

ON THE CALCULATION OF THE SPECTRA OF BURNSIDE TAMBARA FUNCTORS

HIROYUKI NAKAOKA

ABSTRACT. For a finite group G , a Tambara functor on G is regarded as a G -bivariant analog of a commutative ring. In our previous article, we consider a G -bivariant analog of the ideal theory for Tambara functors. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G = \mathbb{Z}/q\mathbb{Z}$.

1. INTRODUCTION AND PRELIMINARIES

A *Tambara functor* is firstly defined by Tambara [8] in the name ‘TNR-functor’, to treat the multiplicative transfers of Green functors. (For the definitions of Green and Mackey functors, see [1].) Later it is used by Brun [2] to describe the structure of Witt-Burnside rings.

For a finite group G , a Tambara functor is also regarded as a G -bivariant analog of a commutative ring, as seen in [9]. As such, for example a G -bivariant analog of the fraction ring was considered in [3], and a G -bivariant analog of the semigroup-ring construction was discussed in [5] and [6], with relation to the Dress construction [7].

In this analogy, we considered a G -bivariant analog of the ideal theory for Tambara functors in our previous article [4]. In this article, we will demonstrate calculations of spectra of Burnside Tambara functors, when $G = \mathbb{Z}/q\mathbb{Z}$ for some prime number q .

Throughout this article, the unit of a finite group G will be denoted by e . Abbreviately we denote the trivial subgroup of G by e , instead of $\{e\}$. $H \leq G$ means H is a subgroup of G . $G\text{set}$ denotes the category of finite G -sets and G -equivariant maps. If $H \leq G$ and $g \in G$, then ${}^gH = gHg^{-1}$ denotes the conjugate ${}^gH = gHg^{-1}$.

A ring is assumed to be commutative, with an additive unit 0 and a multiplicative unit 1. A ring homomorphism preserves 0 and 1.

For any category $\mathcal{C}$ and any pair of objects X and Y in $\mathcal{C}$, the set of morphisms from X to Y in $\mathcal{C}$ is denoted by $\mathcal{C}(X, Y)$.

First we briefly recall the definition of a Tambara functor and its ideal.

Definition 1.1. ([8]) A *Tambara functor* T on G is a triplet $T = (T^*, T_+, T_\bullet)$ of two covariant functors

$$T_+ : G\text{set} \rightarrow \text{Set}, \quad T_\bullet : G\text{set} \rightarrow \text{Set}$$

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and one contravariant functor

$$T^*: {}_G\text{set} \rightarrow \text{Set}$$

which satisfies the following. Here Set is the category of sets.

(1) $T^\alpha = (T^*, T_+)$ is a Mackey functor on G .

(2) $T^\mu = (T^*, T_\bullet)$ is a semi-Mackey functor on G .

Since T^α, T^μ are semi-Mackey functors, we have $T^*(X) = T_+(X) = T_\bullet(X)$ for each $X \in \text{Ob}({}_G\text{set})$. We denote this by $T(X)$.

(3) (Distributive law) If we are given an exponential diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{\lambda} & Z \\ f \downarrow & & \text{exp} & & \downarrow \rho \\ Y & \xleftarrow{q} & & & B \end{array}$$

in ${}_G\text{set}$, then

$$\begin{array}{ccccc} T(X) & \xleftarrow{T_+(p)} & T(A) & \xrightarrow{T^*(\lambda)} & T(Z) \\ T_\bullet(f) \downarrow & & \circlearrowleft & & \downarrow T_\bullet(\rho) \\ T(Y) & \xleftarrow{T_+(q)} & & & T(B) \end{array}$$

is commutative.

If $T = (T^*, T_+, T_\bullet)$ is a Tambara functor, then $T(X)$ becomes a ring for each $X \in \text{Ob}({}_G\text{set})$. For each $f \in {}_G\text{set}(X, Y)$,

- $T^*(f): T(Y) \rightarrow T(X)$ is a ring homomorphism.
- $T_+(f): T(X) \rightarrow T(Y)$ is an additive homomorphism.
- $T_\bullet(f): T(X) \rightarrow T(Y)$ is a multiplicative homomorphism.

$T^*(f), T_+(f), T_\bullet(f)$ are often abbreviated to $f^*, f_+, f_\bullet$.

In this article, a *Tambara functor* always means a Tambara functor on some finite group G .

Example 1.2. If we define Ω by

$$\Omega(X) = K_0({}_G\text{set}/X)$$

for each $X \in \text{Ob}({}_G\text{set})$, where the right hand side is the Grothendieck ring of the category of finite G -sets over X , then Ω becomes a Tambara functor on G . This is called the *Burnside Tambara functor*. For each $f \in {}_G\text{set}(X, Y)$,

$$f_\bullet: \Omega(X) \rightarrow \Omega(Y)$$

is the one determined by

$$f_\bullet(A \xrightarrow{p} X) = (\Pi_f(A) \xrightarrow{\varpi} Y) \quad (\forall (A \xrightarrow{p} X) \in \text{Ob}({}_G\text{set}/X)),$$

where $\Pi_f(A)$ and ϖ is

$$\Pi_f(A) = \left\{ (y, \sigma) \left| \begin{array}{l} y \in Y, \\ \sigma: f^{-1}(y) \rightarrow A \text{ a map of sets,} \\ p \circ \sigma = \text{id}_{f^{-1}(y)} \end{array} \right. \right\},$$

$$\varpi(y, \sigma) = y.$$

G acts on $\Pi_f(A)$ by $g \cdot (y, \sigma) = (gy, {}^g\sigma)$, where ${}^g\sigma$ is the map defined by

$${}^g\sigma(x) = g\sigma(g^{-1}x) \quad (\forall x \in f^{-1}(gy)).$$

Definition 1.3. Let T be a Tambara functor. For each $f \in {}_G\text{set}(X, Y)$, define $f_! : T(X) \rightarrow T(Y)$ by

$$f_!(x) = f_\bullet(x) - f_\bullet(0)$$

for any $x \in T(X)$.

Remark 1.4. ([4]) Let T be a Tambara functor. We have the following for any $f \in {}_G\text{set}(X, Y)$.

- (1) $f_!$ satisfies $f_!(x)f_!(y) = f_!(xy)$ for any $x, y \in T(X)$.
- (2) If f is surjective, then we have $f_! = f_\bullet$.
- (3) If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \xi \downarrow & \square & \downarrow \eta \\ X & \xrightarrow{f} & Y \end{array}$$

is a pull-back diagram, then $f'_! \xi^* = \eta^* f_!$ holds.

- (4) If

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{\lambda} & Z \\ f \downarrow & & \text{exp} & & \downarrow \rho \\ Y & \xleftarrow{\varpi} & & & \Pi \end{array}$$

is an exponential diagram, then $\varpi_+ \rho_! \lambda^* = f_! p_+$ holds.

Definition 1.5. ([4]) Let T be a Tambara functor. An *ideal* $\mathcal{J}$ of T is a family of ideals $\mathcal{J}(X) \subseteq T(X)$ ($\forall X \in \text{Ob}({}_G\text{set})$) satisfying

- (i) $f^*(\mathcal{J}(Y)) \subseteq \mathcal{J}(X)$,
- (ii) $f_+(\mathcal{J}(X)) \subseteq \mathcal{J}(Y)$,
- (iii) $f_!(\mathcal{J}(X)) \subseteq \mathcal{J}(Y)$

for any $f \in {}_G\text{set}(X, Y)$. These conditions also imply

$$\mathcal{J}(X_1 \amalg X_2) \cong \mathcal{J}(X_1) \times \mathcal{J}(X_2)$$

for any $X_1, X_2 \in \text{Ob}({}_G\text{set})$.

Obviously when G is trivial, this definition of an ideal agrees with the ordinary definition of an ideal of a commutative ring.

Remark 1.6. For any ideal $\mathcal{J} \subseteq T$, we have $\mathcal{J}(\emptyset) = T(\emptyset) = 0$.

Definition 1.7. ([4]) An ideal $\mathfrak{p} \subsetneq T$ is *prime* if for any transitive $X, Y \in \text{Ob}({}_G\text{set})$ and any $a \in T(X)$, $b \in T(Y)$,

$$\langle a \rangle \langle b \rangle \subseteq \mathfrak{p} \Rightarrow a \in \mathfrak{p}(X) \text{ or } b \in \mathfrak{p}(Y)$$

is satisfied. Remark that the converse always holds.

An ideal $\mathfrak{m} \subsetneq T$ is *maximal* if it is maximal with respect to the inclusion of ideals not equal to T . A maximal ideal is always prime.

Definition 1.8. ([4]) For any Tambara functor T on G , define $\text{Spec}(T)$ to be the set of all prime ideals of T . For each ideal $\mathcal{J} \subseteq T$, define a subset $V(\mathcal{J}) \subseteq \text{Spec}(T)$ by

$$V(\mathcal{J}) = \{\mathfrak{p} \in \text{Spec}(T) \mid \mathcal{J} \subseteq \mathfrak{p}\}.$$

Remark 1.9. ([4]) For any Tambara functor T , we have the following.

- (1) $V(\mathcal{J}) = \emptyset$ if and only if $\mathcal{J} = T$.
- (2) $V(\mathcal{J}) = \text{Spec}(T)$ if and only if $\mathcal{J} \subseteq \bigcap_{\mathfrak{p} \in \text{Spec}(T)} \mathfrak{p}$.

Remark 1.10. ([4]) For any Tambara functor T , the family $\{V(\mathcal{J}) \mid \mathcal{J} \subseteq T \text{ is an ideal}\}$ forms a system of closed subsets of $\text{Spec}(T)$. Thus $\text{Spec } \Omega$ becomes a topological space.

2. SOME PROPOSITIONS

Proposition 2.1. *Let T be a Tambara functor. Suppose we are given a family of ideals indexed by the set of finite non-empty transitive G -sets*

$$(2.1) \quad \{\mathcal{J}(X_0) \subseteq T(X_0) \mid \emptyset \neq X_0 \in \text{Ob}(G\text{set})\}.$$

For any $X \in \text{Ob}(G\text{set})$, take its orbit decomposition $X = \coprod_{1 \leq i \leq s} X_i$ and put

$$\mathcal{J}(X) = \mathcal{J}(X_1) \times \cdots \times \mathcal{J}(X_s) \subseteq T(X).$$

(We used the identification $T(X) \cong \prod_{1 \leq i \leq s} T(X_i)$.) Then the following are equivalent.

(1) $\mathcal{J} = \{\mathcal{J}(X)\}_{X \in \text{Ob}(G\text{set})}$ is an ideal of T .

(2) The family (2.1) satisfies

(i) $f^*(\mathcal{J}(Y_0)) \subseteq \mathcal{J}(X_0)$

(ii) $f_+(\mathcal{J}(X_0)) \subseteq \mathcal{J}(Y_0)$

(iii) $f_\bullet(\mathcal{J}(X_0)) \subseteq \mathcal{J}(Y_0)$

for any transitive $X_0, Y_0 \in \text{Ob}(G\text{set})$ and any $f \in G\text{set}(X_0, Y_0)$

Proof. Remark that for any non-empty transitive $X_0, Y_0 \in \text{Ob}(G\text{set})$ and any $f \in G\text{set}(X_0, Y_0)$, we have $f_\bullet = f_!$. Obviously, (1) implies (2). We will show the converse.

Assume (2) holds. It suffices to show $\mathcal{J}$ satisfies (i), (ii), (iii) in Definition 1.5 for any $f \in G\text{set}(X, Y)$.

First, we reduce to the case where Y is transitive. Take the orbit decomposition $Y = \coprod_{1 \leq j \leq t} Y_j$, put

$$X_j = f^{-1}(Y_j), \quad f_j = f|_{X_j} : X_j \rightarrow Y_j,$$

and suppose (i), (ii), (iii) in Definition 1.5 holds for each f_j . Since we have commutative diagrams

$$\begin{array}{ccccc} T(X) & \xrightarrow{\cong} & \prod_j T(X_j) & & T(Y) & \xrightarrow{\cong} & \prod_j T(Y_j) & & T(X) & \xrightarrow{\cong} & \prod_j T(X_j) \\ f_+ \downarrow & \circ & \downarrow \Pi_j f_{j+} & , & f^* \downarrow & \circ & \downarrow \Pi_j f_j^* & , & f \downarrow & \circ & \downarrow \Pi_j f_j! \\ T(Y) & \xrightarrow{\cong} & \prod_j T(Y_j) & & T(X) & \xrightarrow{\cong} & \prod_j T(X_j) & & T(Y) & \xrightarrow{\cong} & \prod_j T(Y_j) \end{array}$$

under the canonical identification, we obtain

$$\begin{aligned} f_+(\mathcal{J}(X)) &= \prod_i f_{j+}(\mathcal{J}(X_j)) \subseteq \prod_j \mathcal{J}(Y_j) = \mathcal{J}(Y), \\ f^*(\mathcal{J}(Y)) &= \prod_i f_j^*(\mathcal{J}(Y_j)) \subseteq \prod_j \mathcal{J}(X_j) = \mathcal{J}(X), \\ f_!(\mathcal{J}(X)) &= \prod_i f_{j!}(\mathcal{J}(X_j)) \subseteq \prod_j \mathcal{J}(Y_j) = \mathcal{J}(Y). \end{aligned}$$

Now it remains to show in the case Y is transitive. If $X = \emptyset$, then there is nothing to show. Otherwise, take the orbit decomposition $X = \coprod_{1 \leq i \leq s} X_i$ and put $f_i = f|_{X_i}: X_i \rightarrow Y$. Remark that in this case, we have $f_\bullet = f_!$. By assumption, each f_i satisfies

$$\begin{aligned} f_{i+}(\mathcal{J}(X_i)) &\subseteq \mathcal{J}(Y), \\ f_i^*(\mathcal{J}(Y)) &\subseteq \mathcal{J}(X_i), \\ f_{i\bullet}(\mathcal{J}(X_i)) &\subseteq \mathcal{J}(Y). \end{aligned}$$

Under the identification $T(X) \cong \prod_{1 \leq i \leq s} T(X_i)$, we obtain $f^*(\mathcal{J}(Y)) \subseteq \mathcal{J}(X_1) \times \cdots \times \mathcal{J}(X_s) = \mathcal{J}(X)$. Moreover, for any $x \in \mathcal{J}(X)$, under the identification

$$\begin{aligned} \mathcal{J}(X) &= \mathcal{J}(X_1) \times \cdots \times \mathcal{J}(X_s) \\ x &= (x_1, \dots, x_s), \end{aligned}$$

we have

$$\begin{aligned} f_+(x) &= f_{1+}(x_1) + \cdots + f_{s+}(x_s) \in \mathcal{J}(Y), \\ f_\bullet(x) &= f_{1\bullet}(x_1) \cdot \cdots \cdot f_{s\bullet}(x_s) \in \mathcal{J}(Y). \end{aligned}$$

Thus it follows $f_+(\mathcal{J}(X)) \subseteq \mathcal{J}(Y)$, $f_\bullet(\mathcal{J}(X)) \subseteq \mathcal{J}(Y)$. □

Corollary 2.2. *To give an ideal $\mathcal{J}$ of a Tambara functor T on G is equivalent to give a family of ideals indexed by $\mathcal{O}_G$*

$$\{\mathcal{J}(G/H) \subseteq T(G/H)\}_{H \in \mathcal{O}(G)}$$

satisfying

- (i) $\text{res}_K^H(\mathcal{J}(G/H)) \subseteq \mathcal{J}(G/K)$
- (ii) $\text{ind}_K^H(\mathcal{J}(G/K)) \subseteq \mathcal{J}(G/H)$
- (iii) $\text{jnd}_K^H(\mathcal{J}(G/K)) \subseteq \mathcal{J}(G/H)$
- (iv) $c_{g,H}(\mathcal{J}(G/H)) \subseteq \mathcal{J}(G/gH)$

for any $K \leq H \leq G$ and $g \in G$. In particular, $\mathcal{J}(G/H) \subseteq T(G/H)$ is $N_G(H)/H$ -invariant.

By construction, for ideals $\mathcal{J}, \mathcal{J}' \subseteq T$, we have

$$\mathcal{J} \subseteq \mathcal{J}' \Leftrightarrow \mathcal{J}(G/H) \subseteq \mathcal{J}'(G/H) \ (\forall H \in \mathcal{O}(G)).$$

Corollary 2.3. *When $G = \mathbb{Z}/q\mathbb{Z}$ where q is a prime number, then to give an ideal $\mathcal{J}$ of T is equivalent to give*

- a G -invariant ideal $\mathcal{J}(G/e) \subseteq T(G/e)$,
- an ideal $\mathcal{J}(G/G) \subseteq T(G/G)$,

satisfying

- (i) $\pi^*(\mathcal{J}(G/G)) \subseteq \mathcal{J}(G/e)$,
- (ii) $\pi_+(\mathcal{J}(G/e)) \subseteq \mathcal{J}(G/G)$,
- (iii) $\pi_\bullet(\mathcal{J}(G/e)) \subseteq \mathcal{J}(G/G)$,

where $\pi: G/e \rightarrow G/G$ is the unique constant map.

Remark 2.4. (Corollary 4.5 in [4]) An ideal $\mathcal{J} \subseteq T$ is prime if and only if for any transitive $X, Y \in \text{Ob}({}_G\text{set})$ and any $a \in T(X), b \in T(Y)$, the following two conditions become equivalent.

- (1) $a \in T(X)$ or $b \in T(Y)$.
- (2) For any $C \in \text{Ob}({}_G\text{set})$ and for any pair of diagrams in ${}_G\text{set}$

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

$$(v_!w^*(a)) \cdot (v'_!w'^*(b)) \in \mathcal{J}(C) \text{ is satisfied.}$$

Note that (1) always implies (2).

By the following lemma, it is enough to check (2) only when C, D, D' are transitive.

Lemma 2.5. Let $\mathcal{J} \subseteq T$ be an ideal. Condition (2) in Remark 2.4 is equivalent to the following.

- (2)' For any transitive $C \in \text{Ob}({}_G\text{set})$ and for any pair of diagrams in ${}_G\text{set}$

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y$$

where D and D' are transitive, $(v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \in \mathcal{J}(C)$ is satisfied.

Proof. It suffices to show (2)' implies (2). Assume (2)' holds, take any $C \in \text{Ob}({}_G\text{set})$ and

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

with not necessarily transitive C, D, D' .

Let $C = \coprod_{a \leq i \leq m} C_i$ be the orbit decomposition, and put

$$\begin{aligned} D_i &= v^{-1}(C_i) \quad , \quad D'_i = v'^{-1}(C_i), \\ v_i &= v|_{D_i}: D_i \rightarrow C_i \quad , \quad v'_i = v'|_{D'_i}: D'_i \rightarrow C_i, \\ w_i &= w|_{D_i}: D_i \rightarrow X \quad , \quad w'_i = w'|_{D'_i}: D'_i \rightarrow Y. \end{aligned}$$

Then we have $v_!w^*(a) = (v_1!w_1^*(a), \dots, v_m!w_m^*(a))$, where

$$v_i!w_i^*(a) = \begin{cases} v_{i\bullet}w_i^*(a) & \text{if } D_i \neq \emptyset \\ 0 & \text{if } D_i = \emptyset. \end{cases}$$

Similarly for b . In any case, $(v_i!w_i^*(a)) \cdot (v'_i!w'_i{}^*(b)) \in \mathcal{J}(C_i)$ ($1 \leq i \leq m$) follows from (2)', which means

$$(v_!w^*(a)) \cdot (v'_!w'^*(b)) \in \mathcal{J}(C).$$

□

Proposition 2.6. *Let T be a Tambara functor, and $\mathfrak{p} \subseteq T$ be a prime ideal. Let $T(G/e)^G$ denote the subring of G -invariant elements in $T(G/e)$:*

$$T(G/e)^G = \{x \in T(G/e) \mid gx = x \ (\forall g \in G)\}$$

Similarly for $\mathfrak{p}(G/e)^G$:

$$\mathfrak{p}(G/e)^G = \mathfrak{p}(G/e) \cap T(G/e)^G$$

Then, $\mathfrak{p}(G/e)^G \subseteq T(G/e)^G$ is a prime ideal (in the ordinary ring-theoretic meaning).

Proof. Suppose $a, b \in T(G/e)^G$ satisfies $ab \in \mathfrak{p}(G/e)$. By Lemma 2.5, it suffices to show for any transitive C, D, D' and any pair of diagrams in $G\text{-set}$

$$(2.2) \quad C \xleftarrow{v} D \xrightarrow{w} G/e, \quad C \xleftarrow{v'} D' \xrightarrow{w'} G/e,$$

$(v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \in \mathfrak{p}(C)$ is satisfied. Since D and D' are transitive with trivial stabilizers, we may assume $D = D' = G/e$. Furthermore, modifying v and v' by conjugations, we may assume

$$C = G/H, \quad v = v' = p_e^H: G/H \rightarrow G/e$$

for some $H \leq G$. Thus (2.2) is reduced to the case

$$G/H \xleftarrow{p_e^H} G/e \xrightarrow{w} G/e, \quad G/H \xleftarrow{p_e^H} G/e \xrightarrow{w'} G/e,$$

where w, w' are the multiplication by some $g, g' \in G$. Then we have

$$\begin{aligned} ((p_e^H)_\bullet w^*(a)) \cdot ((p_e^H)_\bullet w'^*(b)) &= (p_e^H)_\bullet ((ga) \cdot (g'b)) \\ &= (p_e^H)_\bullet (ab) \in \mathfrak{p}(G/H). \end{aligned}$$

□

Corollary 2.7. *If $\mathfrak{p} \subseteq \Omega$ is prime, then $\mathfrak{p}(G/e) \subseteq \Omega(G/e)$ is prime.*

Proof. This immediately follows from the fact that $\Omega(G/e) \cong \mathbb{Z}$ has a trivial G -action. □

3. $\text{Spec } \Omega$ FOR $G = \mathbb{Z}/q\mathbb{Z}$

In the following, we assume $G = \mathbb{Z}/q\mathbb{Z}$ for some prime number q , and denote the canonical projection by $\pi = p_e^G: G/e \rightarrow G/G$.

3.1. Structure of Ω .

Proposition 3.1. *For $G = \mathbb{Z}/q\mathbb{Z}$, Burnside Tambara functor has the following structure.*

(1) *There are isomorphisms of rings*

$$\begin{aligned} \Omega(G/e) &\xrightarrow{\cong} \mathbb{Z} \ ; \ \ell G/e \mapsto \ell, \\ \Omega(G/G) &\xrightarrow{\cong} \mathbb{Z}[X]/(X^2 - qX) \ ; \ mG/e + nG/G \mapsto m + nX. \end{aligned}$$

(2) *Under the isomorphisms in (1), the structure morphisms $\pi_+, \pi^*, \pi_\bullet$ are*

$$\begin{aligned} \pi_+ &: \mathbb{Z} \rightarrow \mathbb{Z}[X]/(X^2 - qX) \ ; \ \ell \mapsto \ell X, \\ \pi^* &: \mathbb{Z}[X]/(X^2 - qX) \rightarrow \mathbb{Z} \ ; \ m + nX \mapsto m + qn, \\ \pi_\bullet &: \mathbb{Z} \rightarrow \mathbb{Z}[X]/(X^2 - qX) \ ; \ \ell \mapsto \ell + \frac{\ell^q - \ell}{q} X. \end{aligned}$$

Proof. The only non-trivial part will be

$$\pi_{\bullet}(\ell) = \ell + \frac{\ell^q - \ell}{q}X.$$

This is shown by using the following.

Fact 3.2. (Proposition 4.17 in [4])

The following diagram is commutative.

$$\begin{array}{ccc} & \xrightarrow{\ell \mapsto \ell} & \\ \Omega(G/e) & \xrightarrow{\quad} & \mathbb{Z} \\ \pi_{\bullet} \searrow & \nearrow & \nearrow m \\ & \Omega(G/G) \ni m+nX & \end{array}$$

From this fact, for any $\ell \in \mathbb{Z}$ we have

$$(3.1) \quad \pi_{\bullet}(\ell) = \ell + nX$$

for some $n \in \mathbb{Z}$. Remark that $n \geq 0$ holds if $\ell \geq 0$.

Besides, by the definition of $\pi_{\bullet}$, for any $\ell \in \mathbb{N}_{\geq 0}$ we have

$$\pi_{\bullet}(\coprod_{\ell} G/e \xrightarrow{\nabla} G/e) = \{\sigma \mid \sigma: G/e \rightarrow \coprod_{\ell} G/e, \text{ a section map for } \nabla\},$$

and thus

$$(3.2) \quad \#(\pi_{\bullet}(\ell)) = \ell^q.$$

From (3.1) and (3.2),

$$\pi_{\bullet}(\ell) = \ell + \frac{\ell^q - \ell}{q}X$$

for any $\ell \geq 0$. As for a negative ℓ , since we have

$$\pi_{\bullet}(\ell) = \pi_{\bullet}(-1)\pi_{\bullet}(|\ell|),$$

it will be enough to determine $\pi_{\bullet}(-1)$.

By (3.1), we have $\pi_{\bullet}(-1) = -1 + nX$ for some $n \in \mathbb{Z}$, which should satisfy

$$1 = \pi_{\bullet}(-1)^2 = (-1 + nX)^2 = 1 + n(qn - 2)X.$$

When q is odd, it follows $n = 0$, and $\pi_{\bullet}(-1) = -1$. For $q = 2$, both -1 and $-1 + X$ satisfy $(-1)^2 = (-1 + X)^2 = 1$. However, from the Mackey condition for the pullback

$$\begin{array}{ccc} \coprod_2 G/e & \xrightarrow{\nabla} & G/e \\ \nabla \downarrow & \square & \downarrow \pi \\ G/e & \xrightarrow{\pi} & G/G \end{array},$$

$\pi_{\bullet}(-1)$ should satisfy

$$\pi^* \pi_{\bullet}(-1) = 1,$$

which leads to $\pi_{\bullet}(-1) = -1 + X$.

In any case, we obtain

$$\pi_{\bullet}(\ell) = \ell + \frac{\ell^q - \ell}{q}X \quad (\forall \ell \in \mathbb{Z})$$

for any prime q . □

3.2. Decomposition into fibers. Using the structural isomorphism in Proposition 3.1, we go on to determine $\text{Spec } \Omega$ for $G = \mathbb{Z}/q\mathbb{Z}$. By Corollary 2.7, any prime ideal $\mathfrak{p} \subseteq \Omega$ satisfies $\mathfrak{p}(G/e) = (p)$ for some prime p or $p = 0$. Thus we have a map

$$F: \text{Spec } \Omega \rightarrow \text{Spec } \mathbb{Z} \quad ; \quad \mathfrak{p} \mapsto \mathfrak{p}(G/e).$$

(F will be shown to be continuous after $\text{Spec } \Omega$ is determined.)

Definition 3.3. Let $p \in \mathbb{Z}$ be prime or $p = 0$. We call an ideal $\mathcal{J} \subseteq \Omega$ *is over p* if it satisfies $\mathcal{J}(G/e) = (p)$. A *prime ideal over p* is simply a prime ideal $\mathfrak{p} \subseteq \Omega$ which is over p .

Remark 3.4. By the above arguments, we have

- $F^{-1}((p)) = \{\mathfrak{p} \in \text{Spec } \Omega \mid \text{prime ideal over } p\},$
- $\text{Spec } \Omega = \coprod_{(p) \in \text{Spec } \mathbb{Z}} F^{-1}((p)).$

In the following, we investigate the fibers $F^{-1}((p))$, in the cases $p = 0$, $p = q$, and $p \neq 0, q$.

For each $(p) \in \text{Spec } \mathbb{Z}$, its fiber $F^{-1}((p))$ at least contains one maximal point. In fact, the following was shown in [4].

Fact 3.5. (Corollary 4.42 in [4])

$$\text{Spec } \Omega \supseteq \{\mathcal{J}_{(p)} \mid p \in \mathbb{Z} \text{ is prime}\} \cup \{\mathcal{J}_{(0)}\} \cup \{(0)\}.$$

Here, for each ideal $I \subseteq \Omega(G/e)$, ideal $\mathcal{J}_I \subseteq \Omega$ is defined by

$$\mathcal{J}_I(G/e) = I, \quad \mathcal{J}_I(G/G) = (\pi^*)^{-1}(I).$$

$\mathcal{J}_I$ is the largest one, among all ideals $\mathcal{J} \subseteq \Omega$ satisfying $\mathcal{J}(G/e) = I$.

Under the isomorphism in Proposition 3.1, for any $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} \mathcal{J}_{(\ell)}(G/e) &= (\ell) \subseteq \mathbb{Z}, \\ \mathcal{J}_{(\ell)}(G/G) &= \{m + nX \in \mathbb{Z}[X]/(X^2 - qX) \mid m + qn \in (\ell)\} \\ &= \{k\ell + n(X - q) \in \mathbb{Z}[X]/(X^2 - qX) \mid k, n \in \mathbb{Z}\} \\ &= (\ell, X - q) \subseteq \mathbb{Z}[X]/(X^2 - qX). \end{aligned}$$

In this article, we denote $\mathcal{J}_{(p)}$ by $\mathfrak{m}_p$. For any prime $p \neq 0$, $\mathfrak{m}_p$ is a maximal ideal of Ω . Namely it is a closed point in $\text{Spec } \Omega$, while $\mathfrak{m}_0 = \mathcal{J}_{(0)}$ is not. (For this reason, we prefer to use $\mathcal{J}_{(0)}$ rather than $\mathfrak{m}_0$ only for $p = 0$.)

On the other hand, (0) is the smallest ideal of Ω , namely the generic point in $\text{Spec } \Omega$. We have inclusions

$$(0) \subsetneq \mathcal{J}_{(0)} \subsetneq \mathfrak{m}_p$$

for any prime $p \in \mathbb{Z}$.

3.3. The smallest ideal over p .

Proposition 3.6. *For a prime $p \in \mathbb{Z}$ or $p = 0$, the smallest ideal $I_p \subseteq \Omega$ over p is given by the following.*

- (1) When $p \neq q$ (including the case $p = 0$),

$$I_p(G/G) = (p) \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

- (2) When $p = q$,

$$I_q(G/G) = (qX, X - q) = (q^2, X - q) \subseteq \mathbb{Z}[X]/(X^2 - qX).$$

Proof. (1) $(p) \subseteq I_p(G/e)$ follows from

$$\begin{aligned} p &= \left(p + \frac{p^q - p}{q}X\right) - \frac{p^q - p}{pq} \cdot pX \\ &= \pi_{\bullet}(p) - \frac{p^q - p}{pq} \pi_{+}(p). \end{aligned}$$

To show the converse, it suffices to show that

$$\mathcal{J}(G/e) = (p) \subseteq \mathbb{Z} \text{ and } \mathcal{J}(G/G) = (p) \subseteq \mathbb{Z}[X]/(X^2 - qX)$$

in fact form an ideal $\mathcal{J}$ of Ω . By Corollary 2.3, this is equivalent to show

$$\begin{aligned} \pi^*((p)) &\subseteq (p), \\ \pi_+((p)) &\subseteq (p), \\ \pi_{\bullet}((p)) &\subseteq (p). \end{aligned}$$

However, these immediately follow from

$$\pi^*(p) = p \in (p)$$

and

$$\begin{aligned} \pi_+(\ell p) &= \ell pX \in (p) \\ \pi_{\bullet}(\ell p) &= \ell p + \frac{\ell^q p^q - \ell p}{q}X \in (p) \end{aligned}$$

for any $\ell \in \mathbb{Z}$. (Remark that π^* is a ring homomorphism.)

(2) $(qX, X - q) \subseteq I_q(G/e)$ follows from

$$qX = \pi_+(q)$$

and

$$X - q = q^{q-1}X - \left(q + \frac{q^q - q}{q}X\right) = \pi_+(q^{q-1}) - \pi_{\bullet}(q).$$

To show the converse, it suffices to show

$$\begin{aligned} \pi^*((q^2, X - q)) &\subseteq (q), \\ \pi_+((q)) &\subseteq (qX, X - q), \\ \pi_{\bullet}((q)) &\subseteq (qX, X - q). \end{aligned}$$

These follow from

$$\pi^*(q^2) = q^2, \quad \pi^*(X - q) = 0 \in (q),$$

and

$$\begin{aligned} \pi_+(\ell q) &= \ell qX \in (qX) \\ \pi_{\bullet}(\ell q) &= \ell(q - X) + \ell^q q^{q-1}X \in (q - X, qX) \end{aligned}$$

for any $\ell \in \mathbb{Z}$. □

3.4. All ideals over p .

For $p \neq 0$, ideals $\mathcal{I} \subseteq \Omega$ over p are only I_p and $\mathfrak{m}_p$.

Claim 3.7. *When $p \in \mathbb{Z}$ is prime ($\neq 0$), then there is no ideal between $I_p \subsetneq \mathfrak{m}_p$.*

Proof. It suffices to show that there is no element $f \in \Omega(G/G)$ satisfying

$$(3.3) \quad I_p(G/G) \subsetneq I_p(G/G) + (f) \subsetneq (p, X - q).$$

By $f \in (p, X - q)$, it should be of the form $f = kp + n(X - q)$ for some $k, n \in \mathbb{Z}$.

(1) When $p \neq q$, (3.3) is equal to

$$(p) \subsetneq (p, f) \subsetneq (p, X - q).$$

This will mean the existence of $n \in \mathbb{Z}$ satisfying $(p) \subsetneq (p, n(X - q)) \subsetneq (p, X - q)$. However, since

$$(p, n(X - q)) = \begin{cases} (p) & \text{if } p|n \\ (p, X - q) & \text{if } p \nmid n \end{cases},$$

there should not exist such n .

(2) When $p = q$, (3.3) is equal to

$$(q^2, X - q) \subsetneq (q^2, X - q, f) \subsetneq (q, X - q).$$

This will mean the existence of $k \in \mathbb{Z}$ satisfying

$$(q^2, X - q) \subsetneq (q^2, X - q, kq) \subsetneq (q, X - q).$$

However, since

$$(q^2, X - q, kq) = \begin{cases} (q^2, X - q) & \text{if } q|k \\ (q, X - q) & \text{if } q \nmid k \end{cases},$$

there should not exist such k . □

On the other hand for $p = 0$, there are many ideals between $(0) \subsetneq \mathcal{I}_{(0)}$.

Claim 3.8. *If we define $\mathcal{I}_{(0;n)} \subseteq \Omega$ by*

$$\mathcal{I}_{(0;n)}(G/e) = (0), \quad \mathcal{I}_{(0;n)}(G/G) = n(X - q),$$

then $\mathcal{I}_{(0;n)} \subseteq \Omega$ forms an ideal for each $n \in \mathbb{Z}$. Indeed, these are exactly the all ideals $\mathcal{I} \subseteq \Omega$ over 0:

$$\{\mathcal{I} \subseteq \Omega \text{ ideal} \mid \mathcal{I}(G/e) = (0)\} = \{\mathcal{I}_{(0;n)} \mid n \in \mathbb{Z}\}$$

Proof. Any ideal between $(0) \subsetneq (X - q)$ in $\mathbb{Z}[X]/(X^2 - qX)$ is of the form $(n(X - q))$ for some $n \in \mathbb{Z}$. Since $\mathcal{I}_{(0;n)}(G/e) = (0)$ and $\mathcal{I}_{(0;n)}(G/G) = (n(X - q))$ satisfy

$$\pi^*(n(X - q)) = 0, \quad \pi_+(0) = 0, \quad \pi_\bullet(0) = 0,$$

$\mathcal{I}_{(0;n)} \subseteq \Omega$ gives an ideal for each $n \in \mathbb{Z}$. □

3.5. Criterion to be prime. Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Now we give a criterion for an ideal $\mathcal{J} \subseteq \Omega$ over p to be prime.

Proposition 3.9. *Let $p \in \mathbb{Z}$ be a prime or $p = 0$. Let $\mathcal{J} \subseteq \Omega$ be an ideal over p , not equal to $\mathfrak{m}_p$. Then $\mathcal{J}$ is not prime if and only if one of the following conditions is satisfied.*

(c1) *There exist $a, b \in \mathfrak{m}_p(G/G)$ satisfying*

$$a \notin \mathcal{J}(G/G), \quad b \notin \mathcal{J}(G/G), \quad ab \in \mathcal{J}(G/G).$$

(c2) *There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying*

$$a \notin \mathcal{J}(G/G), \quad \pi_\bullet(b) \notin \mathcal{J}(G/G), \quad a \cdot (\pi_\bullet(b)) \in \mathcal{J}(G/G).$$

(Only here, we use the notation $\mathfrak{m}_0 = \mathcal{J}_{(0)}$ for the consistency.) In particular, if $\mathcal{J}(G/G) \subseteq \Omega(G/G)$ is prime, then $\mathcal{J} \subseteq \Omega$ is prime.

More explicitly, these can be written as follows.

(c1)' *There exist $k, n, k', n' \in \mathbb{Z}$ satisfying*

$$\begin{aligned} kp + n(X - q) &\notin \mathcal{J}(G/G), \quad k'p + n'(X - q) \notin \mathcal{J}(G/G), \\ kk'p^2 + ((n'k + nk')p + nn'q)(X - q) &\in \mathcal{J}(G/G). \end{aligned}$$

(c2)' *There exist $k, n, l \in \mathbb{Z}$ satisfying*

$$\begin{aligned} kp + n(X - q) &\notin \mathcal{J}(G/G), \quad \ell + \frac{\ell^q - \ell}{q}X \notin \mathcal{J}(G/G), \\ kp(\ell + \frac{\ell^q - \ell}{q}X) + n\ell(X - q) &\in \mathcal{J}(G/G). \end{aligned}$$

Proof. By Lemma 2.5, $\mathcal{J} \subseteq \Omega$ is not prime if and only if there exist transitive $X, Y \in \text{Ob}(\mathcal{G}\text{set})$ and $a \in \Omega(X), b \in \Omega(Y)$ satisfying $a \notin \mathcal{J}(X), b \notin \mathcal{J}(Y)$ and

$$(\diamond) \quad (v_\bullet w^*(a)) \cdot (v'_\bullet w'^*(b)) \in \mathcal{J}(C) \text{ for any}$$

$$C \xleftarrow{v} D \xrightarrow{w} X, \quad C \xleftarrow{v'} D' \xrightarrow{w'} Y,$$

with C, D, D' transitive.

We may consider this condition in the following three cases.

- (1) $X = Y = G/e$.
- (2) $X = Y = G/G$.
- (3) $X = G/G, Y = G/e$.

(1) If $X = Y = G/e$, then $(\diamond)$ is reduced to

$$ab \in \mathcal{J}(G/e) = (p),$$

which implies automatically a or b is in $\mathcal{J}(G/e)$. Thus we can exclude this case.

(2) If $X = Y = G/G$, then condition $(\diamond)$ is equivalent to

$$\begin{aligned} ab \in \mathcal{J}(G/G), \quad \pi^*(a)\pi^*(b) &\in \mathcal{J}(G/G), \\ (\pi_\bullet \pi^*(a)) \cdot b \in \mathcal{J}(G/G), \quad a \cdot (\pi_\bullet \pi^*(b)) &\in \mathcal{J}(G/G), \\ (\pi_\bullet \pi^*(a)) \cdot (\pi_\bullet \pi^*(b)) &\in \mathcal{J}(G/G). \end{aligned}$$

Since $\mathcal{J}(G/e) = (p)$ is prime, it follows that $\pi^*(a)$ or $\pi^*(b)$ is in $\mathcal{J}(G/e)$. Thus we may assume $\pi^*(a) \in (p)$, namely $a \in \mathfrak{m}_p(G/G)$. Then the above conditions are reduced to

$$ab \in \mathcal{J}(G/G), \quad a \cdot (\pi_\bullet \pi^*(b)) \in \mathcal{J}(G/G).$$

The existence of such a and b can be divided into the following two cases. Remark that $\pi^*(b) \notin \mathcal{J}(G/e)$ will imply $b \notin \mathcal{J}(G/G)$.

(2-1) (the case $\pi^*(b) \notin (p)$)

There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/G)$ satisfying

$$\begin{aligned} a &\notin \mathcal{J}(G/G), \quad \pi^*(b) \notin \mathcal{J}(G/e), \\ ab &\in \mathcal{J}(G/G), \quad a \cdot (\pi_\bullet \pi^*(b)) \in \mathcal{J}(G/G). \end{aligned}$$

(2-2) (the case $\pi^*(b) \in (p)$)

There exist $a, b \in \mathfrak{m}_p(G/G)$ satisfying

$$a \notin \mathcal{J}(G/G), \quad b \notin \mathcal{J}(G/G), \quad ab \in \mathcal{J}(G/G).$$

(3) If $X = G/G$ and $Y = G/e$, then for $a \in \Omega(G/G)$ and $b \in \Omega(G/e)$ which are not in $\mathcal{J}$, condition $(\diamond)$ is reduced to

$$(\pi^*(a)) \cdot b \in \mathcal{J}(G/e), \quad a \cdot (\pi_\bullet(b)) \in \mathcal{J}(G/G).$$

Since $b \notin \mathcal{J}(G/e) = (p)$, the condition $(\pi^*(a)) \cdot b \in \mathcal{J}(G/e)$ is equivalent to $\pi^*(a) \in \mathcal{J}(G/e)$, namely to $a \in \mathfrak{m}_p(G/G)$. The existence of such a and b can be divided into the following two cases. Remark that $\pi_\bullet(b) \notin \mathcal{J}(G/G)$ will imply $b \notin \mathcal{J}(G/e)$.

(3-1) (the case $\pi_\bullet(b) \notin \mathcal{J}(G/G)$)

There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying

$$a \notin \mathcal{J}(G/G), \quad \pi_\bullet(b) \notin \mathcal{J}(G/G), \quad a \cdot (\pi_\bullet(b)) \in \mathcal{J}(G/G).$$

(3-2) (the case $\pi_\bullet(b) \in \mathcal{J}(G/G)$)

There exist $a \in \mathfrak{m}_p(G/G)$ and $b \in \Omega(G/e)$ satisfying

$$a \notin \mathcal{J}(G/G), \quad b \notin \mathcal{J}(G/e), \quad \pi_\bullet(b) \in \mathcal{J}(G/G).$$

Note that, in (3-2), the conditions for a and b are completely separated. Moreover since $\mathcal{J}(G/G) \subsetneq \mathfrak{m}_p(G/G)$, such a always exists. Thus (3-2) is reduced to the following.

(3-2)' There exists $b \in \Omega(G/e)$ satisfying

$$b \notin \mathcal{J}(G/e) \text{ and } \pi_\bullet(b) \in \mathcal{J}(G/G).$$

However, this never happens. Indeed, since we have

$$\pi^* \pi_\bullet(\ell) = \ell^q$$

for any $\ell \in \Omega(G/e)$, we obtain

$$\pi_\bullet(\ell) \Rightarrow \pi^* \pi_\bullet(b) \in \mathcal{J}(G/e) \Rightarrow \ell \in \mathcal{J}(G/e).$$

By the arguments so far, $\mathcal{J} \subseteq \Omega$ is not prime if and only if one of (2-1), (2-2), (3-1) is satisfied. Furthermore, we see (2-1) implies (3). Indeed if a and b satisfy (2-1), then $a \in \Omega(G/G)$ and $b' = \pi^*(b) \in \Omega(G/e)$ satisfy

$$\begin{aligned} a &\notin \mathcal{J}(G/G), \quad b' \notin \mathcal{J}(G/e), \\ a \cdot (\pi_\bullet(b')) &\in \mathcal{J}(G/G), \quad \pi^*(a) \cdot b' = \pi^*(ab) \in \mathcal{J}(G/e). \end{aligned}$$

Thus, we can conclude that $\mathcal{J} \subseteq \Omega$ is not prime if and only if one of (2-2), (3-1) is satisfied. These are respectively the conditions (c1), (c2) in the statement of the proposition.

The latter part can be shown easily by using $\mathfrak{m}_p(G/G) = (p, X - q)$. An easy observation $X(X - q) = 0$ will help the calculation. $\square$

3.6. Determine each fiber. Proposition 3.9 enables us to determine the structure of $\text{Spec } \Omega$.

Corollary 3.10. *Let $p \in \mathbb{Z}$ be a prime or $p = 0$. In each fiber $F^{-1}((p))$ over p , we have the following.*

(1) (the case $p \neq q, 0$)

If $p \neq 0$ is a prime other than q , then $I_p \subseteq \Omega$ in Proposition 3.9 is prime. For this reason, in the rest we denote I_p by $\mathfrak{p}_p$.

(2) (the case $p = q$)

$I_q \subseteq \Omega$ is not prime.

(3) (the case $p = 0$)

$\mathcal{J}_{(0;n)} \subseteq \Omega$ in Claim 3.8 is prime if and only if $n = 0$ or $n = \pm 1$.

Proof. (1) It suffices to show that either of (c1)', (c2)' does not occur. Remark that we have $\mathfrak{p}_p(G/G) = (p)$.

(c1)' For any k, n, k', n' , since

$$\begin{aligned} kp + n(X - q) \notin \mathfrak{p}_p(G/G) &\Leftrightarrow p \nmid n, \\ k'p + n'(X - q) \notin \mathfrak{p}_p(G/G) &\Leftrightarrow p \nmid n', \\ kk'p^2 + ((n'k + nk')p + nn'q)(X - q) \in \mathfrak{p}_p(G/G) &\Leftrightarrow p \mid nn', \end{aligned}$$

these never happens simultaneously.

(c2)' For any $k, n, l \in \mathbb{Z}$, since

$$\begin{aligned} kp + n(X - q) \notin \mathfrak{p}_p(G/G) &\Leftrightarrow p \nmid n, \\ \ell + \frac{\ell^q - \ell}{q}X \notin \mathfrak{p}_p(G/G) &\Leftrightarrow p \nmid \ell, \\ kp(\ell + \frac{\ell^q - \ell}{q}X) + n\ell(X - q) \in \mathfrak{p}_p(G/G) &\Leftrightarrow p \mid n\ell, \end{aligned}$$

these never happens simultaneously.

(2) We show (c1) holds for I_q . Remark that we have $I_q(G/G) = (qX, X - q)$.

For $a = b = X \in \mathfrak{m}_q(G/G)$, we have

$$a = b \notin I_q(G/G) \quad \text{and} \quad ab = qX \in I_q(G/G).$$

Thus I_q is not prime.

(3) We already know $(0) \subseteq \Omega$ and $\mathcal{J}_{(0)} \subseteq \Omega$ are prime. It suffices to show $\mathcal{J}_{(0;n)} \subseteq \Omega$ is not prime for $n \notin \{-1, 0, 1\}$. We show (c2) holds for these n . Remark that we have $\mathcal{J}_{(0;n)}(G/G) = (n(X - q))$.

For $a = X - q \in \Omega(G/G)$ and $b = n \in \Omega(G/e)$, we have

$$\begin{aligned} a &\notin \mathcal{J}_{(0;n)}(G/G), \\ \pi_\bullet(b) &= n + \frac{n^q - n}{q}X \notin \mathcal{J}_{(0;n)}(G/G), \\ (X - q) \cdot (\pi_\bullet(b)) &= n(X - q) \in \mathcal{J}_{(0;n)}(G/G). \end{aligned}$$

Thus $\mathcal{J}_{(0;n)}$ is not prime for $n \notin \{-1, 0, 1\}$. $\square$

3.7. Total picture. As a consequence, $\text{Spec } \Omega$ can be determined as

$$\begin{aligned} \text{Spec } \Omega &= (\{(0)\} \cup \{\mathcal{J}(0)\}) \cup \{\mathfrak{m}_q\} \\ &\cup (\{\mathfrak{p}_p \mid p \in \mathbb{Z} \text{ is prime, } p \neq q\} \cup \{\mathfrak{m}_p \mid p \in \mathbb{Z} \text{ is prime, } p \neq q\}). \end{aligned}$$

Inclusions are

$$\begin{aligned} (0) &\subsetneq \mathcal{J}(0) \subsetneq \mathfrak{m}_q \\ \mathfrak{p}_p &\subsetneq \mathfrak{m}_p \quad (p \neq q). \end{aligned}$$

Especially the dimension of $\text{Spec } \Omega$ is 2.

$\mathfrak{m}_q$ and $\mathfrak{m}_p$'s are the closed points, and (0) is the generic point in $\text{Spec } \Omega$. If we represent the points in $\text{Spec } \Omega$ by their closures, $\text{Spec } \Omega$ with fibration F can be depicted as follows. It can be also easily seen that F is continuous.

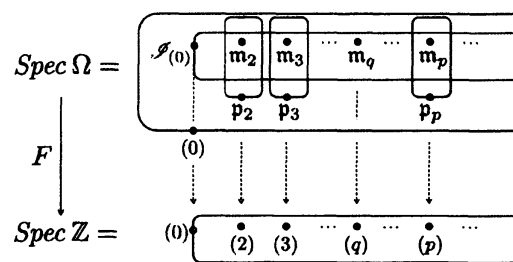


FIGURE 1. $\text{Spec } \Omega$ for $G = \mathbb{Z}/q\mathbb{Z}$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KAGOSHIMA UNIVERSITY, 1-21-35 KORIMOTO, KAGOSHIMA, 890-0065 JAPAN

E-mail address: nakaoka@sci.kagoshima-u.ac.jp