

Alexander Polynomials of Two-Bridge Links

Taizo KANENOBU

Kobe University

Hartley [H] gave a necessary condition for a polynomial to be the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link. He showed how the coefficients of the polynomial may be read straight from the extended diagram, which is derived from Schubert's normal form of a two-bridge knot or link, and showed the following theorem: If $\Delta(t) = \sum_{i=0}^n (-1)^i a_i t^i$, where $a_i > 0$, is the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link, then for some integer s , $a_0 < a_1 < \dots < a_s = a_{s+1} = \dots = a_{n-s} > \dots > a_n$. On the other hand, using surgery techniques, Bailey [B] presented an algorithm for calculating the Alexander polynomial of a two-bridge link from Conway's diagram. As a corollary to this he proved a conjecture of Kidwell about the linking complexity or geometric intersection numbers of a link in the special case of two-bridge links.

The main results of this paper are Theorems 1 and 3, the former

provides another algorithm for calculating the Alexander polynomial of a two-bridge link putting every two-bridge link in the special form of Conway's diagram. The latter gives some necessary conditions for a polynomial to be the Alexander polynomial of a two-bridge link. These conditions are analogous to Hartley's theorem above. Theorem 2 and Corollary 1 also give some properties of the Alexander polynomial of a two-bridge link, including the Torres condition [T]. Corollary 2 is a conjecture of Kidwell in the case of two-bridge links.

In Section 2, we show some lemmas for Theorems 1 and 2 using Fox's free differential calculus. In Section 3, we summarize some properties of two-bridge links. In Section 4, we state the above-mentioned results. In Section 5, we prove Theorem 3.

1. Preliminaries

In this paper, a link L will mean a piecewise linear embedding of two oriented circles K_1 and K_2 in the 3-sphere S^3 . Two links L and L' are called equivalent, if there is an orientation preserving autohomeomorphism of S^3 , which maps L onto L' . The Alexander polynomial $\Delta(x,y)$ of L is an element of the polynomial ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}] = \Lambda$, and is determined only up to multiplication by a unit $\pm x^i y^j$. Let $G = \pi_1(S^3 - L)$, and let G' be its commutator subgroup. Then $\Lambda = \mathbb{Z}[G/G']$; the basis $\{x, y\}$ of G/G' is always taken to be represented by the meridians of K_1 and K_2 respectively. We will calculate the Alexander polynomial of a link by using Fox's free differential calculus,

see [F], [T].

Throughout this paper, we will often abbreviate a polynomial $f(x,y)$ in Λ to f and will use the following notation;

$$F_n(x,y) = \begin{cases} \sum_{i=0}^{n-1} (xy)^i & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} (xy)^i & \text{if } n < 0. \end{cases}$$

In the figures of this paper we will use a tangle [C], which is a portion of the link diagram containing two arcs. An integral tangle, which is represented by a circle labeled " i " or " $-i$ ", where i is a non-negative integer, is a 2-braid with i or $-i$ crossings, in the manner indicated in Fig. 1.

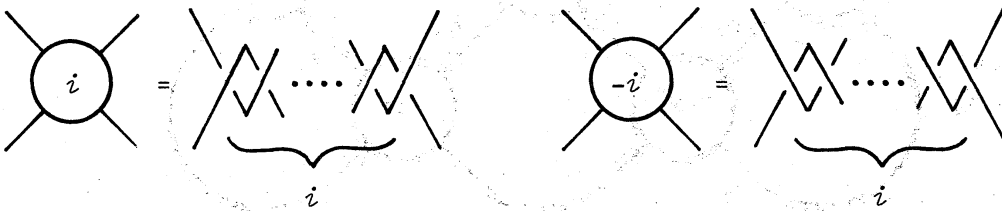


Fig. 1

2. Lemmas

Lemma 1. Let $L(q,r,s,t)$ be a link as shown in Fig. 2, where T is any tangle. Let $\Delta^{(q,r,s,t)}$ be the Alexander polynomial of $L(q,r,s,t)$. If we set $\Delta = \Delta^{(q,r,s,t)}$, $\Delta_0 = \Delta^{(q,r,0,0)}$ and $\Delta_{00} = \Delta^{(0,0,0,0)}$, then

$$(2.1) \quad \Delta = \{s(x-1)(y-1)F_t + 1\}\Delta_0 + \frac{F_t}{F_r}(xy)^r(\Delta_0 - \Delta_{00}),$$

where $r \neq 0$.

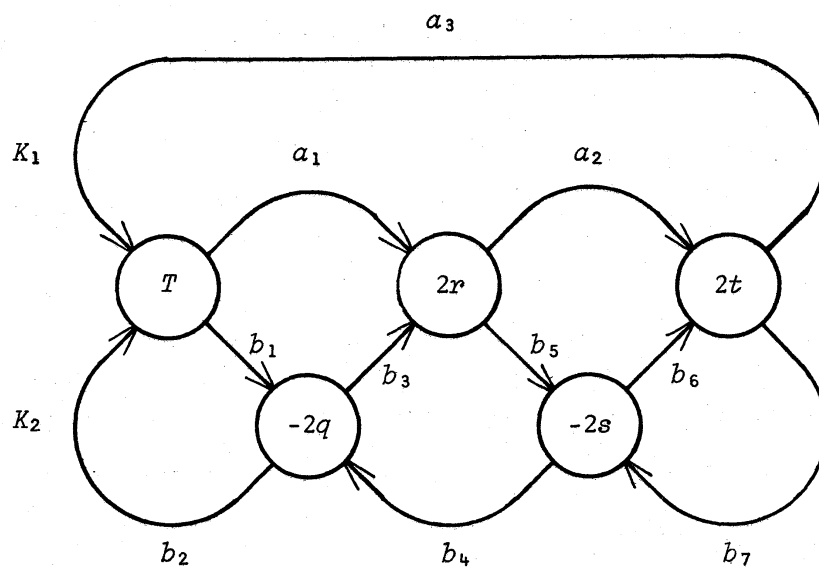


Fig. 2

Proof. We have a presentation of $\pi_1(S^3 - L(q,r,s,t))$ as follows:

generators; $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7,$

$$c_i \quad (7 \leq i \leq n+1),$$

relations; (i) $b_3 = (b_1^{-1}b_2)^q b_1 (b_2^{-1}b_1)^q,$

$$(ii) \quad b_4 = (b_1^{-1}b_2)^q b_2 (b_2^{-1}b_1)^q,$$

$$(iii) \quad a_2 = (b_3 a_1)^r a_1 (a_1^{-1} b_3^{-1})^r,$$

$$(iv) \quad b_5 = (b_3 a_1)^r b_3 (a_1^{-1} b_3^{-1})^r,$$

$$(v) \quad b_6 = (b_5^{-1} b_4)^s b_5 (b_4^{-1} b_5)^s,$$

$$(vi) \quad b_7 = (b_5^{-1} b_4)^s b_4 (b_4^{-1} b_5)^s,$$

$$(vii) \quad a_3 = (b_6 a_2)^t a_2 (a_2^{-1} b_6^{-1})^t,$$

$$(viii) \quad b_7 = (b_6 a_2)^t b_6 (a_2^{-1} b_6^{-1})^t,$$

$$(ix_j) \quad S_j = 1 \quad (5 \leq j \leq n),$$

where c_i and $S_j = 1$ are obtained in the tangle T , so

S_j is a word in $a_1, a_2, b_1, b_2, c_7, \dots, c_{n+1}$.

(i) and (ii), (iii) and (iv), (v) and (vi), (vii) and (viii)

imply

$$(ii') \quad b_4 = b_3 b_1^{-1} b_2,$$

$$(iv') \quad b_5 = b_3 a_1 a_2^{-1},$$

$$(vi') \quad b_7 = b_6 b_5^{-1} b_4,$$

$$(viii') \quad b_7 = b_6 a_2 a_3^{-1},$$

respectively, and we eliminate (ii), (iv), (vi) and (viii).

Using (vi') and (viii'), we have

$$(x) \quad b_5^{-1} b_4 = a_2 a_3^{-1}$$

and we eliminate b_7 . Substituting (iv') and (x) in (v), we have

$$(v') \quad b_6 = (a_2 a_3^{-1})^s b_3 a_1 a_2^{-1} (a_3 a_2^{-1})^s$$

and we eliminate (v). Substituting (v') in (vii), we have $R_1 = 1$, where

$$R_1 = \{(a_2 a_3^{-1})^s b_3 a_1 (a_2^{-1} a_3)^s\}^t a_2 \{(a_3^{-1} a_2)^s a_1^{-1} b_3^{-1} (a_3 a_2^{-1})^s\}^t a_3^{-1}$$

and we eliminate b_6 . Substituting (ii') and (iv') in (x), we have $R_3 = 1$, where

$$R_3 = a_3 a_1^{-1} b_1^{-1} b_2$$

and we eliminate b_4 and b_5 .

Hence $\pi_1(S^3 - L(q, r, s, t))$ is presented by

generators; $a_1, a_2, a_3, b_1, b_2, b_3, c_7, \dots, c_{n+1}$,

relators; $R_1, R_2, R_3, R_4, S_5, \dots, S_n$,

where

$$R_2 = (b_3 a_1)^r a_1 (a_1^{-1} b_3^{-1})^r a_2^{-1}$$

and

$$R_4 = (b_1^{-1} b_2)^q b_1 (b_2^{-1} b_1)^q b_3^{-1},$$

which come from (iii) and (i) respectively.

From this presentation we have the Alexander matrix of $L(q, r, s, t)$ as follows:

$$\begin{array}{c}
R_1 \\
R_2 \\
R_3 \\
R_4 \\
S_5 \\
\vdots \\
S_n
\end{array}
\left[\begin{array}{cccccc|cc}
b_3 & a_2 & a_3 & a_1 & b_2 & b_1 & c_7 & \dots & c_{n+1} \\
(1-x)F_t & f & g & (1-x)yF_t & 0 & 0 & & & \\
(1-x)F_r & -1 & 0 & (y-1)F_r+1 & 0 & 0 & & & \\
0 & 0 & 1 & -1 & y^{-1} & y^{-1} & & & \\
-1 & 0 & 0 & 0 & q(y^{-1}-1) & -q(y^{-1}-1)+1 & & & \\
\hline
0 & 0 & \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \beta_{57} & \dots & \beta_{5,n+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \alpha_{n4} & \beta_{n7} & \dots & \beta_{n,n+1}
\end{array} \right]$$

where $f = s(x-1)(y-1)F_t + (xy)^t$ and $g = -s(x-1)(y-1)F_t - 1$.

Add the third column to the second column, the first column multiplied by $-y$ to the fourth column, and the fifth column to the sixth column, and delete the last column. Then we have an $n \times n$ matrix as follows:

$$(2.2) \quad \left[\begin{array}{cccccc|c}
(1-x)F_t & (xy-1)F_t & g & 0 & 0 & 0 & \\
(1-x)F_r & -1 & 0 & (xy)^r & 0 & 0 & \\
0 & 1 & 1 & -1 & y^{-1} & 0 & \\
-1 & 0 & 0 & y & q(y^{-1}-1) & 1 & \\
\hline
0 & A_1 & A_1 & A_2 & A_3 & A_3+A_4 & B
\end{array} \right]$$

where 0 is a zero column vector of dimension $n-4$, A_i is the

column vector $\begin{bmatrix} \alpha_{51} \\ \vdots \\ \alpha_{ni} \end{bmatrix}$ and B is the matrix $\begin{bmatrix} \beta_{57} & \dots & \beta_{5n} \\ \vdots & & \vdots \\ \beta_{n7} & \dots & \beta_{nn} \end{bmatrix}$.

Let $\Phi^{(q,r,s,t)}$ be the determinant of (2.2). Then

$$(2.3) \quad \Phi^{(q,r,s,t)} = (z-1)\Delta^{(q,r,s,t)},$$

where z is x or y according as c_{n+1} is represented by the

meridian of K_1 or K_2 [T, p.61]. Let C be the $(n-2) \times (n-2)$ matrix obtained from (2.2) by deleting the first and second row, and first and second column. The expansion of (2.2) according to the first row gives us

$$(2.4) \quad \mathbb{H}^{(q,r,s,t)} = \{s(x-1)(y-1)F_t + 1\}\mathbb{H}^{(q,r,0,0)} + \varphi^{(q,r,s,t)}.$$

$$\text{Here } \varphi^{(q,r,s,t)} = (1-x)F_t \begin{vmatrix} -1 & 0 & (xy)^r & 0^T \\ 1 & & & \\ 0 & & C & \\ A_1 & & & \end{vmatrix} \\ - (xy-1)F_t \begin{vmatrix} (1-x)F_r & 0 & (xy)^r & 0^T \\ 0 & & & \\ -1 & & C & \\ 0 & & & \end{vmatrix},$$

where 0^T is a zero row vector of dimension $n-4$. Since

$$\begin{vmatrix} -1 & 0 & (xy)^r & 0^T \\ 1 & & & \\ 0 & & C & \\ A_1 & & & \end{vmatrix} = \begin{vmatrix} -1 & 0 & (xy)^r & 0^T \\ 0 & & & \\ 0 & & C & \\ 0 & & & \end{vmatrix}, \quad \text{we have}$$

$$\varphi^{(q,r,s,t)} = (xy)^r F_t \begin{vmatrix} x-1 & 0 & 1 & 0^T \\ 0 & & & \\ xy-1 & & C & \\ 0 & & & \end{vmatrix}.$$

On the other hand $\Phi^{(q,r,0,0)} = - \begin{vmatrix} (1-x)F_r & -1 & (xy)^r & 0^T \\ 0 & & & \\ -1 & & c & \\ 0 & & & \end{vmatrix}.$

Since $L(q,0,0,0)$ is equivalent to $L(0,0,0,0)$, $\Phi^{(q,0,0,0)} = \Phi^{(0,0,0,0)}$ by (2.3), so we have

$$\Phi^{(q,r,0,0)} - \Phi^{(0,0,0,0)} = F_r \begin{vmatrix} x-1 & 0 & 1-xy & 0^T \\ 0 & & & \\ -1 & & c & \\ 0 & & & \end{vmatrix}.$$

Since it is easily seen that

$$\begin{vmatrix} x-1 & 0 & 1 & 0^T \\ 0 & & & \\ xy-1 & & c & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} x-1 & 0 & 1-xy & 0^T \\ 0 & & & \\ -1 & & c & \\ 0 & & & \end{vmatrix}$$

we have $\Phi^{(q,r,s,t)} = (xy)^{\frac{rF_t}{F_r}} (\Phi^{(q,r,0,0)} - \Phi^{(0,0,0,0)})$ if $r \neq 0$.

Thus it follows from (2.4) and (2.3), we obtain (2.1). $\square$

Lemma 2. Besides the notation in Lemma 1, let $\Delta'_0 = \Delta^{(q,r,0,t)}$ and $\Delta^{(t_0)} = \Delta^{(q,r,s,t_0)}$. Then

$$(2.5) \quad \Delta = s(x-1)(y-1)F_t \Delta_0 + \Delta'_0;$$

$$(2.6) \quad \Delta^{(t)} = F_t \Delta^{(1)} - xy F_{t-1} \Delta_0;$$

$$(2.7) \quad \Delta^{(t)} + xy \Delta^{(t-2)} = (1+xy) \Delta^{(t-1)}.$$

Remarks. (1) In the above notation $\Delta^{(t)} = \Delta$ and $\Delta^{(0)} = \Delta_0$.

(2) (2.7) is a special case of Conway's result [C, p.338], see also [K, p.462].

Proof. Putting $s = 0$ in (2.1) we have

$$\Delta'_0 = \Delta_0 + \frac{F_t}{F_r}(xy)^r(\Delta_0 - \Delta_{00}).$$

Combining this formula with (2.1) we obtain (2.5).

Next, from (2.1) we have

$$\begin{aligned}\Delta^{(t)} - \Delta_0 &= F_t \{s(x-1)(y-1)\Delta_0 + (xy)^r \frac{1}{F_r}(\Delta_0 - \Delta_{00})\} \\ &= F_t(\Delta^{(1)} - \Delta_0).\end{aligned}$$

Since $1 - F_t = -xyF_{t-1}$, we obtain (2.6).

Finally, using (2.6) we have

$$\begin{aligned}\Delta^{(t)} - \Delta^{(t-1)} &= (F_t - F_{t-1})\Delta^{(1)} - xy(F_{t-1} - F_{t-2})\Delta_0 \\ &= (xy)^{t-1}(\Delta^{(1)} - \Delta_0).\end{aligned}$$

Hence

$$\Delta^{(t)} - \Delta^{(t-1)} = xy(\Delta^{(t-1)} - \Delta^{(t-2)}),$$

and (2.7) follows. $\square$

3. Two-bridge links

According to Conway's presentation [C], every two-bridge link can be put in the form as shown in Fig. 3. It will be denoted by $C(a_1, a_2, \dots, a_n)$ including the indicated orientation of each component. The diagram is slightly different in the cases $n = 2k$ and $n = 2k+1$, as indicated in Fig. 3. From this projection we can see that a two-bridge link is a link with two components each of which is a trivial knot. Moreover a two-bridge link is interchangeable, that is, there is an isotopy of S^3 which interchanges the two components. This follows immediately from Schubert's normal form [Sc], or Bailey [B, p.48] also proves this using Conway's diagram.

Let $\alpha (> 0)$ and β be the coprime integers computed from the continued fraction:

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

Then α is even and $0 < |\beta| < \alpha$. The two-fold cover of S^3 branched over this link is the lens space $L(\alpha, \beta)$, see [C], [S]. This link is equivalent to the link with Schubert's normal form (α, β) , denoted by $S(\alpha, \beta)$, endowed with suitable orientations. According to Schubert [Sc, p.144], $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{2\alpha}$. Furthermore, if $\beta' \equiv \beta + \alpha \pmod{2\alpha}$ or $\beta\beta' \equiv \alpha + 1 \pmod{2\alpha}$, then $S(\alpha, \beta)$ differs from $S(\alpha, \beta')$ only by the orientation of one of the components (cf. [S, p.7]).

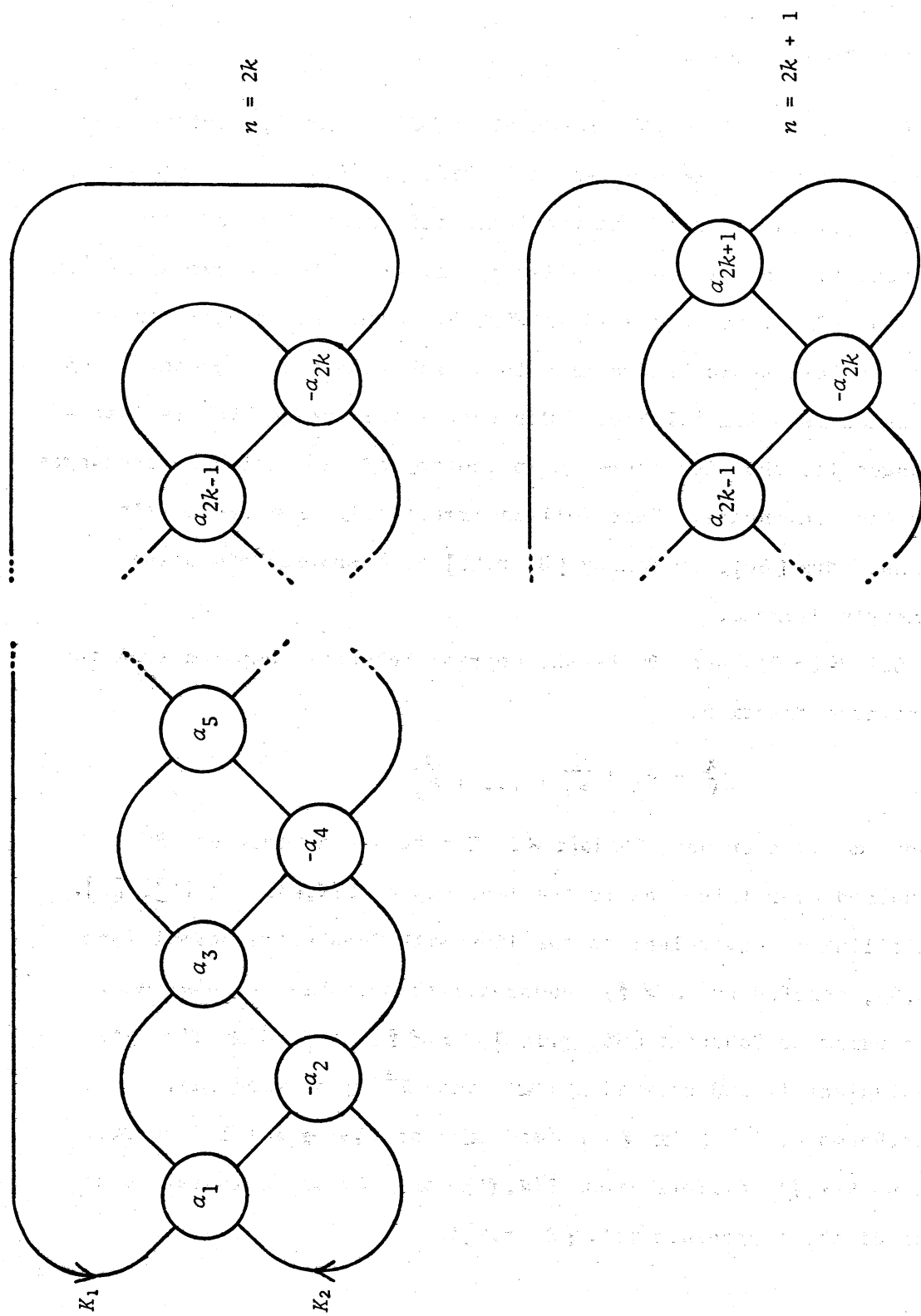


Fig. 3

We can obtain easily another continued fraction:

$$\frac{\alpha}{\beta} = 2b_1 + \frac{1}{2b_2} + \dots + \frac{1}{2b_m},$$

where m is odd. $C(2b_1, 2b_2, \dots, 2b_m)$ is then equivalent to $C(a_1, a_2, \dots, a_n)$ and will be denoted by $D(b_1, b_2, \dots, b_m)$. In the following we will consider every two-bridge link putting in this form (cf. [S, p.13]).

4. Main theorems

From Lemma 1, we have

Theorem 1. Let $L_0 = D(0)$ and for $n \geq 1$ let $L_n = D(p_1, q_1, p_2, q_2, \dots, p_{n-1}, q_{n-1}, p_n)$, where $\prod_{i=1}^n p_i \prod_{j=1}^{n-1} q_j \neq 0$. Let $\Delta_n(x, y)$

be the polynomial inductively defined as follows:

$$\Delta_0 = 0;$$

$$(4.1) \quad \Delta_1 = F_{p_1};$$

$$\Delta_n = \{q_{n-1}(x-1)(y-1)F_{p_n} + 1\}\Delta_{n-1} + (xy)^{p_{n-1}} \frac{F_{p_n}}{F_{p_{n-1}}}(\Delta_{n-1} - \Delta_{n-2}),$$

for $n \geq 2$.

Then $\Delta_n(x, y)$ is the Alexander polynomial of L_n .

In the following, by the Alexander polynomial of a two-bridge link we mean the polynomial defined in Theorem 1 and we will use the following notation besides that in Theorem 1. Let $\Delta_n^{(p)}$ be the Alexander polynomial of $D(p_1, q_1, p_2, q_2, \dots, p_{n-1}, q_{n-1}, p)$; thus $\Delta_n^{(p_n)} = \Delta_n$ and $\Delta_n^{(0)} = \Delta_{n-1}$. Let $l_n = \sum_{i=1}^n p_i$, that is the

linking number of L_n . Let $\tilde{l}_n = \sum_{i=1}^n |p_i|$.

From Lemma 2, we have

Theorem 2.

$$(4.2) \quad \Delta_n = q_{n-1}(x-1)(y-1)F_{p_n}\Delta_{n-1} + \Delta_{n-1}^{(p_{n-1} + p_n)};$$

$$(4.3) \quad \Delta_n^{(p)} = F_{p_n}\Delta_n^{(1)} - xyF_{p-1}\Delta_{n-1};$$

$$(4.4) \quad \Delta_n^{(p)} + xy\Delta_n^{(p-2)} = (1 + xy)\Delta_n^{(p-1)}.$$

Using (4.4) or Theorem 1 we can show easily the following each formula by induction on n .

Corollary 1.

$$(4.5) \quad \Delta_n(x, y) = \Delta_n(y, x);$$

$$(4.6) \quad \Delta_n(x, y) \equiv F_{l_n}(x, y) \pmod{(x-1)(y-1)};$$

$$(4.7) \quad \Delta_n(x, y) = (xy)^{l_n-1} \Delta_n(x^{-1}, y^{-1}).$$

The fact that a two-bridge link is interchangeable assures us of (4.5). From (4.6), we have immediately

$$(4.8) \quad \Delta_n(x, 1) = F_{l_n}(x, 1).$$

(4.7) and (4.8) constitute the Torres conditions [T] for two-bridge links.

Definition 1. Let $f(x, y)$ be a polynomial in Λ . If $f(x, y) \neq 0$, then $\deg_x f = (\text{maximum } x\text{-power of any term of } f) \text{ minus } (\text{minimum } x\text{-power of any term of } f)$. If $f(x, y) = 0$, then

$\deg_x f = -1$. We define $\deg_y f$ in the same way.

Definition 2. $\Lambda^{+1}(r,s)$ denotes the set of all polynomials $f(x,y) = \sum_{r \leq i, j \leq s} a_{ij} x^i y^j$ in Λ satisfying the following conditions.

(i) $\deg_x f = \deg_y f = s-r$.

(ii) Let $M(f) = \begin{bmatrix} a_{sr} & \dots & a_{ss} \\ \vdots & & \vdots \\ a_{rr} & \dots & a_{rs} \end{bmatrix}$ and $W(f) = \begin{bmatrix} a_{rr} & \dots & a_{rs} \\ \vdots & & \vdots \\ a_{sr} & \dots & a_{ss} \end{bmatrix}$.

Then both $M(f)$ and $W(f)$ are symmetric matrices.

(iii) $a_{ij} \geq 0$ if $i+j$ is even and $a_{ij} \leq 0$ if $i+j$ is odd.

(iv) Let $b_{ij} = a_{i+r, j+r}$. Then

$$|b_{k,0}| \leq |b_{k-1,1}| \leq \dots \leq |b_{k-u,u}|,$$

and $|b_{k,0}| \leq |b_{k+1,1}| \leq \dots \leq |b_{k+v,v}|$

for $0 \leq k \leq s-r$, where $u = \left[\frac{k}{2}\right]^*$ and $v = \left[\frac{k-s+r}{2}\right] + 1$.

Furthermore $\Lambda^{-1}(r,s)$ denotes the set of all polynomials $f(x,y)$ in Λ such that $-f(x,y) \in \Lambda^{+1}(r,s)$.

Theorem 3. For $n \geq 1$, $\Delta_n \in \Lambda^{\mathcal{E}_n}(r_n, s_n)$, where $\mathcal{E}_n =$

$$\prod_{i=1}^n \frac{p_i}{|p_i|} \prod_{j=1}^{n-1} \frac{q_j}{|q_j|}, \quad r_n = \frac{l_n - \tilde{l}_n}{2} \quad \text{and} \quad s_n = \frac{l_n + \tilde{l}_n}{2} - 1.$$

Note that $r_n \leq 0 \leq s_n$, $r_n - r_{n-1} = \frac{p_n - |p_n|}{2}$ and $s_n - s_{n-1} = \frac{p_n + |p_n|}{2}$. The proof of Theorem 3 will be given in Section 5.

*) $[]$ denotes the Gaussian symbol.

Concerning the reduced Alexander polynomial of a two-bridge link, Theorem 3 yields the weaker result than that of Hartley stated in the beginning of this paper.

For the sake of Corollary 2 below, we need some preliminaries.

Definition 3. Let $L = K_1 \cup K_2$ be a link and S be a Seifert surface for K_1 with S and K_2 in general position. If $\delta_S = 2(\text{genus of } S) \text{ plus (the number of times } K_2 \text{ intersects } S)$, then $\delta_1 = \min_S \delta_S$ is the linking complexity of K_2 with K_1 . We define δ_2 in the same way. We call the ordered pair (δ_1, δ_2) the linking complexity of the link L .

This definition follows Bailey [B, p.45], see also [K].

Proposition 1. (Kidwell) If $\Delta(x,y)$ is the Alexander polynomial of a link L with linking complexity (δ_1, δ_2) , then $\delta_1 - 1 \geq \deg_x \Delta(x,y)$.

Proof. See [B, p.46]. $\square$

Corollary 2. Let (δ_1, δ_2) be the linking complexity of L_n . Then

$$(4.9) \quad \delta_1 = \delta_2;$$

$$(4.10) \quad \deg_x \Delta_n(x,y) + 1 = \delta_1 = \tilde{\delta}_n.$$

Remark. The first equality of (4.10) is Proposition 6 of [B, p.57].

Proof. (4.9) follows from interchangeability of a two-bridge link or (4.10). For (4.10), from the diagram of L_n , we see that $\delta_1 \leq \tilde{\lambda}_n$. By Theorem 3, $\deg_x \Delta_n + 1 = \tilde{\lambda}_n$ and by Proposition 1, $\delta_1 \geq \deg_x \Delta_n + 1$. $\square$

5. Proof of Theorem 3

In this section we use the following trivial lemma without mention.

Lemma 3. Let $f \in \Lambda^\varepsilon(r, s)$ and $g \in \Lambda^\varepsilon(r-k, s+k)$ ($k \geq 0$). Then $f + g \in \Lambda^\varepsilon(r-k, s+k)$.

Lemma 4. Let $f \in \Lambda^\varepsilon(r, s)$. Then

$$F_n f \in \begin{cases} \Lambda^\varepsilon(r, s+n-1) & \text{if } n > 0, \\ \Lambda^{-\varepsilon}(r+n, s-1) & \text{if } n < 0, \end{cases}$$

$$G_n f \in \Lambda^{(-1)^{n-1}\varepsilon}(r, s+n-1) \text{ if } n > 0,$$

where $G_n(x, y) = x^{n-1} F_n(x^{-1}, y)$.

Proof. We show that $f \in \Lambda^{+1}(r, s)$ implies $F_n f \in \Lambda^{+1}(r, s+n-1)$ if $n > 0$. The other case can be proved similarly.

It is clear that $F_n f$ satisfies the conditions (i), (ii), (iii) and the first inequality of (iv) in Definition 2. The second inequality of (iv) can be reduced to Sublemma below. $\square$

Sublemma. Let $f(x) = \sum_{i=0}^n a_i x^i$, where $a_i = a_{n-i}$ and $0 < a_0 \leq a_1 \leq \dots \leq a_{[n/2]}$. Let $(\sum_{j=0}^m x^j)f(x) = \sum_{k=0}^{m+n} b_k x^k$. Then $b_k = b_{m+n-k}$ and $0 < b_0 \leq b_1 \leq \dots \leq b_{[(m+n)/2]}$.

Proof. We proceed by induction on n . For $n = 0, 1$, the sublemma is trivial. Assume the sublemma proved for polynomials of degree $< n$. Write $f(x) = a_0 \sum_{i=0}^n x^i + xg(x)$, where $g(x) = \sum_{j=0}^{n-2} a_{j+1} x^j$. Let $(\sum_{j=0}^m x^j)g(x) = \sum_{i=0}^{m+n-2} c_i x^i$. Then $c_i = c_{m+n-2-i}$ and $0 < c_0 \leq c_1 \leq \dots \leq c_{[(m+n-2)/2]}$ by inductive hypothesis.

Thus if $(\sum_{j=0}^m x^j)f(x) = \sum_{k=0}^{m+n} b_k x^k$, it is easy to see that $b_k = b_{m+n-k}$ and $0 < b_0 \leq b_1 \leq \dots \leq b_{[(m+n)/2]}$. $\square$

Lemma 5. If $\Delta_{n-1} \in \Lambda^{-\varepsilon}(r, s-1)$ and $\Delta_n^{(1)} \in \Lambda^{\varepsilon}(r, s)$, then

$$\Delta_n^{(p)} \in \begin{cases} \Lambda^{\varepsilon}(r, s+p-1) & \text{if } p > 0, \\ \Lambda^{-\varepsilon}(r+p, s-1) & \text{if } p < 0. \end{cases}$$

Proof. (4.2) in Theorem 2 states that $\Delta_n^{(p)} = F_p \Delta_n^{(1)} - xyF_{p-1} \Delta_{n-1}$. The case $p = 1$ is the hypothesis. If $p \geq 2$, then using Lemma 4, $F_p \Delta_n^{(1)} \in \Lambda^{\varepsilon}(r, s+p-1)$ and $-xyF_{p-1} \Delta_{n-1} \in \Lambda^{\varepsilon}(r+1, s+p-2)$. Thus $\Delta_n^{(p)} \in \Lambda^{\varepsilon}(r, s+p-1)$. If $p \leq -1$, then $F_p \Delta_n^{(1)}, -xyF_{p-1} \Delta_{n-1} \in \Lambda^{-\varepsilon}(r+p, s-1)$, so $\Delta_n^{(p)} \in \Lambda^{-\varepsilon}(r+p, s-1)$. $\square$

Lemma 6. Let $\Delta_n^{<m>}$ be the Alexander polynomial of $D(p_1, q_1, \dots, p_{n-m}, q_{n-m}, 1, q_{n-m+1}, 1, \dots, q_{n-1}, 1)$. Then we have

$$(5.1) \quad \Delta_n^{<m>} = G_{m+1} \Delta_{n-m} - xy G_m \Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1) \sum_{k=1}^m (q_{n-k}+1) G_k \Delta_{n-k},$$

where the last term denotes zero if $m = 0$.

Proof. We prove (5.1) by induction on m . For $m = 0$, it is clear that $\Delta_n^{<0>} = \Delta_n$. Assume that (5.1) proved for $m-1$.

Substituting $p_{n-m+1} = 1$ in $\Delta_n^{<m-1>}$ we have

$$\Delta_n^{<m>} = G_m \Delta_{n-m+1}^{(1)} - xy G_{m-1} \Delta_{n-m+1}^{(0)} + (x-1)(y-1) \sum_{k=1}^{m-1} (q_{n-k}+1) G_k \Delta_{n-k}.$$

By (4.2), $\Delta_{n-m+1}^{(1)} = q_{n-m}(x-1)(y-1) \Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)}$. Thus we have

$$\begin{aligned} \Delta_n^{<m>} &= G_m \{ -(x-1)(y-1) \Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)} \} - xy G_{m-1} \Delta_{n-m} \\ &\quad + (x-1)(y-1) \sum_{k=1}^m (q_{n-k}+1) G_k \Delta_{n-k}. \end{aligned}$$

By (4.4), $\Delta_{n-m}^{(p_{n-m}+1)} = (xy+1) \Delta_{n-m} - xy \Delta_{n-m}^{(p_{n-m}-1)}$. Thus we have

$$\begin{aligned} \Delta_n^{<m>} &= \{ (x+y) G_m - xy G_{m-1} \} \Delta_{n-m} - xy G_m \Delta_{n-m}^{(p_{n-m}-1)} \\ &\quad + (x-1)(y-1) \sum_{k=1}^m (q_{n-k}+1) G_k \Delta_{n-k}. \end{aligned}$$

Since $(x+y) G_m - xy G_{m-1} = G_{m+1}$, we have (5.1). $\square$

Now we are in position to prove Theorem 3. We use induction on n . For $n = 1$, the theorem is clear. Assume the theorem proved for Δ_k , where $1 \leq k \leq n-1$. Without loss of generality

we may suppose that $q_{n-1} < 0$. By Lemma 5 we have only to prove for the case $p_n = 1$. Then there exists an integer m such that:

$$(I) \ 1 \leq m \leq n-1, \quad p_{n-m+1} = p_{n-m+2} = \dots = p_{n-1} = 1, \quad p_{n-m} \neq 1$$

and $q_{n-m}, q_{n-m+1}, \dots, q_{n-1} < 0$,

$$(II) \ 1 \leq m \leq n-2, \quad p_{n-m} = p_{n-m+1} = p_{n-m+2} = \dots = p_{n-1} = 1,$$

$q_{n-m}, q_{n-m+1}, \dots, q_{n-1} < 0$ and $q_{n-m-1} > 0$, or

$$(III) \ m = n-1, \quad p_1 = p_2 = \dots = p_{n-1} = 1, \quad q_1, q_2, \dots, q_{n-1} < 0.$$

To prove Theorem 3, it suffices to prove that $\Delta_{n-m} \in \Lambda^{\epsilon}(r, s)$ implies $\Delta_n \in \Lambda^{(-1)^m \epsilon}(r, s+m)$, where by Lemma 6

$$(5.2) \quad \Delta_n = G_{m+1} \Delta_{n-m} - xy G_m \Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1) \sum_{k=1}^m (q_{n-k}+1) G_k \Delta_{n-k}.$$

By Lemma 4, we have

$$(5.3) \quad G_{m+1} \Delta_{n-m} \in \Lambda^{(-1)^m \epsilon}(r, s+m).$$

By inductive hypothesis, $\Delta_{n-k} \in \Lambda^{(-1)^{m-k} \epsilon}(r, s+m-k)$ for $1 \leq k \leq m$. Then by Lemma 4, $G_k \Delta_{n-k} \in \Lambda^{(-1)^{m-1} \epsilon}(r, s+m-1)$; hence we obtain

$$(5.4) \quad (x-1)(y-1) \sum_{k=1}^m (q_{n-k}+1) G_k \Delta_{n-k} \begin{cases} = 0 & \text{if } q_{n-k} = -1 \text{ for any } k, \\ \in \Lambda^{(-1)^m \epsilon}(r, s+m) & \text{otherwise.} \end{cases}$$

Case (I). If $p_{n-m} \neq 1$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{\epsilon}(r, s-1) & \text{if } p_{n-m} \geq 2, \\ \Lambda^{\epsilon}(r-1, s) & \text{if } p_{n-m} \leq -1. \end{cases}$$

Thus, using Lemma 4, we have

$$(5.5) \quad -xyG_m \Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{(-1)^m \varepsilon}_{(r+1, s+m-1)} & \text{if } p_{n-m} \geq 2, \\ \Lambda^{(-1)^m \varepsilon}_{(r, s+m)} & \text{if } p_{n-m} \leq -1. \end{cases}$$

Case (II). If $p_{n-m} = 1$ and $q_{n-m-1} > 0$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} = \Delta_{n-m-1} \in \Lambda^{\varepsilon}_{(r, s-1)}.$$

Thus, using Lemma 4, we have

$$(5.6) \quad -xyG_m \Delta_{n-m}^{(p_{n-m}-1)} \in \Lambda^{(-1)^m \varepsilon}_{(r+1, s+m-1)}.$$

Case (III). Since $m = n-1$ and $p_1 = 1$, we have

$$(5.7) \quad -xyG_m \Delta_{n-m}^{(p_{n-m}-1)} = 0.$$

From (5.2) ~ (5.7), we have $\Delta_n \in \Lambda^{(-1)^m \varepsilon}_{(r, s+m)}$. This completes the proof of Theorem 3.

APPENDIX

Alexander Polynomials of Two-Bridge Links of 10 Crossings

For every two-bridge link of 10 crossings in the table of Conway [C, p.353], we list the Alexander polynomial. Two-bridge links are presented by Conway's notation; $p_1 p_2 \dots p_n$ denotes a two-bridge link with the notation $C(p_1, p_2, \dots, p_n)$ in this paper. The Alexander polynomial is abbreviated in the same manner as Rolfsen's table [R, Appendix C].

60

10

1
1
1
1
1

5122

-2 1
-2 3 -2
-2 3 -2
1 -2

73

1
1 -1 1
1 -1 1
1 -1 1
1

442

-2 3
-2 5 -2
3 -2

622

-1 2
-1 3 -1
-1 3 -1
2 -1

424

-1 3
-2 4 -1
-1 4 -2
3 -1

55

1
1 -1 1
1 -1 1 -1 1
1 -1 1
1

4213

-1 2
-1 3 -3 2
2 -3 3 -1
2 -1

523

1
2 -2 1
2 -3 2
1 -2 2
1

41212

1 -2 1
1 -3 3 -2
-2 3 -3 1
1 -2 1

411112

$$\begin{array}{cccc} & 1 & -2 & 1 \\ 1 & -3 & 5 & -2 \\ -2 & 5 & -3 & 1 \\ 1 & -2 & 1 & \end{array}$$

3142

$$\begin{array}{cc} & -3 & 2 \\ -3 & 5 & -3 \\ 2 & -3 & \end{array}$$

343

$$\begin{array}{ccccccc} & & 1 & & & & \\ & 2 & -2 & 1 & & & \\ 1 & -2 & 3 & -2 & 1 & & \\ & 1 & -2 & 2 & & & \\ & & 1 & & & & \end{array}$$

31213

$$\begin{array}{ccc} & -2 & 1 \\ & -4 & 5 & -2 \\ -2 & 5 & -4 \\ 1 & -2 & \end{array}$$

3322

$$\begin{array}{cccc} & 2 & -1 & \\ 2 & -4 & 4 & -1 \\ -1 & 4 & -4 & 2 \\ & -1 & 2 & \end{array}$$

311212

$$\begin{array}{cccc} & 1 & -2 & 1 \\ 1 & -5 & 5 & -2 \\ -2 & 5 & -5 & 1 \\ 1 & -2 & 1 & \end{array}$$

3223

$$\begin{array}{ccccccc} & & 1 & & & & \\ & 2 & -3 & 2 & & & \\ 1 & -3 & 5 & -3 & 1 & & \\ & 2 & -3 & 2 & & & \\ & & 1 & & & & \end{array}$$

3111112

$$\begin{array}{cccc} & 1 & -2 & 1 \\ 1 & -5 & 7 & -2 \\ -2 & 7 & -5 & 1 \\ 1 & -2 & 1 & \end{array}$$

32122

$$\begin{array}{cccc} & 1 & -2 & \\ 1 & -5 & 5 & -2 \\ -2 & 5 & -5 & 1 \\ & -2 & 1 & \end{array}$$

262

$$\begin{array}{cc} -3 & 4 \\ 4 & -3 \end{array}$$

23212

$$\begin{array}{ccc} 2 & -4 & 2 \\ -4 & 7 & -4 \\ 2 & -4 & 2 \end{array}$$

231112

$$\begin{array}{ccc} 2 & -4 & 2 \\ -4 & 9 & -4 \\ 2 & -4 & 2 \end{array}$$

22222

$$\begin{array}{ccc} 1 & -4 & 4 \\ -4 & 9 & -4 \\ 4 & -4 & 1 \end{array}$$

221122

$$\begin{array}{ccc} 2 & -5 & 2 \\ -5 & 9 & -5 \\ 2 & -5 & 2 \end{array}$$

21412

$$\begin{array}{cccc} -1 & 2 & -1 & \\ 2 & -3 & 2 & -1 \\ -1 & 2 & -3 & 2 \\ -1 & 2 & -1 & \end{array}$$

213112

$$\begin{array}{cccc} -1 & 2 & -1 & \\ 2 & -5 & 4 & -1 \\ -1 & 4 & -5 & 2 \\ -1 & 2 & -1 & \end{array}$$

2112112

$$\begin{array}{cccc} 1 & -2 & 1 & \\ 1 & -6 & 7 & -2 \\ -2 & 7 & -6 & 1 \\ 1 & -2 & 1 & \end{array}$$

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Department of Mathematics
 Kobe University
 Nada, Kobe, 657
 Japan