

THE EASTON COLLAPSE AND A SATURATED FILTER

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ABSTRACT. Suppose that there is a huge cardinal. We prove that a two-stage iteration of Easton collapses produces a saturated filter on the successor of a regular cardinal.

1. INTRODUCTION

In the pioneering work [10] Kunen established:

Theorem (Kunen). *Suppose that κ is huge with target λ . Then in some forcing extension $\kappa = \omega_1$, $\lambda = \omega_2$ and ω_1 carries an ω_2 -saturated filter.*

Kunen's forcing has the form $P * \dot{S}(\kappa, \lambda)$, where P forces that $\kappa = \omega_1$ and $\dot{S}(\kappa, \lambda)$ is the Silver collapse introduced in [15]. The poset P is constructed by recursion so that $P * \dot{S}(\kappa, \lambda)$ can be completely embedded into $j(P)$, where $j : V \rightarrow M$ is the original huge embedding. Kunen's construction has since been modified to get models containing filters that are strongly saturated in various senses. We refer the reader to [5] for a comprehensive survey of the development.

In [7] Foreman, Magidor and Shelah proved the following striking result: If λ is supercompact, then the Levy collapse $C(\omega_1, \lambda)$ forces that ($\lambda = \omega_2$ and) ω_1 carries a saturated filter. The hypothesis was later reduced by Todorćević (see [2]) to λ being Woodin, which follows from Kunen's hypothesis as well. In contrast Foreman and Magidor [6] showed that $C(\omega_2, \lambda)$ forces the nonexistence of a saturated filter on ω_2 under PFA.

Let us assume again κ is huge with target λ . Todorćević's result implies that a saturated filter on ω_1 can be forced to exist by the iteration $C(\omega, \kappa) * \dot{C}(\kappa, \lambda)$ as well. What about ω_2 ? Namely we ask:

Question. *Does $C(\omega_1, \kappa) * \dot{C}(\kappa, \lambda)$ force that ω_2 carries an ω_3 -saturated filter?*

One motivation for the question comes from the following unpublished result of Woodin: $C(\omega_1, \kappa) * \dot{C}(\kappa, \lambda)$ forces that an ω_2 -dense filter on ω_2 exists in some inner model. (See [5] for an exposition in the case of ω_1 .) Moreover if the answer is positive, then we would get saturated filters on many cardinals by simply iterating Levy collapses. This would in turn help to simplify Foreman's construction [3, 4] of a model in which every regular uncountable cardinal carries a saturated filter.

In this paper we define a poset $E(\mu, \kappa)$ for a pair of regular cardinals $\mu < \kappa$, and call it the Easton collapse. It is the product of standard collapsing posets with Easton support, and forces $\kappa = \mu^+$ if κ is Mahlo. In place of the original question, we answer the corresponding question for the iteration of Easton collapses:

1991 *Mathematics Subject Classification.* 03E05, 03E35, 03E55.

Partially supported by JSPS Grant-in-Aid for Scientific Research No. 19540112.

Theorem. Suppose that κ is huge with target λ . Let $\mu < \kappa$ be regular. Then $E(\mu, \kappa) * \dot{E}(\kappa, \lambda)$ forces that κ carries a λ -saturated filter.

In §4 we prove our theorem in somewhat refined form.

2. PRELIMINARIES

We refer the reader to [9] for background material.

Throughout the paper we use μ, κ and λ to denote a regular cardinal. Unless otherwise stated it is understood that $\mu < \kappa < \lambda$.

Let P and Q be posets. We say that a map $\pi : P \rightarrow Q$ is a projection if the following hold:

- (1) π is order-preserving, i.e. $p' \leq_P p \rightarrow \pi(p') \leq_Q \pi(p)$,
- (2) $\pi(1_P) = 1_Q$ and
- (3) $q \leq_Q \pi(p) \rightarrow \exists p^* \leq_P p (\pi(p^*) \leq_Q q)$.

Suppose that $\pi : P \rightarrow Q$ is a projection. Then $\text{ran } \pi$ is dense in Q . It is straightforward to check that the map $q \mapsto \sum \{p \in P : \pi(p) \leq q\}$ is a complete embedding of Q into $B(P)$, the completion of P . It is also easy to see that if D is dense open in Q , $\pi^{-1}(D)$ is dense in P . So if $G \subset P$ is generic, $\pi''G$ generates a generic filter over Q . Let $H \subset Q$ be V -generic. In $V[H]$ let P/H be the set $\pi^{-1}(H)$ ordered by $\leq_P$. It is straightforward to check that the map $p \mapsto (\pi(p), \dot{p})$, where $\dot{p}$ is a Q -name with $\pi(p) \Vdash_Q \dot{p} = p$, is a dense embedding of P into $Q * (P/H)$. Finally note that the composition of two projections is a projection.

We say that a cardinal γ is strongly regular if $\gamma^{<\gamma} = \gamma$. A set d of strongly regular cardinals is called Easton if $\sup(d \cap \gamma) < \gamma$ for all regular γ .

Suppose that X be a set of ordinals and P_γ is a poset for $\gamma \in X$. Define

$$\prod_{\gamma \in X}^E P_\gamma = \{p : \text{dom } p \subset X \text{ is Easton} \wedge \forall \gamma \in \text{dom } p (p(\gamma) \in P_\gamma)\}.$$

$\prod_{\gamma \in X}^E P_\gamma$ is ordered coordinatewise: $p' \leq p$ iff $\text{dom } p' \supset \text{dom } p$ and $p'(\gamma) \leq_\gamma p(\gamma)$ for all $\gamma \in \text{dom } p$.

Let $Y \subset X$. Then $\prod_{\gamma \in X}^E P_\gamma$ is canonically isomorphic to $\prod_{\gamma \in Y}^E P_\gamma \times \prod_{\gamma \in X-Y}^E P_\gamma$. Suppose in addition $\pi_\gamma : P_\gamma \rightarrow Q_\gamma$ is a projection for $\gamma \in Y$. Then it is easy to see that the map $p \mapsto \langle \pi_\gamma(p(\gamma)) : \gamma \in \text{dom } p \cap Y \rangle$ is a projection from $\prod_{\gamma \in X}^E P_\gamma$ to $\prod_{\gamma \in Y}^E Q_\gamma$.

We say that P has (κ, κ, μ) -cc if for every $X \in [P]^\kappa$ there is $Y \in [X]^\kappa$ such that every $Z \in [Y]^\mu$ has a common extension. Needless to say, (κ, κ, μ) -cc implies κ -cc. If Q is separative and can be completely embedded into P , then the (κ, κ, μ) -cc of P implies that of Q .

Lemma 1. Suppose that κ is Mahlo and P_γ is a poset of size $< \kappa$ for $\gamma < \kappa$. Then

$$\prod_{\mu \leq \gamma < \kappa}^E P_\gamma \text{ has } (\kappa, \kappa, \mu)\text{-cc.}$$

Proof. Let $\{p_\xi : \xi < \kappa\} \subset \prod_{\mu \leq \gamma < \kappa}^E P_\gamma$. It suffices to find $X \in [\kappa]^\kappa$ and $\delta < \kappa$ such that $\text{dom } p_\xi - \delta$ is mutually disjoint and $p_\xi|_\delta$ is constant for $\xi \in X$.

Since $\text{dom } p_\xi$ is Easton, $\sup(\text{dom } p_\xi \cap \xi) < \xi$ for all regular $\xi < \kappa$. Since κ is Mahlo, we get a stationary $S \subset \kappa$ and $\delta < \kappa$ such that $\text{dom } p_\xi \cap \xi \subset \delta$ for all $\xi \in S$.

Since $\text{dom } p_\xi$ is bounded in κ , $C = \{\zeta < \kappa : \forall \xi < \zeta (\text{dom } p_\xi \subset \zeta)\}$ is club. Note that if $\xi < \zeta$ are both from $S \cap C$, we have $\text{dom } p_\xi \cap \text{dom } p_\zeta = \text{dom } p_\xi \cap \zeta \cap \text{dom } p_\zeta \subset \delta$. Since $|\prod_{\mu \leq \gamma < \delta} P_\gamma| < \kappa$, there is $X \in [S \cap C]^\kappa$ such that $p_\xi|_\delta$ is constant for $\xi \in X$, as desired. $\square$

For $\gamma \geq \mu$ we equip the set ${}^{<\mu}\gamma$ with reverse inclusion. Needless to say, ${}^{<\mu}\gamma$ is μ -closed and forces $|\gamma| = \mu$. Let us sketch a proof of

Lemma 2. *If $\gamma^{<\kappa} = \gamma$, then ${}^{<\mu}\gamma$ is isomorphic to a dense subset of ${}^{<\mu}\kappa \times {}^{<\kappa}\gamma$.*

Proof. Define

$$D = \{(q, r) \in {}^{<\mu}\kappa \times {}^{<\kappa}\gamma : \sup\{\beta + 1 : \beta \in \text{ran } q\} = \text{dom } r\}.$$

It is easy to see that D is dense in ${}^{<\mu}\kappa \times {}^{<\kappa}\gamma$. The following three facts should suffice to construct an isomorphism between ${}^{<\mu}\gamma$ and D by recursion.

First $(\emptyset, \emptyset) \in D$. Second each $(q, r) \in D$ has γ immediate extensions in D . Third if $\langle (q_\alpha, r_\alpha) : \alpha < \delta \rangle$ is a descending sequence in D with $\delta < \mu$, then we have $(\bigcup_{\alpha < \delta} q_\alpha, \bigcup_{\alpha < \delta} r_\alpha) \in D$. $\square$

Corollary 3. *If $\gamma \geq \kappa$ is strongly regular, there is a projection from ${}^{<\mu}\gamma$ to ${}^{<\kappa}\gamma$.*

Let F be a filter on a set. We denote by F^+ the set of F -positive sets ordered by: $X' \leq X$ iff $\exists C \in F (X' \cap C \subset X)$. Then F^+ is a separative poset. We say that F is (κ, κ, μ) -saturated if F^+ has (κ, κ, μ) -cc.

3. THE EASTON COLLAPSE

In this section we define the Easton collapse $E(\mu, \kappa)$ and prove its basic properties.

For a set X of ordinals define

$$E(\mu, X) = \prod_{\mu \leq \gamma \in X}^E {}^{<\mu}\gamma.$$

It is easy to see that $E(\mu, X)$ is μ -directed closed and forces $|\gamma| \leq \mu$ for all strongly regular $\gamma \in X$. $E(\mu, \kappa)$ is a subset of V_κ , hence has size κ if κ is inaccessible. If κ is Mahlo, then $E(\mu, \kappa)$ has κ -cc by Lemma 1, and hence forces $\kappa = \mu^+$. If $\mu < \kappa \leq \nu < \lambda$ are all regular, Corollary 3 provides a projection from $E(\mu, \lambda - \kappa) = \prod_{\kappa \leq \gamma < \lambda}^E {}^{<\mu}\gamma$ to $\prod_{\nu \leq \gamma < \lambda}^E {}^{<\nu}\gamma = E(\nu, \lambda)$.

Here is the main result of this section:

Lemma 4. *Suppose that P has κ -cc and size $\leq \kappa$. Then there is a projection $\pi : P \times E(\kappa, \lambda) \rightarrow P * \dot{E}(\kappa, \lambda)$ such that $\pi(p, q)$ has the form $(p, \dot{q})$, where*

- $\Vdash_P \text{dom } \dot{q} = \text{dom } q$ and
- each $\dot{q}(\gamma)$ depends only on $q(\gamma)$, i.e. if in addition $\pi(p', q') = (p', \dot{q}')$ and $q(\gamma) = q'(\gamma)$, then $\Vdash_P \dot{q}(\gamma) = \dot{q}'(\gamma)$.

Proof. Since P has κ -cc and size $\leq \kappa$, forcing with P does not change the class of (strongly) regular cardinals $\geq \kappa$. If $\gamma \geq \kappa$ is regular and $\Vdash \dot{\alpha} < \gamma$, then there is $\beta < \gamma$ with $\Vdash \dot{\alpha} < \beta$. If $\gamma \geq \kappa$ is strongly regular, there exist exactly γ representatives from the P -names $\dot{\alpha}$ such that $\Vdash \dot{\alpha} < \gamma$. Thus we can take P -names $\dot{\tau}(\xi)$ so that for every strongly regular $\gamma \geq \kappa$

- if $\xi < \gamma$, then $\Vdash \dot{\tau}(\xi) < \gamma$ and

- if $\Vdash \dot{\alpha} < \gamma$, then there is $\xi < \gamma$ with $\Vdash \dot{\alpha} = \dot{\tau}(\xi)$.

For $(p, q) \in P \times E(\kappa, \lambda)$ define

$$\pi(p, q) = (p, \dot{q}),$$

where $\dot{q}$ is a P -name such that

- $\Vdash \text{dom } \dot{q} = \text{dom } q$ and
- $\Vdash \dot{q}(\gamma) = \langle \dot{\tau}(q(\gamma)(\eta)) : \eta \in \text{dom } q(\gamma) \rangle$ for every $\gamma \in \text{dom } q$.

Since P has κ -cc, $\text{dom } q$ remains an Easton subset of $\lambda - \kappa$ after forcing with P . Moreover $\Vdash \dot{q}(\gamma)(\eta) < \gamma$ by $q(\gamma)(\eta) < \gamma$ and the choice of $\dot{\tau}(\xi)$. Thus $\pi(p, q) \in P * \dot{E}(\kappa, \lambda)$.

Claim. π is a projection.

Proof. It is easy to see that π is order-preserving and $\pi(1_P, \emptyset) = (1_P, \emptyset)$.

Now assume $(p, q) \in P \times E(\kappa, \lambda)$ and $(p', \dot{q}') \leq \pi(p, q)$ in $P * \dot{E}(\kappa, \lambda)$. Let $(p, \dot{q}) = \pi(p, q)$. Define

$$p^* = p'.$$

Then $p^* \leq p$ by $(p', \dot{q}') \leq (p, \dot{q})$. It remains to find $q^* \leq q$ in $E(\kappa, \lambda)$ such that $\pi(p^*, q^*) \leq (p', \dot{q}')$ in $P * \dot{E}(\kappa, \lambda)$. Define

$$d^* = \{\gamma : \exists r \in P(r \Vdash \gamma \in \text{dom } \dot{q}')\}.$$

Since P has κ -cc and $\Vdash \dot{q}' \in \dot{E}(\kappa, \lambda)$, d^* is an Easton subset of $\lambda - \kappa$. Moreover $\text{dom } q \subset d^*$ because

$$p' \Vdash \text{dom } q = \text{dom } \dot{q} \subset \text{dom } \dot{q}' \subset d^*.$$

The left equality follows from the definition of $\dot{q}$, the middle inclusion from $(p', \dot{q}') \leq (p, \dot{q})$, and the right inclusion from the definition of d^* .

Fix $\gamma \in d^*$. Since P has κ -cc and $\Vdash \dot{q}' \in \dot{E}(\kappa, \lambda)$, there is $\delta_\gamma^* < \kappa$ such that $\Vdash \gamma \in \text{dom } \dot{q}' \rightarrow \text{dom } \dot{q}'(\gamma) \subset \delta_\gamma^*$. If $\gamma \in \text{dom } q$, then $\text{dom } q(\gamma) \subset \delta_\gamma^*$ because

$$p' \Vdash \text{dom } q(\gamma) = \text{dom } \dot{q}(\gamma) \subset \text{dom } \dot{q}'(\gamma) \subset \delta_\gamma^*.$$

The left equality follows from the definition of $\dot{q}$, the middle inclusion from $(p', \dot{q}') \leq (p, \dot{q})$, and the right inclusion from $p' \Vdash \gamma \in \text{dom } \dot{q}'$ and the choice of δ_γ^* .

Now define q^* with $\text{dom } q^* = d^*$ and $\text{dom } q^*(\gamma) = \delta_\gamma^*$ for every $\gamma \in d^*$ so that

- $q^*(\gamma)(\eta) = q(\gamma)(\eta)$ if $\gamma \in \text{dom } q$ and $\eta \in \text{dom } q(\gamma)$, or else
- $q^*(\gamma)(\eta)$ is the minimal ξ such that $\Vdash \gamma \in \text{dom } \dot{q}' \wedge \eta \in \text{dom } \dot{q}'(\gamma) \rightarrow \dot{q}'(\gamma)(\eta) = \dot{\tau}(\xi)$.

Note that $q^*(\gamma)(\eta) < \gamma$ by $q \in E(\kappa, \lambda)$ in the first case, and by $\Vdash \dot{q} \in \dot{E}(\kappa, \lambda)$ and the choice of $\dot{\tau}(\xi)$ in the second case. Thus $q^* \in E(\kappa, \lambda)$ and $q^* \leq q$.

Let $(p^*, \dot{q}^*) = \pi(p^*, q^*)$. Since $p^* = p'$, it remains to prove that $p' \Vdash \dot{q}^* \leq \dot{q}'$. First recall that

$$\Vdash \text{dom } \dot{q}^* = \text{dom } q^* = d^* \supset \text{dom } \dot{q}'.$$

It remains to prove that for every $\gamma \in d^*$ and $\eta \in \delta_\gamma^*$

$$p' \Vdash \gamma \in \text{dom } \dot{q}' \wedge \eta \in \text{dom } \dot{q}'(\gamma) \rightarrow \dot{q}^*(\gamma)(\eta) = \dot{q}'(\gamma)(\eta).$$

If $\gamma \in \text{dom } q$ and $\eta \in \text{dom } q(\gamma)$, the claim follows from

$$p' \Vdash \dot{q}^*(\gamma)(\eta) = \dot{\tau}(q^*(\gamma)(\eta)) = \dot{\tau}(q(\gamma)(\eta)) = \dot{q}'(\gamma)(\eta).$$

The left equality follows from the definition of $\dot{q}^*$, the middle from that of q^* , and the right from $(p', \dot{q}') \leq (p, \dot{q})$.

In the remaining case the claim follows from

$$\Vdash \gamma \in \text{dom } \dot{q}' \wedge \eta \in \text{dom } \dot{q}'(\gamma) \rightarrow \dot{q}^*(\gamma)(\eta) = \dot{\tau}(q^*(\gamma)(\eta)) = \dot{q}'(\gamma)(\eta).$$

The left equality follows from the definition of $\dot{q}^*$, and the right from that of q^* . $\square$

This completes the proof. $\square$

Remark. Lemma 4 should hold for suitable modifications of the collapses of Levy and Silver. See [13] or [14] for the corresponding lemma for the modified Silver collapse and the resulting model in which a saturated filter exists and Chang's conjecture holds.

In [11] Laver introduced a poset $L(\kappa, \lambda)$, here called the Laver collapse. It is the product of collapsing posets with Easton support and bounded height. Using Kunen's method Laver constructed a forcing of the form $P * \dot{L}(\kappa, \lambda)$, which produces an $(\omega_2, \omega_2, \omega)$ -saturated filter on ω_1 . Although Lemma 4 should hold for a suitable modification of the Laver collapse as well, we need to work with the Easton collapse because a projection, say from $L(\mu, \lambda - \kappa)$ to $L(\kappa, \lambda)$ is not available to us. For the same reason we cannot substitute the collapses of Levy or of Silver for the Easton collapse.

For a P -name $\dot{Q}$ for a poset let $T(P, \dot{Q})$ denote the term forcing. It is known that the identity map from $P \times T(P, \dot{Q})$ to $P * \dot{Q}$ is a projection. See [5] for details. In [1] Cummings observed that $T(P, \dot{<\kappa}\gamma)$ is equivalent to $\dot{<\kappa}\gamma$ if P has κ -cc and size $\leq \kappa$, and $\gamma^{<\kappa} = \gamma$. The proof of Lemma 4 shows in effect that $T(P, \dot{E}(\kappa, \lambda))$ is equivalent to $E(\kappa, \lambda)$. To see that the filter in our model is λ -saturated only, it suffices to prove this fact or even Lemma 4 without additional clauses.

4. THE MAIN THEOREM

This section is devoted to the proof of

Theorem. *Suppose that κ is almost huge with target λ and λ is Mahlo. Let $\mu < \nu$ be both regular with $\mu < \kappa \leq \nu < \lambda$. Then $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ forces that $\mathcal{P}_{\kappa}\nu$ carries a (λ, λ, μ) -saturated normal filter.*

Proof. Let $j : V \rightarrow M$ witness that κ is almost huge with target λ , i.e. $\kappa = \text{crit}(j)$, $\lambda = j(\kappa)$ and $\dot{<\lambda}M \subset M$. Then we have $j(E(\mu, \kappa)) = E(\mu, \lambda)$, which is canonically isomorphic to $E(\mu, \kappa) \times E(\mu, \lambda - \kappa)$. As stated in §3, there is a projection from $E(\mu, \lambda - \kappa)$ to $E(\nu, \lambda)$. Since $E(\mu, \kappa)$ has κ -cc and size κ , there is a projection from $E(\mu, \kappa) \times E(\nu, \lambda)$ to $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ as in Lemma 4. Thus we get a projection $\pi : E(\mu, \lambda) \rightarrow E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ such that $\pi(p)$ has the form $(p|_{\kappa}, \dot{q})$, where $E(\mu, \kappa) \Vdash \text{dom } \dot{q} = \text{dom } p - \nu$ and each $\dot{q}(\gamma)$ depends only on $p(\gamma)$.

Now let $\bar{G} \subset E(\mu, \lambda)$ be V -generic. Then $\pi''\bar{G}$ generates a V -generic filter over $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$, say $G * H$. We claim that $V[G][H]$ is the desired model. Since $j''G = G \subset \bar{G}$, we can lift $j : V \rightarrow M$ to $j : V[G] \rightarrow M[\bar{G}]$ in $V[\bar{G}]$. Since λ is Mahlo in V , $E(\mu, \lambda)$ has λ -cc in V . Hence we have $\dot{<\lambda}M[\bar{G}] \subset M[\bar{G}]$ in $V[\bar{G}]$ by $\dot{<\lambda}M \subset M$ in V .

Work in $V[G]$. Since $E(\mu, \kappa)$ has size κ in V , λ remains Mahlo and hence $E(\nu, \lambda)$ has λ -cc. Thus a nice $E(\nu, \lambda)$ -name for a subset of $\mathcal{P}_\kappa \nu$ can be viewed as an $E(\nu, \xi)$ -name for some $\xi < \lambda$. So we can list the set of all such names with cofinal repetition as $\{\dot{X}_\xi : \xi < \lambda\}$.

Now work in $V[\bar{G}]$. Since ${}^{<\lambda}M[\bar{G}] \subset M[\bar{G}]$, $E(j(\nu), j(\xi))^{M[\bar{G}]}$ is λ -directed closed for $\xi < \lambda$. So we can define for $\xi < \lambda$

$$r_\xi = \text{the greatest lower bound of } j^{\text{``}}(H \cap E(\nu, \xi)^{V[G]}) \text{ in } E(j(\nu), j(\xi))^{M[\bar{G}]}$$

Note that $\xi < \zeta < \lambda$ implies $r_\zeta \restriction j(\xi) = r_\xi$. Thus we can define a descending sequence $\langle r_\xi^* : \xi < \lambda \rangle$ in $E(j(\nu), j(\lambda))^{M[\bar{G}]}$ by recursion so that

- $r_\xi^* \leq r_\xi$ in $E(j(\nu), j(\xi))^{M[\bar{G}]}$ and
- if $\dot{X}_\xi$ is a $E(\nu, \xi)^{V[G]}$ -name, then r_ξ^* decides $j^{\text{``}}\nu \in j(\dot{X}_\xi)$ in $M[\bar{G}]$.

Define

$$U = \{(\dot{X}_\xi)_H : \xi < \lambda \wedge M[\bar{G}] \models r_\xi^* \Vdash j^{\text{``}}\nu \in j(\dot{X}_\xi)\}.$$

Standard arguments show that U is a $V[G][H]$ -normal ultrafilter on $\mathcal{P}_\kappa \nu^{V[G][H]}$.

Finally we work in $V[G][H]$. Since $E(\mu, \lambda)$ projects down to $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ in V , there is a $E(\mu, \lambda)^V / (G * H)$ -name $\dot{U}$ such that

$$E(\mu, \lambda)^V / (G * H) \Vdash \dot{U} \text{ is a } V[G][H]\text{-normal ultrafilter on } \mathcal{P}_\kappa \nu^{V[G][H]}.$$

Define

$$F = \{X \subset \mathcal{P}_\kappa \nu : E(\mu, \lambda)^V / (G * H) \Vdash X \in \dot{U}\}.$$

Standard arguments show that F is a normal filter on $\mathcal{P}_\kappa \nu$. We claim that F is (λ, λ, μ) -saturated. Standard arguments show that

$$X \mapsto \sum \{p \in E(\mu, \lambda)^V / (G * H) : p \Vdash X \in \dot{U}\}$$

defines a complete embedding of F^+ into $B(E(\mu, \lambda)^V / (G * H))$. So it suffices to prove that $E(\mu, \lambda)^V / (G * H)$ has (λ, λ, μ) -cc. Let $\{p_\xi : \xi < \lambda\} \subset E(\mu, \lambda)^V / (G * H)$. Since $E(\mu, \kappa)$ has κ -cc and forces $\dot{E}(\nu, \lambda)$ to be κ -closed in V , it suffices to find $S \in [\lambda]^\lambda$ such that if $x \in [S]^\mu$ and $\langle p_\xi : \xi \in x \rangle \in V$, $\{p_\xi : \xi \in x\}$ has a common extension in $E(\mu, \lambda)^V / (G * H)$.

Let R be the set of regular cardinals $< \lambda$ in V . Since λ is Mahlo and $E(\mu, \kappa) * \dot{E}(\nu, \lambda)$ has λ -cc in V , R is stationary. As in the proof of Lemma 1 we get a stationary $S \subset R$ such that $\{\text{dom } p_\xi : \xi \in S\}$ forms a Δ -system, say with root d . Moreover we may assume that $p_\xi \restriction d$ is constant and $\text{dom } p_\xi \cap \kappa \subset d$ for $\xi \in S$.

Suppose $x \in [S]^\mu$ and $\langle p_\xi : \xi \in x \rangle \in V$. Define $p = \bigcup_{\xi \in x} p_\xi$. We claim that p is a lower bound of $\{p_\xi : \xi \in x\}$ in $E(\mu, \lambda)^V / (G * H)$. Since $p_\xi \restriction d$ is constant on S , p is a lower bound of $\{p_\xi : \xi \in x\}$ in $E(\mu, \lambda)^V$.

It remains to prove that $\pi(p) \in G * H$. Let $(p \restriction \kappa, \dot{q}) = \pi(p)$ and $(p_\xi \restriction \kappa, \dot{q}_\xi) = \pi(p_\xi)$ for $\xi \in S$. Since $p_\xi \restriction \kappa$ is constant on S , we have $p \restriction \kappa = p_\xi \restriction \kappa$ for every $\xi \in S$. Hence $p \restriction \kappa \in G$ by $(p_\xi \restriction \kappa, \dot{q}_\xi) = \pi(p_\xi) \in G * H$. To see that $\dot{q}_G \in H$, note first that $(\dot{q}_\xi)_G \in H$ by $(p_\xi \restriction \kappa, \dot{q}_\xi) \in G * H$. Since $\text{dom}(\dot{q}_\xi)_G = \text{dom } p_\xi - \nu$, $\{\text{dom}(\dot{q}_\xi)_G : \xi \in S\}$ forms a Δ -system with root $d - \nu$. Moreover $(\dot{q}_\xi)_G \restriction (d - \nu)$ is constant on S . Thus $\dot{q}_G = \bigcup_{\xi \in x} (\dot{q}_\xi)_G$ is the greatest lower bound of $\{(\dot{q}_\xi)_G : \xi \in x\}$ in $E(\nu, \lambda)^{V[G]}$. Therefore $\dot{q}_G \in H$, as desired. $\square$

Remark. For the moment let us assume that κ is huge with target λ . As remarked in §3, our strategy requires forcing with Easton collapses rather than with Laver collapses. This requires in turn invoking an argument of Magidor [8] that involves local master conditions, even under the stronger hypothesis as above. In fact we can dispense with the argument in the case $\nu > \kappa$. Moreover the proof in this case, if modified as in [12], shows that $[\lambda]^\kappa$ carries a (λ, λ, μ) -saturated κ -complete filter in the extension.

In [11] Laver observed that a strong form of Chang's conjecture holds in his model. We do not know whether our model in the case $\nu = \kappa$ satisfies the conjecture.

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