

Eigenfunction Expansions for Symmetric Systems  
of First Order in the Half-Space  $R_+^n$

by

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1. Introduction

Eigenfunction expansion theory by distorted plane waves was initiated by Ikebe [1] and has been investigated by many authors, for example, Shizuta [6], Shenk II [5], Mochizuki [3], Schulenberg and Wilcox [4] and others. We are concerned with stationary problems for symmetric hyperbolic systems with constant coefficients in the half-space  $R_+^n$  and give an expansion theorem by improper eigenfunctions for such a problem. We note that this problem cannot be treated as a perturbation of whole space problem. In fact, our result is a generalization of the sine and cosine transformations in the  $L^2$  space on the positive half-line which are eigenfunction expansions for  $-d^2/dx^2$  in  $(0, \infty)$  with Dirichlet or Neumann conditions at  $x=0$ .

Let  $R^n$  denote the  $n$ -dimensional Euclidean space. Denote by  $x$  the generic point of  $R^n$  and write  $x'=(x_1, \dots, x_{n-1})$ . We shall also denote by  $R_+^n$  the half-space  $\{x=(x', x_n) \in R^n; x_n > 0\}$  and by  $t$

the time variable. Let  $L$  be a first order symmetric hyperbolic operator with constant coefficients:

$$(1) \quad L = I\partial/\partial t - \sum_{j=1}^n A_j \partial/\partial x_j,$$

where  $I$  is the identity matrix of order  $N$  and the  $A_j$  are  $N \times N$  constant Hermitian matrices. We consider the mixed initial and boundary value problem in  $R_+^n$  for the operator  $L$ :

$$(2) \quad \begin{cases} Lu(t,x) = f(t,x), & t > 0, \quad x \in R_+^n, \\ u(0,x) = u_0(x), & x \in R_+^n, \\ Bu(t,x)|_{x_n=0} = 0, & t > 0, \end{cases}$$

where  $u(t,x)$ ,  $f(t,x)$  and  $u_0(x)$  are vector-valued functions whose values lie in the  $N$ -dimensional complex space  $C^N$  and  $B$  is an  $\ell \times N$  constant matrix with rank  $\ell$ . Replacing  $u(t,x)$  and  $f(t,x)$  in (2) by  $e^{ikt}v(x)$  and  $-ie^{ikt}g(x)$ , respectively, we obtain the corresponding stationary problem:

$$(3) \quad \begin{cases} (A - kI)v(x) = g(x), & x \in R_+^n, \\ Bv(x)|_{x_n=0} = 0, \end{cases}$$

where

$$(4) \quad A = i^{-1} \sum_{j=1}^n A_j \partial/\partial x_j.$$

Our aim is to expand an arbitrary function in  $L^2(R_+^n)$  by means of

generalized or improper eigenfunctions for the self-adjoint operator associated with this problem under some suitable conditions for  $L$  (or  $A$ ) and  $B$ .

Let  $p(\lambda, \eta)$  be the characteristic polynomial associated with the operator  $L$ :

$$(5) \quad p(\lambda, \eta) = \det (\lambda I - A(\eta)),$$

where  $\eta$  denotes a generic point of the real dual space  $E^n$  of  $R^n$  by the duality  $x \cdot \eta = x_1 \eta_1 + \dots + x_n \eta_n$  and

$$(6) \quad A(\eta) = \sum_{j=1}^n \eta_j A_j.$$

The polynomials  $p(\lambda, \eta)$  has a factorization

$$(7) \quad p(\lambda, \eta) = Q_1(\lambda, \eta)^{m_1} \dots Q_q(\lambda, \eta)^{m_q},$$

where the factors  $Q_j(\lambda, \eta)$  are distinct homogeneous polynomials in  $(\lambda, \eta)$ , irreducible over the complex number field  $C$ . Since the coefficient of  $\lambda^N$  in  $p(\lambda, \eta)$  is 1, the factors are unique, apart from their order, by requiring the coefficient of the highest power of  $\lambda$  in each  $Q_j(\lambda, \eta)$  be 1. Put

$$(8) \quad Q(\lambda, \eta) = Q_1(\lambda, \eta) \dots Q_q(\lambda, \eta).$$

Definition 1. The operator  $L$  is called uniformly propagative if the roots  $\lambda_j(\eta)$ ,  $1 \leq j \leq \mu$ , of  $Q(\lambda, \eta) = 0$  satisfy the following con-

ditions where  $\mu$  is the order of  $Q(\lambda, \eta)$ : (i) The roots  $\lambda_j(\eta)$  are all distinct for every  $\eta$  with  $|\eta|=1$ . (ii) A root function  $\lambda_j(\eta)$  vanishes for some real  $\eta \neq 0$  if and only if it vanishes identically (see [7]).

Now we state precisely the assumptions that we impose on  $L$  and  $B$ :

(L.1) The operator  $L$  is uniformly propagative.

(L.2) The operator  $A$  is elliptic, i.e.  $p(0, \eta) \neq 0$  for any  $\eta$  in  $\mathbb{E}^n$  with  $|\eta|=1$ .

(L.3) For any real  $\lambda \neq 0$  and any  $\xi \in \mathbb{E}^{n-1}$  the real roots of  $Q(\lambda, \xi, \tau)$   $\neq 0$  with respect to  $\tau$  are at most double and the number of the real double roots for arbitrarily fixed  $(\lambda, \xi) \neq (0, 0)$  is at most one.

(B.1) The boundary matrix  $B$  is minimally conservative, i.e.

$A_n \zeta \cdot \bar{\zeta} = 0$  for any  $\zeta \in \beta = \ker B \subset \mathbb{C}^N$  and if  $\mathcal{E}$  is a subspace of  $\mathbb{C}^N$  such that  $\mathcal{E} \supset \beta$  and  $A_n \zeta \cdot \bar{\zeta} = 0$  for any  $\zeta \in \mathcal{E}$ ,  $\beta = \mathcal{E}$  holds.

Remark 2. The conditions (L.1) and (L.2) imply that the distinct characteristic roots  $\lambda_j(\eta)$ ,  $1 \leq j \leq \mu$ , of the matrix  $A(\eta)$  have constant multiplicities and that  $\mu$  is even. Thus we put  $\mu = 2p$  and can label  $\{\lambda_j(\eta)\}$  in decreasing order:

$$(9) \quad \begin{cases} \lambda_1(n) > \lambda_2(n) > \dots > \lambda_\rho(n) > 0 > \lambda_{\rho+1}(n) > \dots > \lambda_{2\rho}(n), \\ \lambda_{j+\rho}(n) = -\lambda_{\rho-j+1}(-n), \quad 1 \leq j \leq \rho, \quad n \neq 0. \end{cases}$$

Moreover we see that  $N$  is even. Thus we put  $N=2m$ . The condition (B.1) implies that  $\ell=m$ .

Remark 3. The differential operator  $A$  defines an unbounded linear operator  $\mathcal{A}$  in  $L^2(\mathbb{R}_+^n)$  with domain

$$D(\mathcal{A}) = \{v(x) \in C_0^\infty(\overline{\mathbb{R}_+^n}); Bv(x)|_{x_n=0} = 0\}.$$

$\mathcal{A}$  is closable and we denote by  $A$  its closure. Then the condition (B.1) implies that  $A$  is a self-adjoint operator in  $L^2(\mathbb{R}_+^n)$ .

## 2. Eigenfunctions

Let  $G(x, y; \lambda)$  be the Green function for  $(A - \lambda)$ ,  $\text{Im } \lambda \neq 0$ , constructed in [2]. We define projections  $P_j(n)$ ,  $1 \leq j \leq 2\rho$ , by

$$(10) \quad P_j(n) = \begin{cases} \frac{1}{2\pi i} \int_{|\lambda - \lambda_j(n)| = \delta} (\lambda I - A(n))^{-1} d\lambda, & n \neq 0, \\ 0, & n = 0, \end{cases}$$

where  $\delta$  is chosen sufficiently small such that the set  $\{\lambda; |\lambda - \lambda_j(n)| < \delta\}$  contains no roots of  $Q(\lambda, n) = 0$  except  $\lambda_j(n)$ .

Definition 4. Let  $x \in \mathbb{R}_+^n$ ,  $n \in \mathbb{E}^n$  and  $\text{Im } \lambda \neq 0$ . Define

$$(11) \quad \Psi_j(x, n; \lambda) = \overline{\mathcal{F}}_y[G(x, y; \lambda)](n)(\lambda_j(n) - \lambda)P_j(n),$$

$$(12) \quad \psi_j^\pm(x, \eta) = \Psi_j(x, \eta; \lambda_j(\eta) \pm i0), \quad 1 \leq j \leq 2\rho,$$

and

$$(13) \quad \Psi_{j+2\nu\rho}(x, \eta; \lambda) = \frac{\lambda - k_\nu(\xi)}{\lambda - \lambda_j(\eta)} \Psi_j(x, \eta; \lambda) \quad \text{for } \eta \in D_\nu \times E,$$

$$(14) \quad \psi_{j+2\nu\rho}^\pm(x, \eta) = \Psi_{j+2\nu\rho}(x, \eta; k_\nu(\xi) \pm i0), \quad 1 \leq j \leq 2\rho, \\ 1 \leq \nu \leq s \text{ for almost every } \eta \in D_\nu \times E,$$

where the set  $\{k_\nu(\xi)\}_{\nu \in \{j; \xi \in D_j\}}$  is the totality of non-vanishing

zeros of the Lopatinski determinant for the system  $\{A, B\}$  and

$k_i(\xi) \neq k_j(\xi)$  for  $\xi \in D_i \cap D_j$  and  $i \neq j$  (see [8]). Here we define

$G(x, y; \lambda) = 0$  for  $x \in \mathbb{R}_+^n$  and  $y \in \mathbb{R}_+^n$ .

$\psi_j^\pm(x, \eta)$ ,  $\psi_{j+2\nu\rho}^\pm(x, \eta)$  are (improper) eigenfunctions for the operator  $A$ , i.e.

$$(15) \quad \begin{cases} A_x \psi_j^\pm(x, \eta) = \lambda_j(\eta) \psi_j^\pm(x, \eta), \\ B \psi_j^\pm(x, \eta)|_{x_n=0} = 0, \quad 1 \leq j \leq 2\rho, \end{cases}$$

$$(16) \quad \begin{cases} A_x \psi_{j+2\nu\rho}^\pm(x, \eta) = k_\nu(\xi) \psi_{j+2\nu\rho}^\pm(x, \eta), \quad 1 \leq j \leq 2\rho, \quad 1 \leq \nu \leq s, \\ B \psi_{j+2\nu\rho}^\pm(x, \eta)|_{x_n=0} = 0, \quad \text{for almost every } \eta \in D_\nu \times E. \end{cases}$$

### 3. Expansion theorem

Theorem 5. Assume that the conditions (L.1) - (L.3) and (B.1) are satisfied and that  $f \in L^2(\mathbb{R}_+^n)$ .

(i) The expansion formula

$$(17) \quad Pf(x) = \sum_{j=1}^{2\rho} \int_{\mathbb{E}^n} \psi_j^\pm(x, \eta) \hat{f}_j^\pm(\eta) d\eta \\ + \sum_{v=1}^s \sum_{j=1}^{2\rho} \int_{D_v \times \mathbb{E}} \psi_{j+2vp}^\pm(x, \eta) \hat{f}_{j+2vp}^\pm(\eta) d\eta$$

holds, where

$$(18) \quad \hat{f}_j^\pm(\eta) = \int_{\mathbb{R}_+^n} \psi_j^\pm(x, \eta)^* f(x) dx, \quad 1 \leq j \leq 2\rho,$$

$$(19) \quad \hat{f}_{j+2vp}^\pm(\eta) = \int_{\mathbb{R}_+^n} \psi_{j+2vp}^\pm(x, \eta)^* f(x) dx, \quad 1 \leq j \leq 2\rho, \quad 1 \leq v \leq s.$$

Here the above integrals are taken in the sense of limit in the mean and  $P$  is the orthogonal projection onto  $R(A)^{\perp} = N(A)^{\perp}$ .

(ii)  $f \in D(A)$  if and only if  $\lambda_j(\eta) \hat{f}_j^\pm(\eta) \in P_j(\eta) L^2(\mathbb{E}^n)$ ,  $k_v(\xi) \hat{f}_{j+2vp}^\pm(\eta) \in P_j(\eta) L^2(D_v \times \mathbb{E})$ ,  $1 \leq j \leq 2\rho$ ,  $1 \leq v \leq s$ . Then

$$(20) \quad (Af)(x) = \sum_{j=1}^{2\rho} \int_{\mathbb{E}^n} \lambda_j(\eta) \psi_j^\pm(x, \eta) \hat{f}_j^\pm(\eta) d\eta \\ + \sum_{v=1}^s \sum_{j=1}^{2\rho} \int_{D_v \times \mathbb{E}} k_v(\xi) \psi_{j+2vp}^\pm(x, \eta) \hat{f}_{j+2vp}^\pm(\eta) d\eta,$$

$$(21) \quad (Af)_j^{\pm}(\eta) = \lambda_j(\eta) \hat{f}_j^{\pm}(\eta), \quad 1 \leq j \leq 2\rho,$$

$$(22) \quad (Af)_{j+2vp}^{\pm}(\eta) = k_v(\xi) \hat{f}_{j+2vp}^{\pm}(\eta), \quad 1 \leq j \leq 2\rho, \quad 1 \leq v \leq s.$$

Remark 6. (i) The condition (L.2) can be removable. (ii)

We can prove the principles of limiting amplitude and limiting absorption for the operator  $A$  by Theorem 5 and representations of

eigenfunctions (see [9]).

#### 4. Outline of proof

The self-adjoint operator  $\mathbb{A}$  admits a uniquely determined spectral resolution:

$$(23) \quad \mathbb{A} = \int_{-\infty}^{\infty} \lambda dE(\lambda),$$

where  $\{E(\lambda)\}_{-\infty < \lambda < \infty}$  denotes the right-continuous spectral family of  $\mathbb{A}$ . Then it follows from the Stieltjes inversion formula that for  $f \in C_0^\infty(\mathbb{R}_+^n)$  and  $a < b$

$$(24) \quad \left( \{ (E(b) + E(b-0))/2 - (E(a) + E(a-0))/2 \} f, f \right) \\ = \lim_{\varepsilon \rightarrow 0} \pi^{-1} \sum_{j=1}^{2p} \int_{\mathbb{R}^n} d\eta \int_a^b dk \frac{\varepsilon}{(\lambda_j(\eta) - k)^2 + \varepsilon^2} |\hat{f}_j(\eta; k \pm i\varepsilon)|^2,$$

where  $(\cdot, \cdot)$  denotes the inner product of  $L^2(\mathbb{R}_+^n)$  and

$$(25) \quad \hat{f}_j(\eta; \lambda) = \int_{\mathbb{R}_+^n} \psi_j(x, \eta; \lambda)^* f(x) dx, \quad \text{Im } \lambda \neq 0, \quad 1 \leq j \leq 2p.$$

In order to prove the expansion theorem it suffices to show that we can interchange the order of  $\lim_{\varepsilon \rightarrow 0}$  and  $\int_{\mathbb{R}^n} d\eta$  in (24). On the other hand we have

$$(26) \quad \psi_j(x, \eta; \lambda) = (2\pi)^{-n/2} e^{ix \cdot \eta} p_j(\eta) \\ - \frac{1}{i} (2\pi)^{-1/2} \overline{\mathcal{G}_y}, [G(x, y', +0; \lambda)](\xi) A_n p_j(\eta).$$

Thus, the part most involved of our study is to analyse the behav-

ior around the singular points of the second term on the right hand side of (26) where the Lopatinski determinant vanishes. The detailed proof and further results are given in [8].

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