

Non-Monotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators

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§ 1. Introduction.

In the present paper, we consider the existence and regularity of strong solutions of several types of problems for the following abstract equation (E) in a real separable Hilbert space H :

$$(E) \quad \frac{du}{dt}(t) + \partial \varphi^t(u(t)) + B(t, u(t)) \ni f(t),$$

where $f(t)$ is a given function, $\partial \varphi^t$ is the subdifferential of a time-dependent lower semi-continuous convex function φ^t from H into $[0, +\infty]$ with $\varphi^t \not\equiv +\infty$, and where $B(t, \cdot)$ is a possibly non-monotone multi-valued nonlinear operator with $D(B(t, \cdot)) \supset D(\partial \varphi^t)$, which is regarded as a perturbation for $\partial \varphi^t$ in a sense. Here and henceforth we are concerned with strong solutions of (E) in the following sense.

DEFINITION 1.1. A function $u(t)$ is said to be a strong solution of (E) in an open interval I of $\mathbb{R}^1$, if the following properties (i) and (ii) are satisfied.

(i) $u(t)$ is an H -valued absolutely continuous function on any compact subset of I .

(ii) $u(t) \in D(\partial \varphi^t)$ for a.e. $t \in I$ and there exist two functions $g(t), b(t) \in L^2_{loc}(I; H)$ such that $g(t) \in \partial \varphi^t(u(t)), b(t) \in B(t, u(t))$

and $du(t)/dt + g(t) + b(t) = f(t)$ hold for a.e. $t \in I$.

Actually our main concern here is to study the existence of strong solutions for the following three types of problems :

- (I) Cauchy Problems $(E)_0$: For each initial data a given in $B_{\alpha,p}(\partial\varphi^0)$ ($0 < \alpha < 1/2$, $1 \leq p \leq +\infty$), interpolation classes between $\overline{D(\partial\varphi^0)}$ and $D(\partial\varphi^0)$, find a strong solution $u(t)$ of (E) in $(0, +\infty)$ with $u(+0) = a$.
- (II) Periodic Problems $(E)_\pi$: When $\varphi^0(\cdot) = \varphi^T(\cdot)$, find a strong solution $u(t)$ of (E) in $(0, T)$ with $u(0) = u(T)$.
- (III) Almost-Periodic Problems $(E)_{\alpha\pi}$: When $f(t)$ is an H -valued almost-periodic function and φ^t varies almost-periodically with respect to t in a sense, find an H -valued almost-periodic strong solution of (E) in $\mathbb{R}^1$.

When $B(t, \cdot)$ is a monotone-type operator (or $B(t, \cdot) \equiv 0$), many results on the existence, uniqueness and regularity of strong solutions for $(E)_0$ and $(E)_\pi$ have been developed so far. In particular, we here refer to Brézis [9], Watanabe [30], Maruo [21], Attouch-Damlamian [2], Kenmochi [17], Yamada [31] and Yotsutani [33] for $(E)_0$, and Bénéilan-Brézis [4], Nagai [23], Yamada [32] for $(E)_\pi$. On the other hand, when $B(t, \cdot)$ is not monotone, the study for $(E)_0$ has been made recently by several authors under some compactness assumptions on $D(\varphi^t) = \{u \in H ; \varphi^t(u) < +\infty\}$ similar to each other. For example, Attouch-Damlamian [3] and Biroli [5] dealt with the case where $\partial\varphi^t \equiv \partial\varphi$ and $B(t, \cdot)$ belongs to a class of time-dependent upper semi-continuous operators. The case that $\partial\varphi^t \equiv \partial\varphi$ and

$B(t, \cdot) = -\partial\varphi$ was studied by Koi-Watanabe [18], Ishii [16] and the author [24, 25]. Our main distinction here is not only to treat periodic problem $(E)_{\pi}$ but also to allow t -dependence of $D(\varphi^t)$ and to treat a much wider class of perturbations $B(t, \cdot)$ aiming at an applications to Navier-Stokes-type equations.

As for $(E)_{\alpha\pi}$, the study in this direction seems to be very few. When the perturbing term $B(t, \cdot)$ is absent, Biroli [6] studied the case $\varphi^t \equiv \varphi$. In relation to this problem, we also refer to Amerio-Prouse [1] and Foias [11], where the almost-periodic problem for the Navier-Stokes equations in cylindrical domains is treated. We shall discuss abstract problems $(E)_0$, $(E)_{\pi}$, $(E)_{\alpha\pi}$ and their applications for the Navier-Stokes equations in bounded regions with moving boundaries in §2, §3 and §4 respectively.

§ 2. Cauchy Problems.

2.1. Subdifferential operators and interpolation classes.

Let H be a real Hilbert space with the inner product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$, which are often denoted by $(\cdot, \cdot)$ and $|\cdot|$ respectively. We denote by $\Phi(H)$ the family of all lower semi-continuous convex functions φ from H into $(-\infty, +\infty]$ with $\varphi \not\equiv +\infty$. For each $\varphi \in \Phi(H)$, the effective domain $D(\varphi)$ of φ is defined by $D(\varphi) = \{u \in H; \varphi(u) < +\infty\}$ and the subdifferential $\partial\varphi$ of φ is defined by

$$\partial\varphi(u) = \{f \in H; \varphi(v) - \varphi(u) \geq (f, v - u)_H \text{ for all } v \in H\}$$

with domain $D(\partial\varphi) = \{u \in H; \partial\varphi(u) \neq \emptyset\}$.

Then, as is well known, $\partial\varphi$ is a maximal monotone in H , and $\overline{D(\varphi)}$, the closure of $D(\varphi)$ in the H -norm, coincides with $\overline{D(\partial\varphi)}$ (see Brézis [9]).

Let A be a maximal monotone operator in H with domain $D(A)$, and put $J_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$. For each $\alpha \in (0,1)$ and $p \in [1, +\infty]$, intermediate classes $\mathcal{B}_{\alpha,p}(A)$ between $\overline{D(A)}$ and $D(A)$ are defined by

$$\mathcal{B}_{\alpha,p}(A) = \{u \in \overline{D(A)} ; t^{-\alpha} |u - J_t|_H \in L_*^p(0,1)\},$$

where $L_*^p(0,1) = \{f ; |f|_{L_*^p} = (\int_0^1 |f(t)|^p \frac{dt}{t})^{1/p} < +\infty\}$, $1 \leq p < +\infty$, and $L_*^\infty(0,1) = L^\infty(0,1)$.

Then it is shown that $\mathcal{B}_{\alpha,p}(A) \subset \mathcal{B}_{\alpha,q}(A)$ for all $\alpha \in (0,1)$ if $1 \leq p < q \leq \infty$, and that $\mathcal{B}_{\alpha,p}(A) \subset \mathcal{B}_{\beta,q}(A)$ for all $1 \leq p, q \leq \infty$ if $0 < \beta < \alpha < 1$ (see D. Brézis [8]).

2.2. Local existence.

First of all we introduce the following three conditions, which will be assumed throughout this paper.

(A. φ^t) For each $t \in \mathbb{R}^1$, $\varphi^t \in \Phi(H)$ and $\varphi^t \geq 0$. Furthermore, there exist constants $K \geq 0$, $\delta > 0$, $\beta \in [0,1]$, and a continuous monotone increasing function $m(\cdot)$ on $[0, +\infty)$ such that for each $t_0 \in \mathbb{R}^1$ and $x_0 \in D(\varphi^{t_0})$, there exists a function $x(t)$ satisfying

$$(2.1) \quad |x(t) - x_0|_H \leq m(|x_0|_H) |t - t_0| (\varphi^{t_0}(x_0) + K)^\beta,$$

$$(2.2) \quad \varphi^t(x(t)) \leq \varphi^{t_0}(x_0) + m(|x_0|_H) |t - t_0| (\varphi^{t_0}(x_0) + K),$$

for all $t \in [t_0 - \delta, t_0 + \delta]$.

(A.1) For each $t \in \mathbb{R}^1$ and $L \in (0, +\infty)$, the set $\{u \in H ; \varphi^t(u) + |u|_H \leq L\}$ is compact in H .

In what follows, we always assume that $B(t, \cdot)$ is single valued, for the sake of simplicity.

(A.2) For each interval $[a,b]$ in R^1 , the following (i) and (ii) are satisfied.

(i) $B(t, \cdot)$ is measurable in the following sense : If $u(t) \in C([a,b]; H)$, $du(t)/dt \in L^2(a,b; H)$ and there exists a function $g(t) \in L^2(a,b; H)$ with $g(t) \in \partial \varphi^t(u(t))$ for a.e. $t \in [a,b]$, then $B(t, u(t))$ is measurable in $t \in [a,b]$.

(ii) $B(t, \cdot)$ is demiclosed in the following sense : If $u_n \rightarrow u$ in $C([a,b]; H)$, $g_n \rightarrow g$ weakly in $L^2(a,b; H)$ with $g_n(t) \in \partial \varphi^t(u_n(t))$, $g(t) \in \partial \varphi^t(u(t))$ for a.e. $t \in [a,b]$, and if $B(t, u_n(t)) \rightarrow b(t)$ weakly in $L^2(a,b; H)$, then $b(t) = B(t, u(t))$ for a.e. $t \in [a,b]$.

Next, we introduce the following three types of boundedness conditions for $B(t, \cdot)$.

(A.3) There exist a function $M(\cdot) \in \mathcal{M}$ and a constant $k \in [0,1)$ such that

$$(2.3) \quad |B(t, u)|_H^2 \leq k |\partial \varphi^t(u)|_H^2 + M(\varphi^t(u) + |u|_H) \quad \text{for all } t \in R^1 \text{ and } u \in D(\partial \varphi^t).$$

(A.4) _{α} For an exponent $\alpha \in (0, 1/2)$, there exists a function $M(\cdot) \in \mathcal{M}$ such that

$$(2.4) \quad |B(t, u)|_H \leq M(|u|_H) \left\{ \varepsilon |\partial \varphi^t(u)|_H + M\left(\frac{1}{\varepsilon}\right) |\varphi^t(u)|^{\frac{1-\alpha}{1-2\alpha}} + 1 \right\} \\ \text{for all } \varepsilon > 0, t \in R^1 \text{ and } u \in D(\partial \varphi^t).$$

(A.5) There exist a constant $\gamma \in (0, 1)$ and a function $M(\cdot) \in \mathcal{M}$ such that

$$(2.5) \quad |B(t, u)|_H \leq M(|u|_H) \left(|\partial \varphi^t(u)|_H^{1-\gamma} + |\varphi^t(u)|^{1-\gamma} + 1 \right) \\ \text{for all } t \in R^1 \text{ and } u \in D(\partial \varphi^t).$$

Here and henceforth, $\mathcal{M}$ denotes the family of all positive

monotone increasing functions on $[0, +\infty)$, and $\partial \varphi^t$ the minimal section of $\partial \varphi^t$, i.e., $\partial \varphi^t(u)$ is the unique element of least norm in $\partial \varphi^t(u)$.

Then our local existence results are stated as follows according as initial data belong to $D(\varphi^0)$, $\mathcal{B}_{\alpha,p}(\partial \varphi^0)$ ($0 < \alpha < 1/2$), and $\overline{D(\varphi^0)}$.

THEOREM I Let (A, φ^t) , (A.1), (A.2) and (A.3) be satisfied. Let $a \in D(\varphi^0)$ and $f(t) \in L^2_{loc}([0, +\infty); H)$. Then there exists a positive number T depending on $|a|_H$ and $\varphi^0(a)$ such that $(E)_0$ has a strong solution $u(t)$ in $(0, T)$ satisfying

$$(2.6) \quad du(t)/dt, \quad g(t), \quad B(t, u(t)) \in L^2(0, T; H),$$

$$(2.7) \quad \varphi^t(u(t)) \text{ is absolutely continuous on } [0, T].$$

THEOREM II Let (A, φ^t) , (A.1), (A.2) and $(A.4)_\alpha$ be satisfied. Let $a \in \mathcal{B}_{\alpha,p}(\partial \varphi^0)$ with $p \in [1, 2]$ and $f(t) \in L^2_{loc}([0, +\infty); H)$. Then there exists a positive number T depending on $|a|_H$ and $|a|_{\alpha,p}^{(*1)}$ such that $(E)_0$ has a strong solution $u(t)$ in $(0, T)$ satisfying

$$(2.8) \quad t^{\frac{1}{2}-\alpha} du(t)/dt, \quad t^{\frac{1}{2}-\alpha} g(t), \quad t^{\frac{1}{2}-\alpha} B(t, u(t)) \in L^2(0, T; H),$$

$$(2.9) \quad t^{-\alpha} |u(t) - a|_H, \quad t^{\frac{1}{2}-\alpha} |\varphi^t(u(t))| \in L^q_*(0, T) \text{ for all } q \in [2, \infty].$$

THEOREM III Let (A, φ^t) , (A.1), (A.2) and (A.5) be satisfied. Let $a \in \overline{D(\varphi^0)}$ and $f(t) \in L^2_{loc}([0, +\infty); H)$. Then there exists a positive number T depending on $|a|_H$ such that $(E)_0$ has a strong solution $u(t)$ in $(0, T)$ satisfying (2.8) with $\alpha = 0$ and

$$(2.10) \quad \varphi^t(u(t)) \in L^1(0, T), \quad t \varphi^t(u(t)) \in L^\infty(0, T).$$

(*1) $|a|_{\alpha,p} = |t^{-\alpha} |a - J_t a|_H|_{L^p_*}, \quad J_t = (I + t \partial \varphi^0)^{-1}.$

sketch of proof. These theorems can be proved in much the same way, so we here give a sketch of the proof for Theorem II.

Let X_S^α be a Banach space with the norm

$$\|u\|_{\alpha,S} = \left(\int_0^S t^{1-2\alpha} |u(t)|_H^2 dt \right)^{1/2}, \quad 0 < \alpha < 1/2.$$

For each $h(t) \in X_S^\alpha$, let us consider the equation:

$$(E)_O^* \begin{cases} (2.11) & du_h(t) + \mathcal{A}^t(u_h(t)) = -h(t) + f(t), \quad 0 < t < S, \\ (2.12) & u_h(0) = a. \end{cases}$$

Then, under assumption $(A.\mathcal{A}^t)$, it is known that $(E)_O^*$ has a unique strong solution $u_h(t)$ in $(0,S)$ (see [31] and [33]).

Therefore, for each $a \in \mathcal{B}_{\alpha,p}(\mathcal{A}^0)$, $S \in (0,+\infty)$ and $f(t) \in L^2(0,S;H)$, we can define an operator $\mathbb{E}_{a,f,S}$ from X_S^α into $C([0,S];H)$ by $\mathbb{E}_{a,f,S}(h) = u_h$. Furthermore we introduce another operator $\mathbb{B}_{a,f,S}$ by $\mathbb{B}_{a,f,S}(h)(t) = B(t, \mathbb{E}_{a,f,S}(h)(t)) = B(t, u_h(t))$. Then, making good use of the nonlinear interpolation theory introduced by D. Brézis [8] and energy estimates for $(E)_O^*$, under assumptions $(A.\mathcal{A}^t)$ and $(A.4)_\alpha$, we find that for an appropriate positive number R and a sufficiently small S , $\mathbb{B}_{a,f,S}$ maps the set $K_{S,R}^\alpha = \{u \in X_S^\alpha; \|u\|_{\alpha,S} \leq R\}$ into itself. In order to obtain energy estimates for $(E)_O^*$, we much rely on the following proposition.

PROPOSITION 2.1. Let $(A.\mathcal{A}^t)$ be satisfied and $u(t)$ be a continuous function on $[a,b]$ such that the set $\mathcal{L} = \{t \in [a,b]; du(t)/dt, d\mathcal{A}^t(u(t))/dt \text{ exist and } u(t) \in D(\mathcal{A}^t)\}$. Then

$$\begin{aligned} & \left| \frac{d}{dt} \mathcal{A}^t(u(t)) - (g, \frac{du}{dt}(t))_H \right| \\ & \leq m(|u(t)|_H) |g|_H (\mathcal{A}^t(u(t)) + K)^\beta + m(|u(t)|_H) (\mathcal{A}^t(u(t)) + K) \end{aligned}$$

holds for all $t \in \mathcal{L}$ and $g \in \mathcal{A}^t(u(t))$.

Furthermore, by using energy estimates and a compactness argument (Ascoli's theorem), we deduce the following continuity of $\mathbb{E}_{a,f,S}$ and $\mathbb{B}_{a,f,S}$.

LEMMA 2.2. Let $(A.4^t), (A.1), (A.2)$ and $(A.4)$ be satisfied. If $h^n \rightarrow h$ weakly in X_S^α as $n \rightarrow +\infty$, then $\mathbb{E}_{a,f,S}(h^n) \rightarrow \mathbb{E}_{a,f,S}(h)$ in $C([0,S];H)$ and $\mathbb{B}_{a,f,S}(h^n) \rightarrow \mathbb{B}_{a,f,S}(h)$ weakly in X_S^α as $n \rightarrow +\infty$.

Thus, for a sufficiently small S , $\mathbb{B}_{a,f,S}$ is a weakly continuous mapping from the weakly compact convex set $K_{S,R}^\alpha$ into itself. Then, by Schauder's fixed-point theorem, there exists an element $b = \mathbb{B}_{a,f,S}(b)$, i.e., $\mathbb{E}_{a,f,S}(b) = u$ satisfies

$$\begin{cases} du(t)/dt + \mathcal{A}^t(u(t)) + b(t) = f(t) & \text{for a.e. } t \in (0,S), \\ b(t) = B(t, u(t)) & \text{for a.e. } t \in (0,S), \\ u(0) = a. \end{cases}$$

That is to say, $u(t)$ is the desired local strong solution of $(E)_0$ in $(0,S)$.

As for the cases $a \in D(\mathcal{A}^0)$ and $a \in \overline{D(\mathcal{A}^0)}$, we can apply the same idea as above by replacing X_S^α by $L^2(0,S;H)$ and $X_S^0 = \{ u ; (\int_0^S |u(t)|_H^2 dt)^{1/2} + (\int_0^S |u(t)|_H^{(2+\gamma)/2} dt)^{1/2} < +\infty \}$ (γ is the exponent appearing in (A.5)) respectively.

REMARK 2.3. When $B(t, \cdot)$ is a multi-valued operator, as a matter of course, $\mathbb{B}$ becomes a multi-valued mapping. In this case, however, instead of Schauder's theorem, we can rely on Fan's fixed-point theorem for upper semi-continuous multi-valued mappings (see [3], [5] and [10]).

2.3. Global existence.

Firstly we give a sufficient condition which guarantees that every local strong solutions can be continued globally to $(0, +\infty)$.

THEOREM IV Let $(A. \varphi^t)$, (A.1), (A.2) and the following (A.6) be satisfied.

(A.6) There exist constants $\alpha > 0$, $C \geq 0$, $k \in [0, 1)$ and a function $M(\cdot) \in \mathcal{M}$ such that

$$(2.11) \quad (-g - B(t, u), u)_H + \alpha \varphi^t(u) \leq C(|u|_H^2 + 1) \\ \text{for all } t \in (0, +\infty), u \in D(\partial \varphi^t) \text{ and } g \in \partial \varphi^t(u),$$

$$(2.12) \quad |B(t, u)|_H^2 \leq k |\partial \varphi^t(u)|_H^2 + M(|u|_H) (\varphi^t(u) + 1)^2 \\ \text{for all } t \in (0, +\infty) \text{ and } u \in D(\partial \varphi^t).$$

Let $f(t) \in L_{loc}^2([0, +\infty); H)$ with $\|f\|_{2, \infty} = \sup_{t > 0} \int_t^{t+1} |f(s)|_H^2 ds < +\infty$.

Then every local strong solution of $(E)_0$ can be continued globally to $(0, +\infty)$ as a strong solution of $(E)_0$.

Proof. Let $u(t)$ be a strong solution of $(E)_0$ in $(0, S)$.

Then it is easy to see that (2.11) gives a priori bounds for

$$\max_{0 \leq t \leq S} |u(t)|_H + \int_0^S \varphi^t(u(t)) dt. \quad \text{Hence, by virtue of Proposition}$$

2.1 and Gronwall's inequality, multiplying (1.1) by $g(t) = -du(t)/dt - B(t, u(t)) + f(t) \in \partial \varphi^t(u(t))$, we can obtain a priori bounds for $\varphi^t(u(t))$. Then the assertion of the theorem follows from Theorem I.

When condition (A.6) is absent, it is known that there are some cases where if a and $f(t)$ satisfy certain conditions, then the corresponding local strong solution $u(t)$ of $(E)_0$ blows up in a finite time T_m , i.e., $|u(t)|_H \rightarrow +\infty$, $\varphi^t(u(t)) \rightarrow +\infty$, etc. as $t \rightarrow T_m$ (see, e.g., Fujita [12], Tsutsumi [29],

Ishii [16] and the author [25]). In such cases, however, it is quite often possible to continue local solutions globally if their data a and $f(t)$ are sufficiently small. This is also the case with our situation. To illustrate this, we introduce the following condition.

(A.7) The following (i) and (ii) are satisfied.

(i) $\varphi^t(0) = 0$ for all $t \in \mathbb{R}^1$,

(ii) There exist positive constants k, α, C_1, p and functions $\ell(\cdot) \in \mathcal{M}$, $\ell_1(\cdot), \ell_2(\cdot) \in \mathcal{M}_0 = \{\ell(\cdot) \in \mathcal{M}; \ell(r) \rightarrow 0 \text{ as } r \rightarrow 0\}$ such that

$$(2.13) \quad |B(t, u)|_H^2 \leq \{k + \ell_1(\varphi^t(u))\} |\partial \varphi^t(u)|_H^2 + \ell(\varphi^t(u)), \quad 0 \leq k < 1, \\ \text{for all } t \in \mathbb{R}^1 \text{ and } u \in D(\partial \varphi^t),$$

$$(2.14) \quad (-g - B(t, u), u)_H + \alpha \varphi^t(u) \leq \ell_2(\varphi^t(u)) \cdot \varphi^t(u) \\ \text{for all } t \in \mathbb{R}^1 \text{ and } u \in D(\partial \varphi^t),$$

$$(2.15) \quad C_1 |u|_H^p \leq \varphi^t(u), \quad 1 < p < +\infty, \text{ for all } u \in D(\partial \varphi^t).$$

Then we have the following stability result.

LEMMA 2.4. Let $(A. \varphi^t)$, (A.1), (A.2) and (A.7) be satisfied.

Then there exist positive number N and r_0 such that for every $r \in (0, r_0)$, if $|a|_H + \varphi^0(a) \leq r^{1/(p-1)}$ and $\|f\|_{2, \infty} \leq r$, then every strong solution $u(t)$ of $(E)_0$ in $(0, T)$ enjoys the a priori estimate $\max \{|u(t)|_H + \varphi^t(u(t)); 0 \leq t \leq T\} \leq N r^{1/(p-1)}$ independent of T .

From this lemma, the following global extension result is derived.

THEOREM V Let (A.7) and all assumptions in Theorem I (resp. II or III) be satisfied. Then there exists a (sufficiently small) positive number r such that if $|a|_H + \varphi^0(a) \leq r$ (resp. $|a|_H + \|a\|_{\alpha,p} \leq r$ or $|a|_H \leq r$) and $\|f\|_{2,\infty} \leq r^{p-1}$, then $(E)_0$ has a global strong solution in $(0, +\infty)$, i.e., the assertion of Theorem I (resp. II or III) holds true with $T = +\infty$.

2.4. Application.

Let $Q = \bigcup_{t \in \mathbb{R}} Q(t) \times \{t\}$ be a non-cylindrical domain in $\mathbb{R}_x^n \times \mathbb{R}_t^1$ which is smooth in (x, t) in the following sense.

(A.Q) For each $t \in \mathbb{R}^1$, $Q(t)$ is a bounded domain in $\mathbb{R}_x^n$ of C^3 -class, and there exists a C^3 -diffeomorphism $\Psi : Q_0 := \overline{Q(0)} \times \mathbb{R}^1 \rightarrow \overline{Q}$ with $\Psi(x, t) = (F(x, t), t)$ (level preserving) satisfying

- (i) $F(x, 0) = x$ for all $x \in Q(0)$,
- (ii) $\sup \{D^m F(x, t) ; m = 0, 1, 2, 3, (x, t) \in Q_0, D = \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\} < +\infty$,
- (iii) $\inf \{ \det(\frac{\partial F^i}{\partial x_j}(x, t)) ; (x, t) \in Q_0 \} > 0$.

Let us now consider the following initial-boundary value problem for the Navier-Stokes equation in $Q_+ = \bigcup_{t>0} Q(t) \times \{t\}$:

$$(\text{Pr.NS})_0 \left\{ \begin{array}{ll} (2.16) & \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u = f - \nabla p_* \quad \text{in } Q_+, \\ (2.17) & \operatorname{div} u = 0 \quad \text{in } Q_+, \\ (2.18) & u = 0 \quad \text{on } \Gamma_+ = \bigcup_{t>0} \partial Q(t) \times \{t\}, \\ (2.19) & u = a(x) \quad \text{in } Q(0), \end{array} \right.$$

where the unknown $u(x, t)$ and given $f(x, t)$, $a(x)$ are real n -dimensional vector functions, while the unknown $p_*(x, t)$ is a real scalar function.

This kind of problem has been investigated by several authors : Fujita-Sauer [14], Bock [7], Inoue-Wakimoto [15] and Ôtani-Yamada [27]. These contributions differ in methods and results. Our advantage here, as well as in [27], is that regularity of solution with respect to time t near boundary can be given explicitly. To formulate our results, we shall use the notations:

$$C_{\sigma}^{\infty}(\Omega) = \{u = (u^1, u^2, \dots, u^n); u^i \in C_{\sigma}^{\infty}(\Omega), i=1, 2, \dots, n, \operatorname{div} u = 0\},$$

$$\mathbb{H}(\Omega) = (L^2(\Omega))^n = \{u = (u^1, u^2, \dots, u^n); u^i \in L^2(\Omega), i=1, 2, \dots, n\},$$

$$\mathbb{H}_{\sigma}(\Omega) = \text{the completion of } C_{\sigma}^{\infty}(\Omega) \text{ in the } \mathbb{H}(\Omega)\text{-norm,}$$

$$P_{\Omega} = \text{the orthogonal projection from } \mathbb{H}(\Omega) \text{ onto } \mathbb{H}_{\sigma}(\Omega),$$

$$\mathbb{H}_{\sigma}^1(\Omega) = (H_{\sigma}^1(\Omega))^n, \quad \mathbb{H}^2(\Omega) = (H^2(\Omega))^n, \quad \mathbb{H}_{\sigma}^1(\Omega) = \mathbb{H}_{\sigma}^1(\Omega) \cap \mathbb{H}_{\sigma}(\Omega),$$

$$\begin{aligned} A_{\Omega} &= \text{the Stokes operator} - P_{\Omega} \Delta \text{ with domain } D(A_{\Omega}) \\ &= \mathbb{H}^2(\Omega) \cap \mathbb{H}_{\sigma}^1(\Omega), \end{aligned}$$

$$A_{\Omega}^{\alpha} = \text{the fractional power of } A_{\Omega} \text{ of order } \alpha > 0.$$

Results. (1) The case $n = 2$: Let $a \in D(A_{Q(0)}^{\alpha})$ with $\alpha > 0$

$$\text{and } \|f\|_{2,\infty} = \sup_{t>0} \int_t^{t+1} |f(s)|_{\mathbb{H}(Q(s))}^2 ds < +\infty. \quad \text{Then } (\text{Pr.NS})_{\Omega} \quad (*)_2$$

has a (unique) global strong solution $u(x, t)$.

(2) The case $n = 3$: Let $a \in D(A_{Q(0)}^{\alpha})$ with $\alpha \geq 1/4$ and

$\|f\|_{2,\infty} < +\infty$. Then $(\text{Pr.NS})_{\Omega}$ has a (unique) local strong solution. Moreover, if $|a|_{A_{Q(0)}^{\alpha}}$ and $\|f\|_{2,\infty}$ are sufficiently

small, then the solution can be continued globally. (This result is a natural extension of that of Fujita-Kato [13] for the non-cylindrical case.)

$$(*)_2 \quad u(\cdot, t) \in D(A_{Q(t)}) \text{ for a.e. } t \in \mathbb{R}^1; \quad \partial u / \partial t, \Delta u \in L_{\text{loc}}^2((0, +\infty);$$

$\mathbb{H}(Q(t)))$; and the zero extension $\hat{u}$ of u to $\mathbb{R}_x^n$ satisfies

$$\hat{u} \in C((0, +\infty); \mathbb{H}_{\sigma}^1(\mathbb{R}^n)) \cap C([0, +\infty); \mathbb{H}_{\sigma}(\mathbb{R}^n)), \quad \partial \hat{u} / \partial t \in L_{\text{loc}}^2((0, +\infty); \mathbb{H}_{\sigma}(\mathbb{R}^n)).$$

Sketch of the proof. Let Ω be a bounded auxiliary open ball such that $\bar{Q} \subset \Omega \times \mathbb{R}^1$. Let $H = \mathbb{H}_0(\Omega)$ and put

$$\varphi^t(u) = \begin{cases} \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \left| \frac{\partial u^i}{\partial x_j} \right|^2 dx & \text{if } u \in \mathbb{H}_0^1(\Omega) \text{ and } u = 0 \text{ a.e.} \\ & x \in \Omega \setminus Q(t), \\ + \infty & \text{otherwise,} \end{cases}$$

$$B(t, u) = P_{\Omega} (u \cdot \nabla) u \quad \text{with domain } D(B(t, \cdot)) = D(\partial \varphi^t).$$

Then $(\text{Pr.NS})_0$ can be reduced to the following abstract Navier-Stokes problem in $\mathbb{H}_0(\Omega)$:

$$(\text{ANS})_0 \quad \begin{cases} d\hat{u}(t)/dt + \partial \varphi^t(\hat{u}(t)) + B(t, \hat{u}(t)) \ni P_{\Omega} \hat{f}(t), \\ \hat{u}(0) = \hat{a}, \end{cases}$$

where $\hat{f}(\cdot, t)$ and $\hat{a}(\cdot)$ are zero extensions of $f(\cdot, t)$ and $a(\cdot)$ to Ω .

Then (A.1) and (A.2) are easily verified and $(A.\varphi^t)$ with $K = 0$ and $\beta = 1/2$ ^{is assured} by (A.Q). Since $(B(t, u), u) = 0$,

$(\partial \varphi^t(u), u) = 2 \varphi^t(u)$ for all $u \in D(\partial \varphi^t)$; and since

$$(2.20) \quad |B(t, u)|_H \leq \text{Const.} |u|_H^{1/2} |\varphi^t(u)|^{1/2} |\partial \varphi^t(u)|_H^{1/2} \\ \text{for all } u \in D(\partial \varphi^t), \text{ if } n=2,$$

$$(2.21) \quad |B(t, u)|_H \leq \text{Const.} |\varphi^t(u)|^{3/4} |\partial \varphi^t(u)|_H^{1/2} \\ \text{for all } u \in D(\partial \varphi^t), \text{ if } n=3,$$

(see, e.g., Ladyzhenskaya [19] and Temam [28]),

for the case $n=2$ (resp. $n=3$), we can apply Theorem I with $\alpha > 0$ (resp. $\alpha \geq 1/4$) and Theorem IV (resp. V) for

$(\text{ANS})_0$. Then the desired solution u is given by $u = \hat{u}|_Q$.

REMARK 2.5. As for the case $n=4$, it can be also proved that

if $|a|_{A_{Q(0)}^{1/2}}$ and $\|f\|_{2,\infty}$ are sufficiently small, then $(\text{Pr.NS})_0$

has a (unique) strong ^{global} solution.

§ 3. Periodic Problems.

The same fixed-point method as for $(E)_0$ works well again for this case. Actually, in parallel with Theorems IV and V, we can obtain the following Theorems VI and VII respectively.

THEOREM VI Let $(A.1), (A.2), (A.6)$ and the following $(A.\varphi^t)_\pi$ be satisfied.

$(A.\varphi^t)_\pi$ All conditions in $(A.\varphi^t)$ and the following (i) and (ii) are satisfied.

$$(i) \quad \varphi^0(\cdot) = \varphi^T(\cdot),$$

(ii) There exist positive numbers C_1 and p such that $(1 < p < +\infty)$

$$(3.1) \quad C_1 |u|_H^p \leq \varphi^t(u) \quad \text{for all } t \in [0, T] \text{ and } u \in D(\partial \varphi^t).$$

Then, for every $f \in L^2(0, T; H)$, $(E)_\pi$ has a strong periodic solution $u(t)$ satisfying (2.6) and (2.7).

THEOREM VII Let $(A.1), (A.2)$ and $(A.\varphi^t)_\pi$ with $K=0$, $\beta \in [1/2, 1]$ and $p \in (1, 2]$, and the following (A.8) be satisfied.

(A.8) There exist a function $M(\cdot) \in \mathcal{M}$ and nonnegative numbers α_1, α_2 with $0 \leq \alpha_2 \leq 1$, $2\alpha_1 + \alpha_2 > 1$ such that

$$(3.2) \quad |B(t, u)|_H \leq M(|u|_H) |\varphi^t(u)|^{\alpha_1} |\partial \varphi^t(u)|_H^{\alpha_2} \quad \text{for all } t \in [0, T] \text{ and } u \in D(\partial \varphi^t).$$

Then there exists a (sufficiently small) positive number r such that if $\sup_{1 < t < T} \int_{t-1}^t |f(s)|_H^2 ds \leq r$, then $(E)_\pi$ has a periodic strong solution $u(t)$ satisfying (2.6) and (2.7).

Application. Let us here consider the periodic problem $(Pr.NS)_\pi$ for the Navier-Stokes equation in $Q_T = \bigcup_{0 < t < T} Q(t) \times \{t\}$ with $Q(0) = Q(T)$, i.e., the problem (2.16)-(2.18) with the periodic condition $u(\cdot, 0) = u(\cdot, T)$. (This problem is already studied

in Morimoto [22] (in a class of weak solutions) and in [27].)

Results. (1) The case $n=2$: For every $f(t) \in L^2(0,T; H(Q(t)))$,

$(Pr.NS)_\pi$ has a periodic strong solution (by Theorem VI).

(2) The case $n=3$ or 4 : If $\sup_{1 < t < T} \int_{t-1}^t |f(s)|^2_{H(Q(s))} ds$ is sufficiently

small, then $(Pr.NS)_\pi$ has a (unique) periodic strong solution (by Theorem VII).

Indeed, as for the case $n=4$, we have

$$|B(t,u)|_H \leq \text{Const.} |\varphi^t(u)|^{1/2} |\partial \varphi^t(u)|_H \quad \text{for all } t \in [0,T] \text{ and } u \in D(\partial \varphi^t),$$

which assures (A.8).

§ 4. Almost-Periodic Problems.

Motivation : Let us here reconsider $(Pr.NS)_\pi$. For example,

suppose that $\partial Q(t)$, the boundary of $Q(t)$, is composed of two connected hypersurfaces $\Gamma_1(t)$ and $\Gamma_2(t)$ for each t .

When $\partial Q(t)$ moves as t goes on, it would be natural to suppose that the movements of $\Gamma_i(t)$ are independent. Therefore, when the periodic movements of $\Gamma_i(t)$ are discussed, it is rather reasonable to treat the case where the periods ω_i of the movements of $\Gamma_i(t)$ are different. So, if ω_1/ω_2 is not a rational number, then the movement of $\partial Q(t)$ is no longer periodic, but almost-periodic (more precisely quasi-periodic).

From this point of view, the almost-periodic problem $(E)_{\alpha\pi}$ is regarded as much more important than $(E)_\pi$.

DEFINITION 4.1. (Bohr) A function $v(t) \in C(R^1; H)$ is said to be H-a.p. (H-almost-periodic) if for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$ in R^1 depending on ε such that

$$\sup_{t \in R^1} |v(t+\tau) - v(t)|_H \leq \varepsilon \quad \text{for all } \tau \in \{\tau\}_\varepsilon$$

Here $\{\tau\}_\varepsilon$ is said to be relatively dense if there exists a positive number ℓ_ε (inclusion length) such that for every $r \in R^1$, the corresponding interval $[r, r + \ell_\varepsilon]$ always contains at least one point of $\{\tau\}_\varepsilon$.

Moreover, a function $w(t) \in L^2_{loc}(R^1; H)$ is said to be $S^2(H)$ -a.p. if $\tilde{w}(t) = \{w(t+\eta); \eta \in [0, 1]\}$ is $L^2(0, 1; H)$ -a.p.

It is well known as Bochner's criterion that the almost-periodicity can be characterized as follows :

THEOREM 4.2. Let $v(t) \in C(R^1; H)$. Then $v(t)$ is H-a.p. if and only if for every sequence $\{\ell_n\}$, there exists a subsequence $\{s_n\}$ such that the sequence $\{g(t+s_n)\}$ converges in H uniformly with respect to $t \in R^1$.

Let us here assume that $D(\varphi^t)$ varies almost-periodically in the following sense.

(A. φ^t) _{$\alpha\pi$} For each $t \in R^1$, $\varphi^t \in \Phi(H)$ and $\varphi^t \geq 0$. Furthermore there exist R^1 -almost-periodic functions $h_1(\cdot), h_2(\cdot) \in W^{1,\infty}(R^1)$ and a continuous function $m(\cdot) \in \mathcal{M}$ such that for every $t_0 \in R^1$ $x_0 \in D(\varphi^{t_0})$, there exists a function $x(t)$ on R^1 such that

$$(4.1) \quad |x(t) - x_0|_H \leq m(|x_0|_H) |h_1(t) - h_1(t_0)| (\varphi^{t_0}(x_0) + 1),$$

$$(4.2) \quad \varphi^t(x(t)) \leq \varphi^{t_0}(x_0) + m(|x_0|_H) |h_2(t) - h_2(t_0)| (\varphi^{t_0}(x_0) + 1),$$

for all $t \in R^1$.

In addition, we assume .

(A.9) The following (i)-(iii) are satisfied.

$$(i) \quad \varphi^t(0) = 0 \quad \text{for all } t \in \mathbb{R}^1,$$

$$(ii) \quad C_1 |u|_H^p \leq \varphi^t(u), \quad C_1 > 0, 1 < p < +\infty, \text{ for all } u \in D(\varphi^t),$$

$$(iii) \quad (g_1 - g_2, u_1 - u_2)_H \geq C |u_1 - u_2|_H^2, \quad C > 0, \text{ for all } t \in \mathbb{R}^1 \\ u_i \in D(\varphi^t) \text{ and } g_i \in \varphi^t(u_i)$$

Then, concerning the unperturbed problem $(E)_{\alpha\pi}$ with $B(t, \cdot) \equiv 0$, we have :

THEOREM VIII Let $(A.\varphi^t)_{\alpha\pi}$, (A.1) and (A.9) be satisfied.

Let $f(t)$ be $(*)_2^3 S^2(H)$ -a.p. Then $(E)_{\alpha\pi}$ with $B(t, \cdot) \equiv 0$ has a unique H -almost-periodic strong solution.

Since the unperturbed problem is solved as above, in order to solve $(E)_{\alpha\pi}$, we intend to apply the same fixed-point method as in § 2. Unfortunately, however, in this procedure some difficulties arise. For example, it is difficult to know if $B(h)(t)$ is almost-periodic (in some sense) when $h(t)$ is almost-periodic, and how to take a (weakly) compact set such as $K_{S,R}^\alpha$ where B works. Therefore we here apply another method similar to that in Biroli [6] : Firstly, the existence and (local) uniqueness of bounded solutions are shown. Next, the unique bounded solution is proved to be almost-periodic by using Bochner's criterion. Nevertheless this method requires so restrictive conditions on φ^t and $B(t, \cdot)$ that we give up to present our results in abstract forms. So we here only illustrate this method for the Navier-Stokes problem (Pr.NS) in regions with almost-periodically moving boundaries.

(*) This can be replaced by $S^2(H)$ -a.p. in a weak topology.

$$(\text{Pr.NS}) \begin{cases} (4.3) & \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u = f - \nabla p_* & \text{in } Q, \\ (4.4) & \operatorname{div} u = 0 & \text{in } Q, \\ (4.5) & u = 0 & \text{on } \Gamma = \bigcup_{t \in \mathbb{R}^1} Q(t) \times \{t\}, \end{cases}$$

where $Q(t)$ moves almost-periodically in the following sense.

(A.Q) $_{\alpha\pi}$ All conditions (i)-(iii) of (A.Q) and the following (iv) be satisfied.

(iv) $D^m F(x, t)$ ($m = 0, 1, 2, 3$) are almost-periodic in t uniformly with respect to $x \in \overline{Q(0)}$, i.e., for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$ such that

$$|D^m F(x, t+\tau) - D^m F(x, t)| < \varepsilon \quad \text{for all } \tau \in \{\tau\}_\varepsilon \text{ and all } (x, t) \in \overline{Q(0)} \times \mathbb{R}^1.$$

Then our result is stated as follows.

THEOREM 4.3. Let $n = 2, 3$ or 4 and (A.Q) $_{\alpha\pi}$ be satisfied.

Then there exists a (sufficiently small) positive number r

such that if $\|f\|_{2,\infty} = \sup_{t \in \mathbb{R}^1} \int_{t-1}^t |f(s)|_{\mathbb{H}(Q(s))}^2 ds \leq r$ and $\hat{f}(t)$

is $S^2(\mathbb{H}(\Omega))$ -a.p., then (Pr.NS) has a (unique) strong solution $u(t)$ such that the zero extension $\hat{u}(t)$ of $u(t)$ is $\mathbb{H}_0(\Omega)$ -a.p.

Sketch of proof. If $\|f\|_{2,\infty} = r$ is sufficiently small, then

Theorem V assures that there exist strong solutions $\hat{u}_n(t)$ in $(-n, +\infty)$ of the abstract Navier-Stokes problems in $H = \mathbb{H}_0(\Omega)$:

$$\begin{cases} d\hat{u}_n(t)/dt + \mathcal{A}^t(\hat{u}_n(t)) + B(t, \hat{u}_n(t)) \ni P_\Omega \hat{f}(t) & t \in (-n, +\infty), \\ \hat{u}_n(-n) = 0. \end{cases}$$

Then, by Lemma 2.4, as a limit of $\hat{u}_n(t)$ we can construct a bounded strong solution $\hat{u}(t)$ in $\mathbb{R}^1$ of the abstract Navier-Stokes problem such that

$$(4.6) \quad \sup_{t \in \mathbb{R}^1} \int_{t-1}^t \left| \frac{d\hat{u}}{ds}(s) \right|_H^2 ds < +\infty,$$

$$(4.7) \quad \sup_{t \in \mathbb{R}^1} (|\hat{u}(t)|_H + \varphi^t(\hat{u}(t))) \leq N r.$$

Now we are going to show that this bounded solution $\hat{u}(t)$ is H-a.p. Suppose that $\hat{u}(t)$ is not H-a.p., then by Bochner's criterion and the almost-periodicity of $\hat{f}(t)$ and $F(\cdot, t)$, there exist sequences $\{\ell_j\}$, $\{t_j\}$ and subsequences $\{\ell_{ij}\}$ of $\{\ell_j\}$ ($i=1,2$) such that

$$(4.8) \quad |\hat{u}(t_j + \ell_{1j}) - \hat{u}(t_j + \ell_{2j})|_H \geq \rho > 0 \quad \text{for all } j,$$

$$(4.9) \quad P_\Omega \hat{f}(t + \tau_{ij}) \rightarrow \hat{f}_\ell(t) \quad \text{in } L_{loc}^2(\mathbb{R}^1; H) \quad \text{uniformly in } t \in \mathbb{R}^1 \\ \text{as } j \rightarrow +\infty,$$

$$(4.10) \quad D^m F^k(x, t + \tau_{ij}) \rightarrow D^m F_\ell^k(x, t) \quad \text{uniformly in } (x, t) \in Q(0) \times \mathbb{R}^1 \\ \text{as } j \rightarrow +\infty, \\ \text{for } m=0,1,2,3 \quad k=1,2,\dots,n.$$

where we put $\tau_{ij} = t_j + \ell_{ij}$.

Put $Q_\ell(t) = \bigcup_{x \in Q(0)} F_\ell(x, t)$ and $u_{ij}(t) = \hat{u}(t + \tau_{ij})$. Then

$Q_\ell(t)$ forms another smooth non-cylindrical domain. Moreover, from (4.6), (4.7) and (A.1), there exist subsequences $\{u_{ij}(t)\}$ of $\{u_{ij}(t)\}$ such that $u_{ij}(t)$ converge to $u_i(t)$ which satisfy

$$(4.11) \quad \left(\frac{du_i}{dt}(t), \phi(t) \right) + (\nabla u_i(t), \nabla \phi(t)) - ((u_i(t) \cdot \nabla) \phi(t), u_i(t)) \\ = (\hat{f}_\ell(t), \phi(t)) \quad \text{for a.e. } t \in \mathbb{R}^1 \text{ and all } \phi(t) \in \mathcal{H}_\sigma^1(Q_\ell(t)).$$

Then, putting $\phi(t) = w(t) = u_1(t) - u_2(t)$ in (4.11), we have

$$(4.12) \quad \frac{1}{2} \frac{d}{dt} |w(t)|^2 + |\nabla w(t)|^2 \leq -((w(t) \cdot \nabla) u_1(t), w(t)) \\ \leq \text{Const. } |\nabla w(t)|^2 |\nabla u_1(t)|.$$

That is to say, by (4.7), for a sufficiently small r ,

$$\frac{d}{dt} |w(t)|^2 + |\nabla w(t)|^2 \leq 0.$$

Hence $|w(t)|$ is monotone decreasing and

$$\int_{t_1}^{t_2} |\nabla w(t)|^2 dt \leq |w(t_1)|^2 - |w(t_2)|^2 \quad \text{for all } t_1 \text{ and } t_2 .$$

Since $|w(t_i)|$ are bounded, letting $t_1 \rightarrow -\infty$, we find that

$|\nabla w(t)| \rightarrow 0$, i.e., $|w(t)| \rightarrow 0$ as $t \rightarrow -\infty$. Thus we have

$|w(0)| \leq \lim_{t \rightarrow -\infty} |w(t)| = 0$, which contradicts (4.8).

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