$

where

$$m(\xi, \eta) = \frac{|\xi\eta|}{|\xi + \eta|^2 |\xi - \eta|^2}.$$

Since the weight function m is singular on the diagonal line $\eta = \xi$, we introduce a *Whitney type decomposition*. Let $\mathcal{D}_+ = \{\tau_k^j = [k2^{-j}, (k+1)2^{-j}] \mid j \in \mathbb{Z}, 0 \leq k \in \mathbb{Z}\}$. For $\tau_k^j, \tau_\ell^j \in \mathcal{D}_+$, we define a binary relation

$$\tau_k^j \sim \tau_\ell^j \Leftrightarrow \begin{cases} \ell - k = -2, 2, 3 & \text{if } k \text{ is even,} \\ \ell - k = -3, -2, 2 & \text{if } k \text{ is odd.} \end{cases}$$

Then, we have the following Whitney-type decomposition of $\mathbb{R}_+ \times \mathbb{R}_+$;

$$\sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j: \tau_\ell^j \sim \tau_k^j} \mathbf{1}_{\tau_k^j}(\xi) \mathbf{1}_{\tau_\ell^j}(\eta) = 1, \quad (\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(\xi, \xi) \mid \xi \in \mathbb{R}_+\}.$$

Since $|\xi\eta| \leq (\xi + \eta)^2/2$ for any $(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+$, one sees that

$$m(\xi, \eta) \leq \frac{1}{2|\xi - \eta|^2} \leq \frac{1}{2|\tau_k^j|^2}$$

for any $(\xi, \eta) \in \tau_k^j \times \tau_\ell^j$ with $\tau_k^j \sim \tau_\ell^j$. We hence obtain

$$\begin{aligned}
& \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \\
&= \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \mathbf{1}_{\tau_k^j}(\xi) \mathbf{1}_{\tau_\ell^j}(\eta) d\xi d\eta \\
&\leq \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\tau_k^j|^{-\frac{2}{3p-2}} \int_{\tau_k^j} |\hat{f}(\xi)|^{(\frac{3p}{2})'} d\xi \int_{\tau_\ell^j} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\eta.
\end{aligned}$$

We choose a slightly larger interval containing τ_k^j and τ_ℓ^j but still of length comparable to τ_k^j . More specifically, it is enough to take $\tau_m^{j-3} \in \mathcal{D}_+$ so that $\tau_k^j, \tau_\ell^j \subset \tau_m^{j-3}$. Then we obtain

$$\begin{aligned}
(1.5) \quad & \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \\
&\leq C \sum_{\tau_k^j \in \mathcal{D}_+} |\tau_k^j|^{-\frac{2}{3p-2}} \left(\int_{\tau_k^j} |\hat{f}(\xi)|^{(\frac{3p}{2})'} d\xi \right)^2 \\
&\leq C \sum_{\tau_k^j \in \mathcal{D}_+} |\tau_k^j|^{-\frac{2}{3p-2}} \|\hat{f}\|_{L^{(\frac{3p}{2})'}(\tau_k^j)}^{\frac{6p}{3p-2}} \\
&= C \|f\|_{\dot{M}_{\frac{3p}{2}, 2(\frac{3p}{2})'}}^{2(\frac{3p}{2})'}.
\end{aligned}$$

By (1.4) and (1.5), we have (1.3), which completes the proof.

§ 2. Application

As an application of the refined Stein-Tomas estimate (1.3), we prove the existence of a minimal non-scattering solution to the generalized Korteweg-de Vries equation

$$\begin{aligned}
(\text{gKdV}) \quad & \begin{cases} \partial_t u + \partial_x^3 u = -\partial_x(|u|^{2\alpha} u) & t, x \in \mathbb{R}, \\ u(0, x) = u_0(x) & x \in \mathbb{R}, \end{cases}
\end{aligned}$$

where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ is a given data, and $\alpha > 0$ is a constant. More precisely, in the *mass-subcritical* range $\alpha < 2$, we show existence of a threshold solution to (gKdV) which lies on the boundary of the set of scattering solutions and the set of non-scattering solutions in the framework of the scale critical space $\hat{L}^\alpha$.

Since the nonlinear term is focusing, (gKdV) admits a soliton solution

$$Q_c(t, x) = c^{\frac{1}{\alpha}} Q(c(x - c^2 t)),$$

where $Q(x)$ is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$ and $c > 0$ is a parameter describing amplitude and propagating speed of soliton.

For the mass critical case $\alpha = 2$, it is known that Q is orbitally unstable [15]. Hence on the analogy of the result by Kenig-Merle [9], we expect that Q will lie on the boundary between the scattering and the non-scattering sets. Concerning this, Martel-Merle-Nakanishi-Raphaël [16] classified the dynamics of solution into three cases (blow-up, soliton, away from soliton) in the small neighborhood of Q . On the other hand, for the mass sub-critical case $\alpha < 2$, it is known that Q is stable [2]. Hence, we see that Q does not belong to the boundary between the scattering and the non-scattering sets. Indeed, the conservation of energy and Stein-Tomas estimate (1.2) yield $d_+ < c_\alpha \ell(Q)$ for some $c_\alpha < 1$, where d_+ is a threshold number defined by (2.1) below ¹. As for the mass-subcritical nonlinear Schrödinger equation, the first author [17, 18] treated a minimization problem similar to (gKdV) in a framework of weighted space and showed existence of a threshold solution which is smaller than ground state solutions.

To state the result, we introduce several notations. We say that solution to (gKdV) scatters in $\hat{L}^\alpha$ forward in time if $\lim_{t \rightarrow +\infty} e^{t\partial_x^3} u(t)$ exists in $\hat{L}^\alpha$. We define a *forward scattering set* $\mathcal{S}_+$ as follows

$$\mathcal{S}_+ := \left\{ u_0 \in \hat{L}^\alpha \left| \begin{array}{l} \text{a solution } u(t) \text{ to (gKdV) with } u|_{t=0} = u_0 \\ \text{scatters in } \hat{L}^\alpha \text{ forward in time} \end{array} \right. \right\}.$$

A *backward scattering set* $\mathcal{S}_-$ is defined in a similar way. Thus, we consider the minimization problem for

$$(2.1) \quad \begin{aligned} d_+ &= d_+(\sigma, M) := \inf \{ \ell(u_0) \mid u_0 \in B_M \setminus \mathcal{S}_+ \}, \\ \ell(u) &= \ell_\sigma(u) := \inf_{\xi \in \mathbb{R}} \| e^{-ix\xi} u \|_{\hat{M}_{2,\sigma}^\alpha}, \end{aligned}$$

where $M > 0$ is a parameter and $B_M := \{ f \in \hat{L}^\alpha \mid \|f\|_{\hat{L}^\alpha} \leq M \}$ is a ball.

Furthermore, for the nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} i\partial_t v - \partial_x^2 v = |v|^{2\alpha} v & t, x \in \mathbb{R}, \\ v(0, x) = v_0(x) & x \in \mathbb{R}, \end{cases}$$

we define

$$(2.2) \quad d_{\text{NLS}} = d_{\text{NLS}}(\sigma, M) := \inf \{ \ell(v_0) \mid v_0 \in B_M \setminus \mathcal{S}_{\text{NLS}} \}$$

with

$$\mathcal{S}_{\text{NLS}} := \left\{ v_0 \in \hat{L}^\alpha \left| \begin{array}{l} \text{a solution } v(t) \text{ to (NLS) with } v|_{t=0} = v_0 \text{ scatters} \\ \text{in } \hat{L}^\alpha \text{ forward and backward in time} \end{array} \right. \right\}.$$

Our second main result is as follows.

¹We prove this assertion in Appendix B.

Theorem 2.1. *Let $3/2 + \sqrt{7/60} < \alpha < 2$ and $\sigma \in (\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M > 0$ so that $B_M \cap \mathcal{S}_+^c \neq \emptyset$. If the assumption*

$$(2.3) \quad d_+ < 2^{1-\frac{1}{\sigma}} \left(\frac{3\sqrt{\pi}\Gamma(\alpha+2)}{2\Gamma(\alpha+\frac{3}{2})} \right)^{\frac{1}{2\alpha}} d_{\text{NLS}},$$

is true, then there exists a special solution $u_c(t)$ to (gKdV) with maximal interval $I_{\max}(u_c) \ni 0$ such that

(i) $u_c(0) \notin \mathcal{S}_+$.

(ii) u_c attains d_+ in such a sense that one of the following two properties holds:

(a) $u_c(0) \in B_M$ and $\ell(u_c(0)) = d_+$.

(b) $u_c(0) \in \mathcal{S}_-$ and scatters backward in time to $u_{c,-}$ satisfying $u_{c,-} \in B_M$ and $\ell(u_{c,-}) = d_+$.

If we impose the additional regularity on the initial data, we are able to prove the existence of minimal non-scattering solution without the assumption (2.3). For fixed $8/5 < \tilde{\alpha} < \alpha$ and $0 < \tilde{s} < 2\alpha + 1$, define $\tilde{B}_M = \{f \in \hat{L}^\alpha \mid \|f\|_{\hat{L}^{\tilde{\alpha}}} + \|f\|_{\dot{H}^{\tilde{s}}} \leq M\}$. It turns out that, as for a minimizing problem for

$$d'_+ = d'_+(\sigma, M) := \inf\{\ell(u_0) \mid u_0 \in \tilde{B}_M \cap \mathcal{S}_+^c\},$$

a minimizer exists *without* the assumption (2.3).

Theorem 2.2. *Let $3/2 + \sqrt{7/60} < \alpha < 2$ and $\sigma \in (\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M > 0$ so that $\tilde{B}_M \cap \mathcal{S}_+^c \neq \emptyset$. Then, there exists a special solution $\tilde{u}_c(t)$ to (gKdV) which attains d'_+ in a similar way to Theorem 2.1.*

To prove Theorem 2.1, we employ the linear profile decomposition, which is roughly speaking a decomposition of a bounded sequence of functions into a sum of characteristic profiles and a remainder by finding weak limit(s) of the sequence modulo deformations. Intuitively, this decomposition is done by a recursive use of a suitable concentration compactness result. Then, to ensure smallness of remainder as the number of detected profiles increases, a decoupling equality, so-called Pythagorean decomposition, plays a crucial role.

Let us now be more precise on the Pythagorean decomposition. Let $\{f_n\}$ be a bounded sequence of $\hat{L}^\alpha$. Since $\hat{L}^\alpha$ is reflexive as long as $1 < \alpha < \infty$, by extracting subsequence, f_n converges to some function $f \in \hat{L}^\alpha$ in weak $\hat{L}^\alpha$ sense. Now we suppose that $f \neq 0$. Then, the Pythagorean decomposition is a decoupling equality of the form

$$(2.4) \quad \|\hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} = \|\hat{f}\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + \|\hat{f} - \hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + o(1)$$

as $n \rightarrow \infty$, It is well known that the above decoupling holds for $\alpha = 2$ and may fail for $\alpha \neq 2$. Due to this reason, we prove the linear profile decomposition in a weaker space $\hat{M}_{2,\sigma}^\alpha$.

The space $\hat{L}^\alpha$ -norm is invariant under the following group actions:

- Translation in Physical side: $(T(y)f)(x) := f(x - y)$.
- Translation in Fourier side: $(P(\xi)f)(x) := e^{-ix\xi}f(x)$.
- Airy flow: $(A(t)f)(x) = (e^{-t\partial_x^3}f)(x)$.
- Dilation (scaling): $(D(h)f)(x) = h^\alpha f(hx)$.

Hence, we define a set of deformations as follows

$$(2.5) \quad G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}.$$

We often identify $\mathcal{G} \in G$ with a corresponding parameter $\Gamma \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if there is no fear of confusion. Let us now introduce a notion of orthogonality between two families of deformations.

Definition 2.3. We say two families of deformations $\{\mathcal{G}_n\} \subset G$ and $\{\tilde{\mathcal{G}}_n\} \subset G$ are *orthogonal* if corresponding parameters $\Gamma_n, \tilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfy

$$(2.6) \quad \lim_{n \rightarrow \infty} \left(\left| \log \frac{h_n}{\tilde{h}_n} \right| + \left| \xi_n - \frac{\tilde{h}_n}{h_n} \tilde{\xi}_n \right| + \left| s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right| (1 + |\xi_n|) \right. \\ \left. + \left| y_n - \frac{h_n}{\tilde{h}_n} \tilde{y}_n - 3 \left(s_n - \left(\frac{h_n}{\tilde{h}_n} \right)^3 \tilde{s}_n \right) (\xi_n)^2 \right| \right) = +\infty.$$

Theorem 2.4 (Linear profile decomposition for real valued functions). *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Let $u = \{u_n\}_n$ be a sequence of real-valued functions in B_M . Then, there exist $\psi^j \in B_M$, $r_n^j \in B_{(2j+1)M}$ and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) parametrized by $\{\Gamma_n^j = (h_n^j, \xi_n^j, s_n^j, y_n^j)\}_n$ such that, extracting a subsequence in n ,*

$$(2.7) \quad u_n = \sum_{j=1}^l \text{Re}(\mathcal{G}_n^j \psi^j) + r_n^l$$

for all $l \geq 1$ and

$$(2.8) \quad \limsup_{n \rightarrow \infty} \left\| e^{-t\partial_x^3} r_n^l \right\|_{L_{t,x}^{3\alpha}} \rightarrow 0$$

as $l \rightarrow \infty$. For all $j \geq 1$,

$$\text{either } \xi_n^j = 0, \quad \forall n \geq 0 \quad \text{or} \quad \xi_n^j \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Moreover, a decoupling inequality

$$(2.9) \quad \limsup_{n \rightarrow \infty} \ell(u_n) \geq \left(\sum_{j=1}^J c_j^{1-\sigma} \ell(\psi^j)^\sigma \right)^{1/\sigma} + \limsup_{n \rightarrow \infty} \ell(r_n^J)$$

holds for all $J \geq 1$, where

$$c_j = \begin{cases} 1 & \text{if } \xi_n^j \equiv 0, \\ 2 & \text{if } \xi_n^j \rightarrow \infty \text{ as } n \rightarrow \infty. \end{cases}$$

Furthermore, it holds that

$$(2.10) \quad c_j \|\psi^j\|_{\hat{L}^\alpha} \leq \limsup_{n \rightarrow \infty} \|u_n\|_{\hat{L}^\alpha}$$

for any j .

Proof. (Outline) To prove Theorem 2.4, we first prove a decomposition of sequence of *complex-valued* functions. The desired decomposition for real-valued functions then follows as a corollary. As in [5], we split the proof into two parts. The first part is the procedure of finding profiles and obtaining pairwise orthogonality between profiles and remainder term. By employing a successive notion of smallness of remainder term, used in [13, 1, 5], this part can be shown in an abstract way. The second part is concentration compactness. Intuitively, the meaning of the concentration compactness is as follows. Let us consider a bonded sequence $\{u_n\}_n \subset X$. Here, X is a Banach space. In addition to the boundedness with respect to X , we make some additional assumption on the sequence. If the additional assumption is so strong that it removes almost all possible deformations for $\{u_n\}_n$ with few exceptions, say G , then we can find a *non-zero* weak limit modulo G . In our case, $X = \hat{M}_{2,\sigma}^\alpha$ and we use

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_n \right\|_{L_{t,x}^{3\alpha}} \geq m$$

as an additional assumption, where m is some positive constant. It will turn out that this assumption removes almost all deformations. The exception is G given in (2.5). This is the reason why we use the set G of deformations in Theorems 2.4. Note that the refined Stein-Tomas estimate (1.3) plays an important rule to show the concentration compactness. See [20, Theorem 4.3] for the detail of the proof. $\square$

The second tool to prove Theorem 2.1 is uniform boundedness of solutions with highly oscillating initial data. The assumption (2.3) is necessary for this boundedness.

Theorem 2.5. *Let $12/7 < \alpha < 2$. Assume (2.3). Let $\phi \in \hat{L}_x^\alpha(\mathbb{R})$ be a complex valued function such that*

$$\ell(\phi) < 2^{1-\frac{1}{\sigma}} d_+.$$

Let $\{\xi_n\}_{n \geq 1} \subset (0, \infty)$ with $\xi_n \rightarrow \infty$ and let $\{t_n\}_{n \geq 1} \subset \mathbb{R}$ be such that $-3t_n\xi_n$ converges to some $T_0 \subset [-\infty, \infty]$. Then for n sufficiently large, a corresponding $\hat{L}^\alpha$ -solution u_n to (gKdV) with the initial condition

$$(2.11) \quad u_n(t_n, x) = A(t_n) \operatorname{Re}[P(\xi_n)\phi(x)]$$

exists globally in time. Moreover, the solution u_n satisfies a uniform space-time bound

$$(2.12) \quad \|u_n\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R}; L_t^{5\alpha}(\mathbb{R}))} + \|u_n\|_{L_{t,x}^{3\alpha}(\mathbb{R}^2)} \leq C,$$

where C is a positive constant depending only on ϕ .

Proof. (Outline) We prove existence of a global solution u_n to (gKdV) by constructing approximating solution via the solution to the one dimensional nonlinear Schrödinger equation (NLS). More precisely, let v_n be a solution of (NLS) with initial (or final) condition

$$(2.13) \quad \begin{cases} v_n(T_0) = P_{|\xi| \leq \xi_n^{1/4}} e^{-iT_0 \partial_x^2} \phi & \text{if } |T_0| < \infty, \\ \lim_{t \rightarrow T_0} \|v_n(t) - P_{|\xi| \leq \xi_n^{1/4}} e^{-it \partial_x^2} \phi\|_{\hat{L}_x^\alpha} = 0 & \text{if } T_0 = \pm\infty, \end{cases}$$

where $P_{|\xi| \leq a} = \mathcal{F}^{-1} \varphi(\xi) \mathcal{F}$ with even bump function φ satisfying $\operatorname{supp} \varphi \subset [-a, a]$. As in [14], we introduce an approximate solution $\tilde{u}_n$ to (gKdV):

$$(2.14) \quad \tilde{u}_n(t, x) := \begin{cases} \operatorname{Re}[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)], & \text{if } |t| \leq \frac{T}{3\xi_n}, \\ e^{-(t - \frac{T}{3\xi_n}) \partial_x^3} \operatorname{Re}[e^{-ix\xi_n - \frac{i}{3} T \xi_n^2} v_n(-T, x + \xi_n T)], & \text{if } t > \frac{T}{3\xi_n}, \\ e^{-(t + \frac{T}{3\xi_n}) \partial_x^3} \operatorname{Re}[e^{-ix\xi_n + \frac{i}{3} T \xi_n^2} v_n(T, x - \xi_n T)], & \text{if } t < -\frac{T}{3\xi_n}, \end{cases}$$

where T is a large parameter independent of n . By applying the long time stability [20, Proposition 3.2] for $\tilde{u}_n$ with a suitable T , we construct a global solution u_n to (gKdV). For the detail of the proof, see [20, Theorem 4.4]. $\square$

We now give an outline of the proof of Theorem 2.1.

Step 1 Take a minimizing sequence $\{u_n\}_n$ as follows;

$$(2.15) \quad u_n \in B_M \setminus \mathcal{S}_+, \quad \ell(u_n) \leq d_+ + \frac{1}{n}.$$

We apply the linear profile decomposition theorem (Theorem 2.4) to the sequence $\{u_n\}_n$. Then, up to subsequence, we obtain a decomposition

$$(2.16) \quad u_n = \sum_{j=1}^l \operatorname{Re}(\mathcal{G}_n^j \psi^j) + r_n^l$$

for $n, l \geq 1$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^j\}_{n,j} = \{\log h_n^j\}_{n,j}, \{t_n^j\}_{n,j}, \{y_n^j\}_{n,j}, \{3\xi_n^j t_n^j\}$, either $x_n^j \equiv 0$, $x_n^j \rightarrow \infty$ as $n \rightarrow \infty$, or $x_n^j \rightarrow -\infty$ as $n \rightarrow \infty$ holds.

Step 2 We show that $\psi^j \equiv 0$ except for at most one j .

Suppose not. Then, by means of (2.9), we have $c_j^{\frac{1}{\sigma}-1} \ell(\psi^j) < d_+$ for all j . Let us define $V_n^j(t, x)$ as follows:

- When $\xi_n \equiv 0$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3 t + t_n^j)$, where $\Psi^j(t)$ is a nonlinear profile associated with $(\operatorname{Re} \psi^j, t_n^j)$, that is,
 - if $t_n^j \equiv 0$ then $\Psi^j(t)$ is a solution to (gKdV) with $\Psi^j(0) = \operatorname{Re} \psi^j$;
 - if $t_n^j \rightarrow \infty$ as $n \rightarrow \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters forward in time to $e^{-t\partial_x^3} \operatorname{Re} \psi^j$;
 - if $t_n^j \rightarrow -\infty$ as $n \rightarrow \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters backward in time to $e^{-t\partial_x^3} \operatorname{Re} \psi^j$;
- When $\xi_n \rightarrow \infty$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi_n^j((h_n^j)^3 t + t_n^j)$, where Ψ_n^j is a solution to (gKdV) with the initial condition

$$\Psi_n^j(t_n^j) = A(t_n^j) \operatorname{Re}(P(\xi_n^j) \psi^j).$$

Here, we define an approximate solution

$$(2.17) \quad \tilde{u}_n^J(t, x) = \sum_{j=1}^J V_n^j(t, x) + e^{-t\partial_x^3} r_n^J.$$

We apply long time stability [20, Proposition 3.2] for $\tilde{u}_n^J$ defined by (2.17) to see that $\|\Psi_n^j\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R}_+; L_t^{5\alpha}(\mathbb{R}))} < \infty$ for sufficiently large n . Then, the scattering criterion in $\hat{L}^\alpha$ implies that $u_n \in \mathcal{S}_+$, which contradicts with the definition of $\{u_n\}_n$.

Step 3 We now see that there exists j_0 such that $c_j^{\frac{1}{\sigma}-1} \ell(\psi^{j_0}) = d_+$. Then, one sees from the definition of $\{u_n\}_n$ and (2.9) that $\psi^j \equiv 0$ for $j \neq j_0$. For simplicity, we drop index j_0 and write

$$u_n = \mathcal{G}_n \psi + r_n, \quad \tilde{u}_n(t) = V_n(t) + e^{-t\partial_x^3} r_n$$

in what follows. Further, we have $\lim_{n \rightarrow \infty} \|r_n\|_{\dot{M}_{2,\sigma}^\alpha} = 0$.

When $|\xi_n| \rightarrow \infty$, as in the previous step, we see from assumption (2.3) and Theorem 2.5 that $u_n \in \mathcal{S}_+$ for large n , a contradiction. Hence, $\xi_n \equiv 0$. Recall that

$$V_n = D(h_n)T(y_n)\Psi((h_n)^3t + t_n)$$

where $\Psi(t)$ is a nonlinear profile associated with (ψ, t_n) . Let us now show that $u_c := \Psi$ is the solution which has the desired property. We have $\Psi(t_n) \notin \mathcal{S}_+$, otherwise $u_n \in \mathcal{S}_+$ for large n by long time stability.

The case $t_n \rightarrow \infty$ ($n \rightarrow \infty$) is excluded since this implies $\Psi(t_n) \in \mathcal{S}_+$. If $t_n \equiv 0$ then $\Psi(0) = \psi$ and so $\ell(\Psi(0)) = d_+$. Finally, if $t_n \rightarrow -\infty$ as $n \rightarrow \infty$ then $\lim_{t \rightarrow -\infty} e^{t\partial_x^3}\Psi(t) = \psi$ and putting $u_{c,-} := \lim_{t \rightarrow -\infty} e^{t\partial_x^3}\Psi(t)$, we have $\ell(u_{c,-}) = d_+$. This completes the proof of Theorem 2.1.

§ Appendix A. Proof of Stein-Tomas estimate

In this appendix, we show the Stein-Tomas estimate (1.2).

Proof. (of Theorem 1.1.) Following [19, Lemma 2.2], we give a direct proof which is based on the fact that the exponents for space-variable and time-variable in the left hand side coincide. The case $p = \infty$ follows from the Hausdorff-Young inequality. Let $p < \infty$. As explained in the introduction, the key point to prove the inequality (1.2) is to reduce the linear form into a bi-linear form;

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L_{t,x}^{3p}}^2 = \left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta \right\|_{L_{t,x}^{\frac{3p}{2}}}.$$

Changing variables by $a = \xi - \eta$ and $b = \xi^3 - \eta^3$, we have

$$\begin{aligned} & \left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta \right\|_{L_{t,x}^{\frac{3p}{2}}} \\ &= \left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} \frac{1}{3|\xi^2 - \eta^2|} da db \right\|_{L_{t,x}^{\frac{3p}{2}}}. \end{aligned}$$

We now use the Hausdorff-Young inequality to deduce that

$$\begin{aligned} & \left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} \frac{1}{3|\xi^2 - \eta^2|} da db \right\|_{L_{t,x}^{\frac{3p}{2}}} \\ & \leq C \left\| |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi) \overline{\hat{f}(\eta)} |\xi^2 - \eta^2|^{-1} \right\|_{L_{a,b}^{(\frac{3p}{2})'}} \\ & \leq C \left\{ \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \right\}^{\frac{1}{(\frac{3p}{2})'}}, \end{aligned}$$

where

$$m(\xi, \eta) = \frac{|\xi\eta|}{|\xi + \eta|^2 |\xi - \eta|^2}.$$

Notice that $3p/2 \geq 2$. From the elementary inequality $\xi\eta \leq (\xi + \eta)^2/2$, we have

$$m(\xi, \eta) \leq \frac{1}{2|\xi - \eta|^2}.$$

Hence we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty m(\xi, \eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \\ & \leq C \int_0^\infty \int_0^\infty \frac{|\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta. \end{aligned}$$

By the Hölder and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{|\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi - \eta|^{\frac{2}{3p-2}}} d\xi d\eta \\ & \leq \left\| |\hat{f}|^{\frac{3p}{3p-2}} \right\|_{L^{\frac{3p-2}{3p-3}}} \left\| (|\xi|^{-\frac{2}{3p-2}} * |\hat{f}|^{\frac{3p}{3p-2}}) \right\|_{L^{3p-2}} \\ & \leq C \left\| \hat{f} \right\|_{L^{p'}}^{\frac{6p}{3p-2}} = C \|f\|_{\dot{L}_x^p}^{\frac{6p}{3p-2}} \end{aligned}$$

as long as $2/(3p-2) < 1$, that is, $p > 4/3$. Combining the above inequalities, we obtain the inequality (1.2). □

§ Appendix B. Upper Bound for d_+

In this appendix, we show the following upper bound for d_+ ;

$$(B.1) \quad d_+ \leq c_\alpha \ell(Q), \quad c_\alpha = \left(\frac{(\alpha+1) \|Q'\|_{L^2}^2}{\|Q\|_{L^{2\alpha+2}}^{2\alpha+2}} \right)^{\frac{1}{2\alpha}} < 1.$$

To show (B.1), we give a criteria for scattering in terms of the energy. Here we remark that an $\hat{L}^\alpha$ -solution has conserved quantities, provided the solution has appropriate regularity. More precisely, when $u_0 \in \hat{L}^\alpha \cap L^2$, a solution $u(t)$ has a conserved mass

$$M[u(t)] := \|u(t)\|_{L^2}^2.$$

Similarly, if $u_0 \in \hat{L}^\alpha \cap \dot{H}^1$ then energy

$$E[u(t)] := \frac{1}{2} \|\partial_x u(t)\|_{L^2}^2 - \frac{1}{2\alpha+2} \|u(t)\|_{L^{2\alpha+2}}^{2\alpha+2}$$

is invariant.

We note that if an $\hat{L}^\alpha$ -solution $u(t)$ scatters (in $\hat{L}^\alpha$ sense) as $t \rightarrow \pm\infty$ and if $u_0 \in \hat{L}^{\alpha_0}$ (resp. if $u_0 \in \dot{H}^\sigma$) then $u(t)$ scatters as $t \rightarrow \pm\infty$ also in $\hat{L}^{\alpha_0}$ sense (resp. $\dot{H}^\sigma$ sense).

Theorem Appendix B.1 ([19]). *Let $8/5 < \alpha < 10/3$. If $u_0 \in \hat{L}^\alpha \cap H^1$ satisfies $u_0 \neq 0$ and $E[u_0] \leq 0$ then $u(t)$ does not scatter as $t \rightarrow \pm\infty$.*

The inequality (B.1) easily follows from Theorem Appendix B.1. Indeed, since $u_0 = c_\alpha Q$ satisfies $E[u_0] = 0$, we have $u_0 \notin \mathcal{S}_+$ by Theorem Appendix B.1. Hence we have (B.1).

Proof. We suppose for contradiction that $u(t)$ scatters to $u_+ \in \hat{L}^\alpha$ as $t \rightarrow \infty$. Since $u_0 \in H^1$, [11] imply that $u(t) \in C(\mathbb{R}; H^1)$. Further, $u(t)$ scatters also in H^1 and so we see that

$$\|\partial_x u(t)\|_{L^2} = \left\| \partial_x e^{t\partial_x^3} u(t) \right\|_{L^2} \rightarrow \|u_+\|_{\dot{H}^1}$$

as $t \rightarrow \infty$.

On the other hand, by the Gagliardo-Nirenberg inequality and mass conservation,

$$\|u(t)\|_{L_x^{2\alpha+2}} \leq C \|u_0\|_{L_x^2}^{\frac{1}{\alpha+1}} \left\| |\partial_x|^{\frac{1}{3\alpha}} u(t) \right\|_{L_x^{3\alpha}}^{\frac{\alpha}{\alpha+1}}.$$

Since $u(t)$ scatters as $t \rightarrow \infty$, as in the proof of [19, Theorem 1.9], we can show that $|\partial_x|^{\frac{1}{3\alpha}} u \in L_{t,x}^{3\alpha}$. Therefore, we can take a sequence $\{t_n\}_n$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ so that $\|u(t_n)\|_{L^{2\alpha+2}} \rightarrow 0$ as $n \rightarrow \infty$. Thus, by conservation of energy,

$$0 \geq E[u_0] = E[u(t_n)] = \frac{1}{2} \|\partial_x u(t_n)\|_{L^2}^2 - \frac{1}{2\alpha+2} \|u(t_n)\|_{L^{2\alpha+2}}^{2\alpha+2} \rightarrow \frac{1}{2} \|u_+\|_{\dot{H}^1}^2$$

as $n \rightarrow \infty$. Hence, $E[u_0] < 0$ yields a contradiction. If $E[u_0] = 0$ then we see that $u_+ = 0$, and so that $\|u_0\|_{L^2} = \|u_+\|_{L^2} = 0$. This contradicts to $u_0 \neq 0$. $\square$

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