

On a sufficient condition for starlikeness

Mamoru Nunokawa

In [1], Miller, Mocanu and Reade proved the following theorem by applying a geometrical properties.

Theorem A. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $E = \{z : |z| < 1\}$, with $f(z)f'(z)/z \neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α -convex in E if and only if

$$\operatorname{Re} \left[(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0 \quad \text{in } E.$$

Then, if $f(z)$ is α -convex in E , then $f(z)$ is starlike in E and moreover, if $1 \leq \alpha$, then $f(z)$ is convex in E .

In this paper, we improve Theorem A by applying the following lemma [2].

Lemma. Let $p(z)$ be analytic in E , $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

$$\operatorname{Re} p(z_0) = 0 \quad \text{and} \quad p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $1 \leq |k|$.

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and suppose that

$$(1) \quad (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \neq il \quad \text{in } E$$

where α is a positive real number and l is also real number and $\sqrt{\alpha(2+\alpha)} \leq |l|$.

Then $f(z)$ is starlike in E .

Proof. If we put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad (p(0) = 1)$$

then we obtain

$$\begin{aligned} (2) \quad & (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \\ &= p(z) + \alpha \frac{zp'(z)}{p(z)} \end{aligned}$$

From (2) and assumption (1), we obtain $p(z) \neq 0$ in E , because if $p(z)$ has zero of order m at $z=\beta$, $|\beta| < 1$ then we can write

$$p(z) = (z-\beta)^m p_1(z)$$

where $p_1(z)$ is analytic in E , $p_1(\beta) \neq 0$ and m is a positive integer.

Then it follows that

$$\begin{aligned} (3) \quad p(z) &+ \alpha \frac{z p'(z)}{p(z)} \\ &= (z-\beta)^m p_1(z) + \frac{\alpha m z}{z-\beta} + \frac{z p_1'(z)}{p_1(z)}. \end{aligned}$$

In a sufficiently small neighborhood of the point $z=\beta$ the right hand side of (3) can take a pure imaginary complex value whose modulus is sufficiently large.

This contradicts the hypothesis of the theorem and therefore $p(z) \neq 0$ in E .

Suppose that there exists a point $z_0 \in E$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

$$\operatorname{Re} p(z_0) = 0, \quad p(z_0) = ia \quad \text{and} \quad p'(z_0) \neq 0,$$

then, from [2], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}$$

and

$$k \leq -\frac{1}{2} \left(|a| + \frac{1}{|a|} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}$$

For the case $\arg p(z_0) = \frac{\pi}{2}$, ($0 < a$), we have

$$p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = i(a + \alpha k)$$

and

$$\begin{aligned} a + \alpha k &\geq \frac{1}{2} \left\{ (2 + \alpha)a + \frac{\alpha}{a} \right\} \\ &\geq \sqrt{\alpha(2 + \alpha)}. \end{aligned}$$

This contradicts assumption of the theorem and for the case $\arg p(z_0) = -\frac{\pi}{2}$, we also have a contradiction by applying the same method as the above.

This completes the proof.

By applying Theorem 1, we have the following very interesting result.

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and suppose that

$$1 + \frac{z f''(z)}{f'(z)} \neq i\ell \quad \text{in } E$$

where ℓ is a real number and $\sqrt{3} \leq |\ell|$.

Then $f(z)$ is starlike in E .

Remark. By applying the same method as the proof of Theorem 1, Theorem 1 holds to be true for the case $\alpha \leq -2$.

References

1. S. S. Miller, P. T. Mocanu and M. O. Reade, All α -con functions are univalent and starlike, Proc. Amer. Math. Soc., 37(2) (1973), 553-554.

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Department of Mathematics

University of Gunma

Arawaki, Maebashi 371-8510

Japan

e-mail : nunokawa@edu.gunma-u.ac.jp