

Loewy Structure as a q -analog of Composition Factors

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Abstract

The Loewy structure of a module can be viewed as a q -analog of its composition factors. From this point of view we define q -composition multiplicity, q -composition length, and the q -Cartan matrix. By way of example, group algebras of finite p -groups and path algebras of finite acyclic quivers are investigated. Some known results for these algebras are stated in terms of the q -Cartan matrix.

Keywords Loewy structure, Composition factor, q -analog, Cartan matrix, Group algebra, the Jennings theorem, Path algebra.

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1 Introduction

The Loewy structure of a module enables us to visualize and gain some intuition about the module. In fact, such visualization is enough to obtain submodules or homological invariants, provided sufficient additional information is available [1, 3]. To establish Loewy structures several studies have been done such as [2, 8, 11]. We explore another way of dealing with Loewy structures in this report.

2 Notation and Terminology

In this section, we define q -composition multiplicity, q -composition length, and the q -Cartan matrix. We follow the notation and terminology of [10] unless otherwise stated. The term module refers to a finitely generated right module.

Definition 2.1. Let V be a module over a right artinian ring. The Jacobson radical of V is denoted by $\text{rad } V$. For an integer $n \geq 0$, the n th radical of V is defined inductively by $\text{rad}^0 V = V$ and

$$\text{rad}^n V = \text{rad}(\text{rad}^{n-1} V)$$

if $n > 0$. We then write

$$\text{rad}_n V = \text{rad}^n V / \text{rad}^{n+1} V$$

and call it the n th radical layer of V . (Note that these are different from the custom.) The decomposition of semisimple modules $\text{rad}_n V$ into simple modules is referred to as the *Loewy structure* of V and may be visualized as (3.2) for instance.

Definition 2.2. Let R be a right artinian ring and $S_1, \dots, S_k$ the representatives of simple R -modules. For an R -module V and its composition series

$$0 = V_0 < V_1 < \dots < V_t = V \quad (t \geq 0)$$

we call

$$c_i(V) := \#\{1 \leq s \leq t \mid V_s/V_{s-1} \cong S_i\} \quad (1 \leq i \leq k)$$

the *composition multiplicity* of S_i in V and $t = c_1(V) + \dots + c_k(V)$ the *composition length* of V . The latter is denoted by $\ell(V)$. We then call its q -analog defined by

$$c_i(V; q) := \sum_{n \geq 0} c_i(\text{rad}_n V) q^n \in \mathbb{Z}[q]$$

the q -composition multiplicity of S_i in V and $c_1(V; q) + \dots + c_k(V; q)$ the q -composition length of V . The latter is denoted by $\ell(V; q)$.

Remark 2.3. The coefficient of q^n in (2.2) represents the number of times that the simple module S_i appears in the decomposition of $\text{rad}_n V$. Hence the Loewy structure of V corresponds to the vector

$$^T(c_1(V; q), \dots, c_k(V; q)).$$

For illustration, see (3.2) and (3.3). Since $c_i(V) = \lim_{q \rightarrow 1} c_i(V; q)$, the Loewy structure of a module can be viewed as a q -analog of its composition factors.

Now it is possible to define a q -analog of anything that is defined in terms of composition multiplicity or composition length. We subsequently define and investigate a q -analog of the Cartan matrix.

Definition 2.4. Under the notation of Definition 2.2, let $P_1, \dots, P_k$ be the projective covers of $S_1, \dots, S_k$. We call

$$C_R := [c_i(P_j)]_{1 \leq i, j \leq k} \quad \text{and} \quad C_R(q) := [c_i(P_j; q)]_{1 \leq i, j \leq k}$$

the *Cartan matrix* of R and the *q -Cartan matrix* of R respectively.

Remark 2.5. Wilson [12] and Fuller [6] give more general definitions for similar concepts and call them *Cartan homomorphisms* for graded algebras and *F -filtered Cartan matrices* respectively in the context of the Cartan determinant conjecture. Bessenrodt and Holm [4] also give essentially the same definition for the factor algebra of a path algebra by a homogeneous ideal and call it a *q -Cartan matrix*.

The q -analogues defined above enable us to deal with Loewy structures algebraically and to ask questions involving terms of matrix theory such as determinant.

3 Group Algebras

Let us show an example that motivates the definition of these q -analogues.

Example 3.1. Let G be a group of order 4 and consider the group algebra FG over a field F of characteristic 2. The composition length is not sufficient to distinguish the difference between the isomorphism classes of groups of order 4; namely $C_2 \times C_2$ and C_4 . The q -composition length, on the other hand, is sufficient to distinguish the difference satisfactorily.



$$\begin{array}{ll}
 \ell(F[C_2 \times C_2]) &= 4 \\
 \ell(F[C_2 \times C_2]; q) &= (1 + q) \times (1 + q)
 \end{array}
 \qquad
 \begin{array}{ll}
 \ell(F[C_4]) &= 4 \\
 \ell(F[C_4]; q) &= 1 + q + q^2 + q^3
 \end{array}$$

In fact, these polynomials are naturally obtained as the generating functions of the dimensions of radical layers. This, of course, does not happen by chance; We restate the Jennings theorem in terms of q -composition length or the q -Cartan matrix and give some remarks.

Theorem 3.2 (Jennings [7, Theorem 3.7]). *Let p be a prime number, G a finite p -group, and F a field of characteristic p . Set*

$$K_n := \{g \in G \mid g - 1 \in \text{rad}^n FG\}$$

for an integer $n \geq 0$. Then K_n/K_{n+1} is an elementary abelian p -group of rank $r_n \geq 0$ and

$$\ell(FG; q) = \prod_{n \geq 1} \left(\frac{1 - q^{np}}{1 - q^n} \right)^{r_n} = \det C_{FG}(q). \quad (3.1)$$

Remark 3.3. Let F be a field of characteristic $p > 0$ and G a finite group such that p divides its order. In modular representation theory of finite groups, it is well-known that the dimension of a projective FG -module is divisible by the order of the Sylow p -subgroup of G [9, Theorem 3.1.26]. The first equality of (3.1) can be viewed as a q -analog of this theorem. It is also well-known that $\det C_{FG}$ is a power of p [9, Lemma 3.6.31]. The second equality of (3.1) can be viewed as a q -analog of this theorem.

Therefore it is expected that some properties of Cartan matrices also hold for q -Cartan matrices. Unfortunately, well-known facts about Cartan matrices of group algebras are no longer hold naïvely for q -Cartan matrices.

For example, the group algebra FA_5 of the alternating group A_5 of degree 5 over an algebraically closed field F of characteristic 2 has the Loewy structures [5, p.52] and q -Cartan matrix below.

$$\begin{array}{cccc}
 & S_1 & S_2 & S_3 \\
 S_2 & \diagdown & | & | \\
 & S_3 & S_1 & S_1 \\
 S_1 & | & | & | \\
 & S_1 & S_3 & S_2 \\
 S_3 & | & | & | \\
 & S_2 & S_1 & S_1 \\
 & | & | & | \\
 & S_1 & S_2 & S_3 \\
 & P_1 & P_2 & P_3 & P_4
 \end{array} \quad (3.2)$$

$$C_{FA_5}(q) = \begin{bmatrix} 1 + 2q^2 + q^4 & q + q^3 & q + q^3 & 0 \\ q + q^3 & 1 + q^4 & q^2 & 0 \\ q + q^3 & q^2 & 1 + q^4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.3)$$

Hence $\ell(P_2; q)$ and $\det C_{FA_5}(q)$ are not divisible by $1 + q$ in $\mathbb{Z}[q]$, unlike the p -group case.

Furthermore, in general q -Cartan matrices of group algebras need not be symmetric as Cartan matrices must [9, Theorem 2.8.21.ii], although it

is the case in (3.3). For example, the group algebra FG of the group G defined by

$$G = \begin{bmatrix} 1 & \mathbb{F}_p \\ 0 & \mathbb{F}_p^\times \end{bmatrix}$$

over a field F of characteristic p has the following q -Cartan matrix.

$$C_{FG}(q) = \begin{bmatrix} 1 + q^{p-1} & q & \cdots & q^{p-2} \\ q^{p-2} & 1 + q^{p-1} & \cdots & q^{p-3} \\ \vdots & \vdots & \ddots & \vdots \\ q & q^2 & \cdots & 1 + q^{p-1} \end{bmatrix} \quad (3.4)$$

An appropriate generalization is expected.

4 Path Algebras

We give a combinatorial interpretation of q -Cartan matrices for path algebras in Theorem 4.2. Let us begin with a simple example to observe how it should look.

Example 4.1. Let F be a field and Q the quiver defined by the following.

$$Q = 1 \leftarrow 2 \leftarrow \cdots \leftarrow k$$

The path algebra of Q over the field F is denoted by FQ . Write S_i for the simple FQ -module that corresponds to a vertex $1 \leq i \leq k$. The projective covers P_i of S_i have the Loewy structures described below.

$$\begin{array}{cccc} & & & S_k \\ & & & | \\ & & & \vdots \\ & & & | \\ & S_2 & & S_2 \\ & | & \cdots & | \\ S_1 & S_1 & & S_1 \\ P_1 & P_2 & \cdots & P_k \end{array}$$

Hence FQ has the following Cartan matrix and q -Cartan matrix.

$$C_{FQ} = \begin{bmatrix} 1 & 1 & & 1 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \quad C_{FQ}(q) = \begin{bmatrix} 1 & q & & q^{k-1} \\ & 1 & \ddots & \\ & & \ddots & q \\ & & & 1 \end{bmatrix}$$

Readers might see in Example 4.1 that the coefficient of q^n of the (i, j) -entry of $C_{FQ}(q)$ agrees with the number of paths from j to i of length n . This phenomenon is generalized as follows. Recall that for a finite quiver Q the matrix with each of its (i, j) -entries equaling the number of arrows from i to j is called the *adjacency matrix* of Q .

Theorem 4.2. *Let F be a field and Q a finite acyclic quiver with vertex set $\{1, \dots, k\}$ and adjacency matrix A . Write S_i and P_i for the simple FQ -modules and their projective covers corresponding to vertices $1 \leq i \leq k$ respectively. Then the q -Cartan matrix $C_{FQ}(q) = [c_i(P_j; q)]$ can be expressed as the power series*

$$C_{FQ}(q) = \sum_{n \geq 0} ({}^T A)^n q^n. \quad (4.1)$$

Remark 4.3. The $q = 1$ case of (4.1) can be found in [10, Corollary I.11.6].

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