

Convolutions and Hölder inequality for certain analytic functions

Junichi Nishiwaki, Shigeyoshi Owa and H. M. Srivastava

Abstract

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclass $\mathcal{MD}(\alpha, \beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ are defined. The object of the present paper is to derive some properties for functions $f(z)$ in the classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ involving their generalized convolution by utilizing methods on the basis of the Hölder inequalities.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Nishiwaki and Owa [2], [4] have considered the subclass $\mathcal{MD}(\alpha, \beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha \leq 0)$ and $\beta (\beta > 1)$. We discuss some properties of functions $f(z)$ belonging to the class $\mathcal{MD}(\alpha, \beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)} = u + iv$ maps $\mathbb{U}$ onto the elliptic domain such that

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta - 1)} v^2 + \frac{\beta + 1}{2}$$

for $\alpha = -1$, and the hyperbolic domain such that

2000 Mathematics Subject Classification: Primary 30C45.

Keywords and Phrases: analytic function, uniformly starlike, Hölder inequality, convolution.

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 - \frac{\alpha^2}{1 - \alpha^2}v^2 > \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $-1 < \alpha \leq 0$.

Recently, Nishiwaki and Owa [2] have given the following coefficient inequality for $f(z)$ belonging to the class $\mathcal{MD}(\alpha, \beta)$.

Lemma 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$(1.1) \quad \sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$, then $f(z) \in \mathcal{MD}(\alpha, \beta)$.

From the above lemma, we easily know

$$\sum_{n=2}^{\infty} \frac{(n - \beta + 1) + |n - \beta - 1| - 2\alpha(n - 1)}{2(\beta - 1)} |a_n| \leq \sum_{n=2}^{\infty} \frac{(n - \beta + 1) + (n + \beta - 3) - 2\alpha(n - 1)}{2(\beta - 1)} |a_n|$$

$$\leq 1$$

for some $\beta(1 < \beta \leq 2)$ and

$$\sum_{n=2}^{\infty} \frac{1}{2} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \sum_{n=2}^{\infty} \frac{1}{2} \{(n + \beta - 3) + (n + \beta - 3) - 2\alpha(n - 1)\} |a_n|$$

$$\leq 1$$

for some $\beta(\beta \geq 2)$. In view of these inequalities, we define the subclass $\mathcal{M}_1(\alpha, \beta)$ of $\mathcal{MD}(\alpha, \beta)$ consisting of functions $f(z)$ which satisfy the condition

$$(1.2) \quad \sum_{n=2}^{\infty} \frac{(n - 1)(1 - \alpha)}{\beta - 1} |a_n| \leq 1$$

for some $\alpha(\alpha \leq 0)$ and $\beta(1 < \beta \leq 2)$, and also the subclass $\mathcal{M}_2(\alpha, \beta)$ of $\mathcal{MD}(\alpha, \beta)$ consisting of functions $f(z)$ which satisfy the condition

$$(1.3) \quad \sum_{n=2}^{\infty} \{n(1 - \alpha) - 3 + \alpha + \beta\} |a_n| \leq 1$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta \geq 2)$.

2 Generalizations of the Convolutions for the classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$

In this section, some convolution properties of $f(z)$ belonging to the classes $\mathcal{M}_1(\alpha, \beta)$ and $\mathcal{M}_2(\alpha, \beta)$ are discussed.

For functions $f_j(z) \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, \dots, m),$$

we define

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m a_{n,j}^{p_j} \right) z^n \quad (p_j > 0).$$

Then $H_m(z)$ denotes the generalization of the convolutions. It was considered by Choi, Kim and Owa [1]. Lately, it was studied by Srivastava and Owa [5] (also see [3]).

For functions $f_j(z) \in \mathcal{A}$, Hölder inequality is given by

$$\sum_{n=2}^{\infty} \left(\prod_{j=1}^m |a_{n,j}| \right) \leq \prod_{j=1}^m \left(\sum_{n=2}^{\infty} |a_{n,j}|^{p_j} \right)^{\frac{1}{p_j}},$$

where $p_j > 1$ and $\sum_{j=1}^m \frac{1}{p_j} \geq 1$.

Our first result for $H_m(z)$ is contained in

Theorem 2.1. *If $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $1 < \beta_j \leq 2$), then $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$ with*

$$\beta^* = 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1 - \alpha)^{s-1}},$$

where $s = \sum_{j=1}^m p_j \geq 1$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Proof. Let $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$, then the inequality (1.2) gives us that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \leq 1 \quad (j = 1, 2, \dots, m),$$

which implies

$$\left\{ \sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta_j - 1} |a_{n,j}| \right\}^{\frac{1}{q_j}} \leq 1$$

with $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$. Applying the Hölder inequality, we have the following inequality

$$\sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \leq 1.$$

Then we have to find the largest β such that

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq 1,$$

that is,

$$\sum_{n=2}^{\infty} \frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq \sum_{n=2}^{\infty} \left\{ \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}} \right\}.$$

Therefore, we need to find the largest β such that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j} \right) \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}} |a_{n,j}|^{\frac{1}{q_j}}$$

which is equivalent to

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \left(\prod_{j=1}^m |a_{n,j}|^{p_j - \frac{1}{q_j}} \right) \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{\frac{1}{q_j}}$$

for all $n \geq 2$. Since

$$\prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}} |a_{n,j}|^{p_j - \frac{1}{q_j}} \leq 1 \quad \left(p_j - \frac{1}{q_j} \geq 0 \right),$$

we see that

$$\prod_{j=1}^m |a_{n,j}|^{p_j - \frac{1}{q_j}} \leq \frac{1}{\prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j - \frac{1}{q_j}}}.$$

This implies that

$$\frac{(n-1)(1-\alpha)}{\beta^* - 1} \leq \prod_{j=1}^m \left(\frac{(n-1)(1-\alpha)}{\beta_j - 1} \right)^{p_j}$$

for all $n \geq 2$. Therefore, β^* should be

$$\beta^* \geq 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1-\alpha)^{s-1} (n-1)^{s-1}} \quad \left(s = \sum_{j=1}^m p_j \right),$$

so that, the right hand side of the last inequality is a decreasing function for $n \geq 2$. This means

$$\begin{aligned}\beta^* &= \max_{n \geq 2} \left\{ 1 + \frac{\prod_{j=1}^m (\beta_j - 1)}{(1 - \alpha)^{s-1} (n - 1)^{s-1}} \right\} \\ &= 1 + \frac{\prod_{j=1}^m (\beta_j - 1)^{p_j}}{(1 - \alpha)^{s-1}}.\end{aligned}$$

This completes the proof of the theorem. □

Example 2.1. Let us define

$$f_j(z) = z + \sum_{n=2}^{\infty} \frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} z^n \quad (|\varepsilon_j| = 1)$$

for each j ($j = 1, 2, 3, \dots, m$). It is easy to see that $f_j(z) \in \mathcal{M}_1(\alpha, \beta_j)$. Then we have

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m \left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} \right)^{p_j} \right) z^n.$$

For this function $H_m(z)$, we calculate that

$$\begin{aligned}& \sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha)}{\beta^* - 1} \right) \left| \prod_{j=1}^m \left(\frac{(\beta_j - 1)\varepsilon_j}{n(n-1)^2(1-\alpha)} \right)^{p_j} \right| \\ &= \sum_{n=2}^{\infty} \frac{1}{n^s (n-1)^{2s-1}} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.\end{aligned}$$

Thus we know that $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$.

Taking $\beta_j = \beta$ ($j = 1, 2, \dots, m$) in Theorem 2.1, we obtain

Corollary 2.1. If $f_j(z) \in \mathcal{M}_1(\alpha, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $1 < \beta \leq 2$), then $H_m(z) \in \mathcal{M}_1(\alpha, \beta^*)$ with

$$\beta^* = 1 + \frac{(\beta - 1)^s}{(1 - \alpha)^{s-1}},$$

where $s = \sum_{j=1}^m p_j \geq 1$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

By using $\mathcal{M}_1(\alpha_j, \beta)$ instead of $\mathcal{M}_1(\alpha, \beta_j)$ in Theorem 2.1, we also derive the next result.

Theorem 2.2. If $f_j(z) \in \mathcal{M}_1(\alpha_j, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha_j \leq 0$, $1 < \beta \leq 2$), then $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$ with

$$\alpha^* = 1 - \frac{\prod_{j=1}^m (1 - \alpha_j)^{p_j}}{(\beta - 1)^{s-1}},$$

where $s = \sum_{j=1}^m p_j \geq 1$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Proof. Using the same method as in the proof of Theorem 2.1, we have

$$\frac{(n-1)(1-\alpha^*)}{\beta-1} \leq \frac{(n-1)^s \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^s},$$

which implies that

$$\alpha^* \geq 1 - \frac{(n-1)^{s-1} \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}}$$

for all $n \geq 2$, so that, the right hand side of the last inequality is a decreasing for $n \geq 2$. This means

$$\begin{aligned} \alpha^* &= \max_{n \geq 2} \left\{ 1 - \frac{(n-1)^{s-1} \prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}} \right\} \\ &= 1 - \frac{\prod_{j=1}^m (1-\alpha_j)^{p_j}}{(\beta-1)^{s-1}}, \end{aligned}$$

which proves the theorem. □

Example 2.2. Let us consider

$$f_j(z) = z + \sum_{n=2}^{\infty} \frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} z^n \quad (|\varepsilon_j| = 1)$$

for each j ($j = 1, 2, 3, \dots, m$). Then we see that $f_j(z) \in \mathcal{M}_1(\alpha_j, \beta)$. Also we have that

$$H_m(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^m \left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} \right)^{p_j} \right) z^n.$$

It follows from the function $H_m(z)$ that

$$\sum_{n=2}^{\infty} \left(\frac{(n-1)(1-\alpha^*)}{\beta-1} \right) \left| \prod_{j=1}^m \left(\frac{(\beta-1)\varepsilon_j}{n(n-1)^2(1-\alpha_j)} \right)^{p_j} \right|$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)^{2s-1}} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.$$

This implies that $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$.

Letting $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) in Theorem 2.2, we obtain

Corollary 2.2. *If $f_j(z) \in \mathcal{M}_1(\alpha, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $1 < \beta \leq 2$), then $H_m(z) \in \mathcal{M}_1(\alpha^*, \beta)$ with*

$$\alpha^* = 1 - \frac{(1 - \alpha)^s}{(\beta - 1)^{s-1}},$$

where $s = \sum_{j=1}^m p_j \geq 1$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Next, for the generalized Hadamard product (or Convolution) of functions in the class $\mathcal{M}_2(\alpha, \beta)$, we also derive

Theorem 2.3. *If $f_j(z) \in \mathcal{M}_2(\alpha, \beta_j)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $\beta_j \geq 2$), then $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ with*

$$\beta^* = 1 + \alpha + \prod_{j=1}^m (\beta_j - 1 - \alpha)^{p_j},$$

where $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Proof. In the same manner as in the proof of Theorem 2.1, we obtain

$$\beta^* + (n-1)(1-\alpha) - 2 \leq \prod_{j=1}^m \{n(1-\alpha) - 3 + \alpha + \beta_j\}^{p_j}.$$

The left hand side of the above inequality is a increasing function for $n \geq 2$. Then we get

$$\beta^* - 1 - \alpha \leq \prod_{j=1}^m \{n(1-\alpha) - 3 + \alpha + \beta_j\}^{p_j}.$$

Also the right hand side of it is a increasing function for $n \geq 2$, so that, we have

$$\beta^* \leq 1 + \alpha + \prod_{j=1}^m (\beta_j - 1 - \alpha)^{p_j}.$$

This completes the proof of the theorem. □

If we take $\beta_j = \beta$ ($j = 1, 2, \dots, m$) in Theorem 2.3, then we obtain

Corollary 2.3. *If $f_j(z) \in \mathcal{M}_2(\alpha, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $\beta \geq 2$), then $H_m(z) \in \mathcal{M}_2(\alpha, \beta^*)$ with*

$$\beta^* = 1 + \alpha + (\beta - 1 - \alpha)^s,$$

where $s = \sum_{j=1}^m p_j \geq 1$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Using $\mathcal{M}_2(\alpha_j, \beta)$ instead of $\mathcal{M}_2(\alpha, \beta_j)$ in Theorem 2.3, we also derive the next result.

Theorem 2.4. *If $f_j(z) \in \mathcal{M}_2(\alpha_j, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha_j \leq 0$, $\beta \geq 2$), then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with*

$$\alpha^* = \max_{n \geq 2} \left\{ 1 - \frac{(\beta - 2) + \prod_{j=1}^m (n(1 - \alpha_j) - 3 + \alpha_j + \beta)^{p_j}}{n - 1} \right\},$$

where $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Proof. By using the same method as in the proof of Theorem 2.1, we see that

$$n - 3 - \alpha^*(n - 1) + \beta \leq \prod_{j=1}^m \{n(1 - \alpha_j) - 3 + \alpha_j + \beta\}^{p_j}$$

which implies that

$$\alpha^* \geq 1 - \frac{(\beta - 2) + \prod_{j=1}^m \{n(1 - \alpha_j) - 3 + \alpha_j + \beta\}^{p_j}}{n - 1}.$$

Therefore, we prove the theorem. □

Finally, taking $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) in Theorem 2.4, we obtain

Corollary 2.4. *If $f_j(z) \in \mathcal{M}_2(\alpha, \beta)$ for each $j = 1, 2, \dots, m$ ($\alpha \leq 0$, $\beta \geq 2$), then $H_m(z) \in \mathcal{M}_2(\alpha^*, \beta)$ with*

$$\alpha^* = 3 - \beta - (\beta - 1 - \alpha)^s,$$

where $s = \sum_{j=1}^m p_j \geq 1 + \frac{2(\beta - 2)}{1 - \alpha}$, $p_j \geq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geq 1$.

Proof. In view of Theorem 2.4, we obtain

$$\alpha^* \geq 1 - \frac{(\beta - 2) + \{n(1 - \alpha) - 3 + \alpha + \beta\}^s}{n - 1}.$$

Let $F(n)$ be the right hand side of the above inequality. Further, let us define $G(n)$ by the numerator of $F'(n)$, so that

$$\begin{aligned} G(n) &= -(n(1 - \alpha) - 3 + \alpha + \beta)^{s-1} \{n(1 - \alpha)(s - 1) - s(1 - \alpha) + 3 - \alpha - \beta\} + (\beta - 2) \\ &\leq -(\beta - 1 - \alpha)^{s-1} \{2(1 - \alpha)(s - 1) - s(1 - \alpha) + 3 - \alpha - \beta\} + (\beta - 2) \\ &\leq 2\beta - 3 - \alpha - s(1 - \alpha) \\ &\leq 0 \quad \left(s \geq 1 + \frac{2(\beta - 2)}{1 - \alpha} \right) \end{aligned}$$

which implies that

$$\begin{aligned} \alpha^* &= \max_{n \geq 2} \left\{ 1 - \frac{(\beta - 2) + \prod_{j=1}^m (n(1 - \alpha_j) - 3 + \alpha_j + \beta)^{p_j}}{n - 1} \right\} \\ &= 3 - \beta - (\beta - 1 - \alpha)^s. \end{aligned}$$

This completes proof of the corollary. □

References

- [1] J. H. Choi, Y. C. Kim and S. Owa, *Generalizations of Hadamard products of functions with negative coefficients*, J. Math. Anal. Appl. **199**(1996), 495-501.
- [2] J. Nishiwaki and S. Owa, *Certain classes of analytic functions concerned with uniformly starlike and convex functions*, Appl. Math. Comp. (to appear).
- [3] J. Nishiwaki and S. Owa, *An application of Hölder inequality for convolution*, (submitted).
- [4] J. Nishiwaki and S. Owa *Some properties of certain analytic functions*, Kokyuroku. (to appear).
- [5] H. M. Srivastava and S. Owa, *Some generalized convolution properties associated with certain subclasses of analytic functions*, J. Inequ. Pure. Appl. Math. **3**. No3. Article42. (2002), 1-13.

Junichi Nishiwaki
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail : jerjun2002@yahoo.co.jp

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail : owa@math.kindai.ac.jp

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
e-mail : hmsri@uvvm.uvic.ca