

ON THE EXTENSIONS OF $\mathcal{W}_{n,A}$ BY $\mathbb{G}_{m,A}$

TSUTOMU SEKIGUCHI*) AND NORIYUKI SUWA†)

1. MOTIVATION

Let $(A, \mathfrak{M})$ be a DVR with $K = Q(A)$ of characteristic 0 and $k = A/\mathfrak{M}$ of characteristic $p > 0$.

The so called Artin-Schreier-Witt exact sequence

$$(1) \quad 0 \rightarrow \mathbb{Z}/p^n \rightarrow W_{n,k} \xrightarrow{F^n - \text{id}} W_{n,k} \rightarrow 0$$

describes any étale p^n -cyclic coverings, where $W_{n,k}$ is the group scheme over k of Witt vectors of length n and F is the Frobenius endomorphism.

On the other hand, when K contains μ_{p^n} , any étale p^n -cyclic coverings are described by the Kummer sequence

$$(2) \quad 0 \rightarrow \mu_{p^n, K} \rightarrow \mathbb{G}_{m, K} \xrightarrow{\theta_{p^n}} \mathbb{G}_{m, K} \rightarrow 0.$$

But we do not like to have two Gods (1) and (2) in the world.

In fact, we can construct a Kummer-Artin-Schreier-Witt exact sequence over DVR $A = \mathbb{Z}_{(p)}[\mu_{p^n}]$:

$$(3) \quad 0 \rightarrow (\mathbb{Z}/p^n)_A \rightarrow \mathcal{W}_n \rightarrow \mathcal{W}_n/(\mathbb{Z}/p^n)_A \rightarrow 0$$

with an exact sequence of Kummer type as the generic fibre:

$$(4) \quad 0 \rightarrow \mu_{p^n, K} \rightarrow (\mathbb{G}_{m, K})^n \rightarrow (\mathbb{G}_{m, K})^n \rightarrow 0$$

and with (1) as the special fibre (cf. [12, 15]).

In $n = 1$ case, the exact sequence (3) is given explicitly as follows:

Let ζ be a primitive p -th root of unity, $\lambda = \zeta - 1$ and $A = \mathbb{Z}_{(p)}[\zeta]$. We define $\mathcal{W}_1$ by the group scheme

$$\mathcal{G}^{(\lambda)} = \text{Spec} A[x, \frac{1}{\lambda x + 1}]$$

with group law $x \cdot y = x + y + \lambda xy$. Then (3) is given by

$$(5) \quad 0 \rightarrow (\mathbb{Z}/p)_A \rightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\Psi} \mathcal{G}^{(\lambda)}/(\mathbb{Z}/p) \cong \mathcal{G}(\lambda^p) \rightarrow 0,$$

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where $\Phi(x) = \frac{1}{\lambda^p} \{(\lambda x + 1)^p - 1\}$. The exact sequence (3) for general n is given by taking suitable extensions step by step starting from the exact sequence (5); that is to say, if we construct $\mathcal{W}_n$ for an n , then the next $\mathcal{W}_{n+1}$ is given by an extension of $\mathcal{W}_n$ by $\mathcal{G}^{(\lambda)}$:

$$(6) \quad 0 \rightarrow \mathcal{G}^{(\lambda)} \rightarrow \mathcal{W}_{n+1} \rightarrow \mathcal{W}_n \rightarrow 0 \in \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

On the other hand, some matters concerning of $\mathcal{G}^{(\lambda)}$ can be calculated by using the exact sequence

$$(7) \quad 0 \rightarrow \mathcal{G}^{(\lambda)} \xrightarrow{\alpha^{(\lambda)}} \mathbb{G}_{m,A} \xrightarrow{r^{(\lambda)}} i_* \mathbb{G}_{A/\lambda} \rightarrow 0,$$

where $i : \text{Spec} A/\lambda \hookrightarrow \text{Spec} A$ is the canonical inclusion, $\alpha^{(\lambda)}(x) = \lambda x + 1$ and $r^{(\lambda)}t \equiv t \pmod{\lambda}$.

In fact, using this exact sequence (7), we can obtain a long exact sequence

$$(8) \quad \begin{aligned} 0 &\rightarrow \text{Hom}(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \rightarrow \text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A}) \rightarrow \text{Hom}(\mathcal{W}_n, i_* \mathbb{G}_{m,A/\lambda}) \\ &\xrightarrow{\partial} \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) \rightarrow \text{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}). \end{aligned}$$

Here we have $\text{Ext}^1(\mathcal{W}_n, \mathbb{G}_{m,A}) = 0$ by Hilbert theorem 90. Therefore for our purpose to search $\mathcal{W}_{n+1}$, to calculate

$$\text{Hom}(\mathcal{W}_n, i_* \mathbb{G}_{m,A/\lambda}) \cong \text{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$$

is important.

Moreover to determine explicitly the quotient $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$ is crucial when we apply our theory to the lifting problems of p^n -cyclic coverings of curves as was expanded by B. Green and M. Matignon [4]. When once we construct the quotient $\mathcal{W}_n/(\mathbb{Z}/p^n)_A$, the next one $\mathcal{W}_{n+1}/(\mathbb{Z}/p^{n+1})_A$ is given in $\text{Ext}^1(\mathcal{W}_n/(\mathbb{Z}/p^n)_A, \mathcal{G}^{(\lambda^p)})$, and it is fixed explicitly by calculating $\text{Hom}(\mathcal{W}_{n,A/\lambda^p}, \mathbb{G}_{m,A/\lambda^p})$.

Our aim of this report is to determine explicitly the groups $\text{Hom}(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$ and $\text{Ext}^1(\mathcal{W}_{n,A/\lambda}, \mathbb{G}_{m,A/\lambda})$.

2. THE STRUCTURE OF $\mathcal{W}_n$

By using the exact sequence (8), for $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathfrak{M} \setminus \{0\}$, $\mathcal{W}_n$ can be written in the form:

$$(9) \quad \mathcal{W}_n = \text{Spec} A[X_0, X_1, \dots, X_{n-1}, \frac{1}{\lambda_1 X_0 + 1}, \frac{1}{\lambda_2 X_1 + D_1(X_0)}, \frac{1}{\lambda_3 X_2 + D_2(X_0, X_1)}, \dots, \frac{1}{\lambda_n X_{n-1} + D_{n-1}(X_0, \dots, X_{n-2})}]$$

where for each $i = 1, 2, \dots, n-1$,

$$D_i : \text{Spec} A / \lambda_{i+1} [X_0, \dots, X_{i-1}, \frac{1}{\lambda_1 X_0 + 1}, \dots, \frac{1}{\lambda_i X_{i-1} + D_{i-1}}] \rightarrow \mathbb{G}_{m,A/\lambda_{i+1}}$$

is a homomorphism and we understand that $D_0 = 1$. The group law of $\mathcal{W}_n$ is that which makes the map

$$\begin{aligned} \mathcal{W}_n &\rightarrow (\mathbb{G}_{m,A})^n \\ \mathbf{x} = (x_0, \dots, x_{n-1}) &\mapsto (\lambda_1 x_1 + 1, \lambda_2 x_1 + D_1(\mathbf{x}), \dots, \lambda_n x_{n-1} + D_{n-1}(\mathbf{x})) \end{aligned}$$

a homomorphism. One can refer to [15] for the details.

3. DEFORMED ARTIN-HASSE EXPONENTIAL SERIES

Let W_n (resp. $\widehat{W}_n$) be the group scheme (resp. the formal group scheme) over $\mathbb{Z}$ of Witt vectors of length n , and W (resp. $\widehat{W}$) the group scheme (resp. the formal group scheme) of Witt vectors over $\mathbb{Z}$, and let $\mathbb{G}_m$ (resp. $\widehat{\mathbb{G}}_m$) be the multiplicative group scheme (resp. the multiplicative formal group scheme) over $\mathbb{Z}$.

We denote the Witt polynomials by

$$\begin{aligned} \Phi_0(\mathbf{X}) &= X_0 \\ \Phi_1(\mathbf{X}) &= X_0^p + pX_1 \\ &\vdots \\ \Phi_n(\mathbf{X}) &= X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n \\ &\vdots \end{aligned}$$

Let F be the Frobenius endomorphism defined by

$$\begin{array}{ccccc} F := \Phi^{-1} \circ \Phi^{(1)} : & W & \xrightarrow{\Phi^{(1)}} & \mathbb{G}_a^\infty & \xleftarrow{\Phi} W \\ & \mathbf{x} & \mapsto & (\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x}), \dots) & \\ & & & (\Phi_0(\mathbf{y}), \Phi_1(\mathbf{y}), \dots) & \longleftarrow \mathbf{y} \end{array}$$

We note that F is also an endomorphism of $\widehat{W}$.

For later use, we define a morphism $[p] : W \rightarrow W$ by

$$[p]\mathbf{b} := (0, b_0^p, b_1^p, \dots)$$

for a vector $\mathbf{b} = (b_0, b_1, \dots) \in W(A)$. Note that if A is an $\mathbb{F}_p$ -algebra, $[p]\mathbf{b}$ is nothing but $p\mathbf{b}$. Moreover, for a vector $\mathbf{a} = (a_0, a_1, \dots) \in W(A)$, we define a map $T_{\mathbf{a}} : W(A) \rightarrow W(A)$ by

$$\Phi_n(T_{\mathbf{a}}\mathbf{b}) = a_0^{p^n} \Phi_n(\mathbf{b}) + pa_1^{p^{n-1}} \Phi_{n-1}(\mathbf{b}) + \dots + p^n a_n \Phi_0(\mathbf{b})$$

for $\mathbf{b} \in W(A)$. Then we can easily see the following.

Lemma 3.1. *Actually, $T_{\mathbf{a}} : W(A) \rightarrow W(A)$ is a well-defined endomorphism.*

Let $\bar{\Lambda}$ denote the Witt vector $(\Lambda, 0, 0, \dots)$ with coefficients in $\mathbb{Z}[\Lambda]$ and $F^{(\Lambda)}$ the endomorphism $F - \bar{\Lambda}^{p-1}$ of the group scheme $W_{\mathbb{Z}[\Lambda]}$.

The so called Artin-Hasse exponential series is given by

$$\begin{aligned} E_p(X) &:= \exp \left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \dots \right) \\ &= e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \dots \in \mathbb{Z}_{(p)}[[X]]. \end{aligned}$$

Now we define a formal power series $E_p(U, \Lambda; X)$ in $\mathbb{Q}[U, \Lambda][[X]]$ by

$$E_p(U, \Lambda; X) := (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} \left(1 + \Lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k} \left(\left(\frac{U}{\Lambda} \right)^{p^k} - \left(\frac{U}{\Lambda} \right)^{p^{k-1}} \right)}.$$

In our argument, one of the crucial points is to decide the integrality of this kind of series. For checking the integrality, Hazewinkel's lemma (cf. [2, §2]) is almost almighty in our case.

Lemma 3.2 ([2, (2.3.3)]). *Let A be an integral domain containing $\mathbb{Z}_{(p)}$, and $\sigma : K = Q(A) \rightarrow K = Q(A)$ be a $\mathbb{Z}_{(p)}$ -algebra homomorphism such that $\sigma(f) \equiv f \pmod{pA}$ for any $f \in A$. Let $d(X) = d_0 X + d_1 X^{p^1} + \dots \in A[\frac{1}{p}][[X]]$. Then*

$$\exp(d(X)) = 1 + d(X) + \frac{1}{2!} d(X)^2 + \dots \in A[[X]]$$

if and only if there exist $b_i \in A$ ($i = 0, 1, \dots$) such that $d_0 = b_0$, and $d_n = b_n + \frac{1}{p} \sigma(d_{n-1}) \in A$ for $n \geq 1$.

By using this lemma, we can see that $E_p(U, \Lambda; X) \in \mathbb{Z}_{(p)}[U, \Lambda][[X]]$. Easily we can see that $E_p(1, 0; X) = E_p(X)$, that is to say, $E_p(U, \Lambda; X)$ gives a deformation of the Artin-Hasse exponential series $E_p(X)$.

Let A be a $\mathbb{Z}_{(p)}$ -algebra, $\lambda \in A$ and $\mathbf{a} = (a_0, a_1, \dots) \in W(A)$. We define a formal power series $E_p(\mathbf{a}, \lambda; X)$ in $A[[X]]$ by

$$\begin{aligned} (10) \quad E_p(\mathbf{a}, \lambda; X) &:= \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{\frac{a_0}{\lambda}} \prod_{k=1}^{\infty} \left(1 + \lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} F^{(\lambda)} \mathbf{a}} \end{aligned}$$

Then the boundary of this power series $E_p(\mathbf{a}, \lambda; X)$ is given by the following.

(11)

$$\begin{aligned} (\partial E_p(\mathbf{a}, \lambda; \cdot))(X, Y) &= \frac{E_p(\mathbf{a}, \lambda; X)E_p(\mathbf{a}, \lambda; Y)}{E_p(\mathbf{a}, \lambda; X + Y + \lambda XY)} \\ &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} F^{(\lambda)} \mathbf{a}} \end{aligned}$$

Now replacing $F^{(\lambda)} \mathbf{a}$ with a Witt vector $\mathbf{b} = (b_0, b_1, \dots)$ in the right hand side of (11), we define a cocycle as follows.

$$(12) \quad F_p(\mathbf{b}, \lambda; X, Y) := \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1} \mathbf{b}}$$

Again using the integrality lemma, we can see that

$$F_p(\mathbf{b}, \lambda; X, Y) \in \mathbb{Z}_{(p)}[\mathbf{b}, \lambda][[X, Y]].$$

4. DETERMINATION OF $\text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A})$ AND $H_0^2(\mathcal{W}_n, \mathbb{G}_{m,A})$

Let A be a $\mathbb{Z}_{(p)}$ -algebra and $\lambda \in A$. By (10) and (11), we can define homomorphisms

$$\xi_0^1 : \text{Ker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) \rightarrow \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; X)$$

and, when λ is nilpotent,

$$\xi_0^1 : \text{Ker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \rightarrow \text{Hom}_{A\text{-gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}); \mathbf{a} \mapsto E_p(\mathbf{a}, \lambda; X).$$

Moreover, by (12), we can define homomorphisms

$$\xi_1^1 : \text{Coker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) \rightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}); \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y)$$

and, when λ is nilpotent,

$$\xi_1^1 : \text{Coker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) \rightarrow H_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}); \mathbf{a} \mapsto F_p(\mathbf{a}, \lambda; X, Y).$$

Under these notations, we gave the result in the one-dimensional case in the previous paper [16] as in the following style.

Theorem 4.1. *Let A be a $\mathbb{Z}_{(p)}$ -algebra and $\lambda \in A$. Then the homomorphisms*

$$\begin{aligned} \xi_0^1 : \text{Ker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) &\rightarrow \text{Hom}_{A\text{-gr}}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}), \\ \xi_1^1 : \text{Coker}(W(A) \xrightarrow{F^{(\lambda)}} W(A)) &\rightarrow H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbb{G}}_{m,A}) \end{aligned}$$

are bijective. Moreover, if λ is nilpotent, the homomorphisms

$$\begin{aligned}\xi_0^1 : \text{Ker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) &\rightarrow \text{Hom}_{A\text{-gr}}(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}), \\ \xi_1^1 : \text{Coker}(\widehat{W}(A) \xrightarrow{F^{(\lambda)}} \widehat{W}(A)) &\rightarrow H_0^2(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A})\end{aligned}$$

are bijective.

For general n , we can consider the both of $\text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A})$, $H_0^2(\mathcal{W}_n, \mathbb{G}_{m,A})$ and $\text{Hom}(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$, $H_0^2(\widehat{\mathcal{W}}_n, \widehat{\mathbb{G}}_{m,A})$, but for simplicity hereafter we treat the first them only.

Next we look at $n = 2$ case. Let $\lambda_1, \lambda_2 \in A$, and assume that λ_1 is nilpotent in A/λ_2 . By (9) and Theorem 4.1, an extension

$$(13) \quad 0 \rightarrow \mathcal{G}^{(\lambda_2)} \rightarrow \mathcal{W}_2 \rightarrow \mathcal{G}^{(\lambda_1)} \rightarrow 0 \quad \in \text{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)})$$

is given by

$$(14) \quad \mathcal{W}_2 = \text{Spec}A[X, Y, \frac{1}{\lambda_1 X + 1}, \frac{1}{\lambda_2 Y + D(X)}]$$

where $D(X) = E_p(\mathbf{a}, \lambda_1; X)$ and $\mathbf{a} \in \widehat{W}(A/\lambda_2)$ with $F^{(\lambda_1)}\mathbf{a} = 0 \in \widehat{W}(A/\lambda_2)$. Now we put $F^{(\lambda_1)}\mathbf{a} = (\lambda_2 a'_0, \lambda_2 a'_1, \dots)$ and $\mathbf{a}' = (a'_0, a'_1, \dots)$. We define an endomorphism $U_2 = U_2(\lambda_1, \lambda_2; \mathbf{a}') : \widehat{W}(A)^2 \rightarrow \widehat{W}(A)^2$ by

$$U_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)} & -T\mathbf{a}' \\ 0 & F^{(\lambda_2)} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)}\alpha - T\mathbf{a}'\beta \\ F^{(\lambda_2)}\beta \end{pmatrix}.$$

For $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \widehat{W}(A)^2$, we define a formal power series $E_p(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y)$ by

$$E_p(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y) := E_p(\alpha, \lambda_1; X) E_p(\beta, \lambda_2; \frac{Y}{D(X)}).$$

When we put $(s, t) := (x_1, y_1) + (x_2, y_2) \in \mathcal{W}_2$, we can easily see that

$$\begin{aligned}s &= x_1 + x_2 + \lambda_1 x_1 x_2, \\ \frac{t}{D(s)} &= \frac{y_1}{D(x_1)} + \frac{y_2}{D(x_2)} + H_1(x_1, x_2) \in \mathcal{G}^{(\lambda_2)},\end{aligned}$$

where

$$\begin{aligned}H_1(x_1, x_2) &= \frac{1}{\lambda_2} \left\{ \frac{D(x_1)D(x_2)}{D(x_1 + x_2 + \lambda_1 x_1 x_2)} - 1 \right\} \\ &= \frac{1}{\lambda_2} \left\{ F_p(F^{(\lambda_1)}\mathbf{a}, \lambda_1; (x_1, x_2)) - 1 \right\}.\end{aligned}$$

Moreover, for $F = F_p(\mathbf{b}, \lambda_1; X_1, X_2)$, we define

$$[p]F := F_p([p]\mathbf{b}, \lambda_1; X_1, X_2),$$

and

$$G_p(\boldsymbol{\delta}, \lambda_2; F) := \prod_{k=1}^{\infty} \left(\frac{1 + (F-1)^{p^k}}{[p]^k F} \right)^{\frac{1}{p^k \lambda_2^{p^k}} \Phi_{k-1} \boldsymbol{\delta}}.$$

Then using again the integrality lemma, we can see that

$$G_p(\boldsymbol{\delta}, \lambda_2; F) \in \mathbb{Z}_{(p)}[\mathbf{a}', \lambda_1, \lambda_2, \boldsymbol{\delta}][[X_1, X_2]].$$

Under these notations, we can show that the boundary of $E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right)$ is given by

(15)

$$\begin{aligned} & (\partial E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; \cdot, \cdot\right))(X_1, Y_1, X_2, Y_2) \\ &= \frac{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X_1, Y_1\right) E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X_2, Y_2\right)}{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1) + (X_2, Y_2)\right)} \\ &= F_p(F^{(\lambda_1)} \boldsymbol{\alpha} - T_{\mathbf{a}'} \boldsymbol{\beta}, \lambda_1; (X_1, X_2)) F_p(F^{(\lambda_2)} \boldsymbol{\beta}, \lambda_2; \left(\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)}\right)) \times \\ & \quad F_p(F^{(\lambda_2)} \boldsymbol{\beta}, \lambda_2; \left(H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)}\right)) G_p(-F^{(\lambda_2)} \boldsymbol{\beta}, \lambda_2; F). \end{aligned}$$

Now arranging the equality (15), we define a cocycle by

$$\begin{aligned} (16) \quad & F_p\left(\begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1), (X_2, Y_2)\right) \\ &= F_p(\boldsymbol{\gamma}, \lambda_1; (X_1, X_2)) F_p(\boldsymbol{\delta}, \lambda_2; \left(\frac{Y_1}{D(X_1)}, \frac{Y_2}{D(X_2)}\right)) \times \\ & \quad F_p(\boldsymbol{\delta}, \lambda_2; \left(H_1, \frac{Y_1}{D(X_1)} + \frac{Y_2}{D(X_2)}\right)) G_p(-\boldsymbol{\delta}, \lambda_2; F). \end{aligned}$$

By the equality (15), we can define the homomorphisms

$$\begin{aligned} \xi_0^2 : \text{Ker} \left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) &\rightarrow \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} &\mapsto E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right) \end{aligned}$$

and

$$\begin{aligned} \xi_1^2 : \text{Coker} \left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) &\rightarrow H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \gamma \\ \delta \end{pmatrix} &\mapsto F_p \left(\begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; (X_1, Y_1), (X_2, Y_2) \right). \end{aligned}$$

Then we can obtain a commutative diagram:

$$\begin{array}{ccccccc} (17) & 0 & \longrightarrow & \text{Ker} F^{(\lambda_1)} & \longrightarrow & \text{Ker} U_2 & \longrightarrow & \text{Ker} F^{(\lambda_2)} & \longrightarrow \\ & & & \xi_0^1 \downarrow & & \xi_0^2 \downarrow & & \xi_0^1 \downarrow & \\ 0 & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} \\ & & & \longrightarrow & \text{Coker} F^{(\lambda_2)} & \longrightarrow & \text{Coker} U_2 & \longrightarrow & \text{Coker} F^{(\lambda_2)} & \longrightarrow & 0 \\ & & & \xi_1^1 \downarrow & & \xi_1^2 \downarrow & & \xi_1^1 \downarrow & \\ & \xrightarrow{\partial} & H_0^2(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{G}^{(\lambda_2)}, \mathbb{G}_{m,A}), \end{array}$$

where the second horizontal line is the exact sequence deduced from (13), and the first horizontal line is the exact sequence defined by the following maps in order:

$$\begin{aligned} \alpha &\mapsto \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, & \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \beta, & \beta &\mapsto T_{a'} \beta, \\ \gamma &\mapsto \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, & \begin{pmatrix} \gamma \\ \delta \end{pmatrix} &\mapsto \delta. \end{aligned}$$

By the commutative diagram (17) and Theorem 4.1, we can show the following.

Theorem 4.2. *The homomorphisms*

$$\xi_0^2 : \text{Ker} \left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) \rightarrow \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A})$$

and

$$\xi_1^2 : \text{Coker} \left(\widehat{W}(A)^2 \xrightarrow{U_2} \widehat{W}(A)^2 \right) \rightarrow H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A})$$

are bijective.

Before we give the final form of the theorem, we will explain the situation by looking at the $n = 3$ case.

By (9), an extension

$$(18) \quad 0 \rightarrow \mathcal{G}^{(\lambda_3)} \rightarrow \mathcal{W}_3 \rightarrow \mathcal{W}_2 \rightarrow 0$$

is given by

$$\mathcal{W}_2 = \text{Spec}A[X, Y, \frac{1}{1 + \lambda_1 X}, \frac{1}{D_1(X) + \lambda_2 Y}]$$

and

$$\mathcal{W}_3 = \text{Spec}A[X, Y, Z, \frac{1}{1 + \lambda_1 X}, \frac{1}{D_1(X) + \lambda_2 Y}, \frac{1}{D_2(X, Y) + \lambda_3 Z}].$$

Moreover, by Theorems 4.1, 4.2, D_1 and D_2 are given by

$$\begin{aligned} D_1(X) &= E_p(\mathbf{a}, \lambda_1; X), \\ D_2(X, Y) &= E_p\left(\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}; X, Y\right), \end{aligned}$$

with

$$\begin{aligned} F^{(\lambda_1)}\mathbf{a} &= \mathbf{0} \in \widehat{W}(A/\lambda_2), \\ U_2\left(\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}\right) &= \begin{pmatrix} F^{(\lambda_1)} & -T\mathbf{a}' \\ 0 & F^{(\lambda_2)} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \mathbf{0} \in \widehat{W}(A/\lambda_3), \end{aligned}$$

where $\mathbf{a}'$ is defined by $(\lambda_2 a'_0, \lambda_2 a'_2, \dots) = F^{(\lambda_1)}\mathbf{a}$. Now we define $\mathbf{b}'$ and $\mathbf{c}'$ by $(\lambda_3 b'_0, \lambda_3 b'_2, \dots) = F^{(\lambda_2)}\mathbf{b} - T\mathbf{a}'\mathbf{c}$ and $(\lambda_3 c'_0, \lambda_3 c'_1, \dots) = F^{(\lambda_3)}\mathbf{c}$.

Again we define a new power series by

$$E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X, Y, Z\right) = E_p(\boldsymbol{\alpha}, \lambda_1; X)E_p(\boldsymbol{\beta}, \lambda_2; \frac{Y}{D_1(X)})E_p(\boldsymbol{\gamma}, \lambda_3; \frac{Z}{D_2(X, Y)}).$$

Then the boundary of this series can be calculated as follows.

$$\begin{aligned} (19) \quad & \left(\partial E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; \cdot, \cdot, \cdot\right) (X_1, Y_1, Z_1, X_2, Y_2, Z_2) \right. \\ & \quad \left. = \frac{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1\right)E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_2, Y_2, Z_2\right)}{E_p\left(\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; (X_1, Y_1, Z_1) + (X_2, Y_2, Z_2)\right)} \right) \end{aligned}$$

$$\begin{aligned}
&= F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2)F_p(F^{(\lambda_2)}\beta, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(F^{(\lambda_2)}\beta, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})E_p(\beta, \lambda_2; H_1)^{-1} \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)}))E_p(\gamma, \lambda_3; H_2)^{-1} \\
&= F_p(F^{(\lambda_1)}\alpha - T_{\mathbf{a}'}\beta - T_{\mathbf{b}'}\gamma, \lambda_1; X_1, X_2) \times \\
&\quad F_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\beta, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\gamma, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)})) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(F^{(\lambda_3)}\gamma, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times \\
&\quad G_p(F^{(\lambda_2)}\beta - T_{\mathbf{b}'}\gamma, \lambda_2; F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2))^{-1} \times \\
&\quad G_p(F^{(\lambda_3)}\gamma, \lambda_3; \lambda_3 H_2 + 1)^{-1}.
\end{aligned}$$

Again arranging the equation (19), we define a cocycle by

$$\begin{aligned}
&F_p\left(\begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2\right) \\
&= F_p(\delta, \lambda_1; X_1, X_2)F_p(\epsilon, \lambda_2; \frac{Y_1}{D_1(X_1)}, \frac{Y_2}{D_1(X_2)}) \times \\
&\quad F_p(\epsilon, \lambda_2; (H_1, \frac{Y_1}{D_1(X_1)} + \frac{Y_2}{D_1(X_2)}))F_p(\zeta, \lambda_3; \frac{Z_1}{D_2(X_1, Y_1)}, \frac{Z_2}{D_2(X_2, Y_2)}) \times \\
&\quad F_p(\zeta, \lambda_3; (H_2, \frac{Z_1}{D_2(X_1, Y_1)} + \frac{Z_2}{D_2(X_2, Y_2)})) \times \\
&\quad G_p(\epsilon, \lambda_2; F_p(F^{(\lambda_1)}\alpha, \lambda_1; X_1, X_2))^{-1}G_p(\zeta, \lambda_3; \lambda_3 H_2 + 1)^{-1}.
\end{aligned}$$

Now we define an endomorphism $U_3 = U(\lambda_1, \lambda_2, \lambda_3, \mathbf{a}', \mathbf{b}', \mathbf{c}') : \widehat{W}(A)^3 \rightarrow \widehat{W}(A)^3$

by

$$U_3 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)} & -T_{\mathbf{a}'} & -T_{\mathbf{b}'} \\ 0 & F^{(\lambda_1)} & -T_{\mathbf{c}'} \\ 0 & 0 & F^{(\lambda_3)} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} F^{(\lambda_1)}\alpha - T_{\mathbf{a}'}\beta - T_{\mathbf{b}'}\gamma \\ F^{(\lambda_1)}\beta - T_{\mathbf{c}'}\gamma \\ F^{(\lambda_3)}\gamma \end{pmatrix}.$$

Then by (19), we can define homomorphisms

$$\begin{aligned} \xi_0^3 : \text{Ker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) &\rightarrow \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &\mapsto E_p\left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X, Y, Z\right), \\ \xi_1^3 : \text{Coker}(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3) &\rightarrow H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix} &\mapsto F_p\left(\begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}; X_1, Y_1, Z_1, X_2, Y_2, Z_2\right). \end{aligned}$$

Then similarly as in the case of $n = 2$, we have a commutative diagram:

$$\begin{array}{ccccccc} (20) & 0 & \longrightarrow & \text{Ker}U_2 & \longrightarrow & \text{Ker}U_3 & \longrightarrow & \text{Ker}F^{(\lambda_3)} & \longrightarrow \\ & & & \xi_0^2 \downarrow & & \xi_0^3 \downarrow & & \xi_0^1 \downarrow & \\ 0 & \longrightarrow & \text{Hom}(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & \text{Hom}(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}) & \xrightarrow{\partial} \\ & & & & & & & & \\ & \longrightarrow & \text{Coker}U_2 & \longrightarrow & \text{Coker}U_3 & \longrightarrow & \text{Coker}F^{(\lambda_3)} & \longrightarrow & 0 \\ & & \xi_1^2 \downarrow & & \xi_1^3 \downarrow & & \xi_1^1 \downarrow & & \\ & \xrightarrow{\partial} & H_0^2(\mathcal{W}_2, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A}) & \longrightarrow & H_0^2(\mathcal{G}^{(\lambda_3)}, \mathbb{G}_{m,A}), & & \end{array}$$

with the second exact horizontal line obtained by the exact sequence (18) and the first horizontal line defined in order as follows:

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, & \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &\mapsto \gamma, & \gamma &\mapsto \begin{pmatrix} T_{\mathbf{b}'}\gamma \\ T_{\mathbf{c}'}\gamma \end{pmatrix}, \\ \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} &\mapsto \begin{pmatrix} \delta \\ \epsilon \\ 0 \end{pmatrix}, & \begin{pmatrix} \delta \\ \epsilon \\ \zeta \end{pmatrix} &\mapsto \zeta. \end{aligned}$$

By the commutative diagram (20) and Theorem 4.2, we have the result in the three-dimensional case.

Theorem 4.3. *The homomorphisms*

$$\xi_0^3 : \text{Ker} \left(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3 \right) \rightarrow \text{Hom}(\mathcal{W}_3, \mathbb{G}_{m,A})$$

and

$$\xi_1^3 : \text{Coker} \left(\widehat{W}(A)^3 \xrightarrow{U_3} \widehat{W}(A)^3 \right) \rightarrow H_0^2(\mathcal{W}_3, \mathbb{G}_{m,A})$$

are bijective.

Then as one can guess the general result, we have the final form as follows.

Let $\mathcal{W}_n$ is a group scheme over A obtained successively by

$$(21) \quad 0 \rightarrow \mathcal{G}^{(\lambda_{i+1})} \rightarrow \mathcal{W}_{i+1} \rightarrow \mathcal{W}_i \rightarrow 0,$$

for $i = 1, 2, \dots, n-1$, where $\mathcal{W}_1 = \mathcal{G}^{(\lambda_1)}$. Then by (9), each $\mathcal{W}_i$ ($i = 1, 2, \dots, n$) is given by

$$\mathcal{W}_i = \text{Spec} A[X_0, X_1, \dots, X_{i-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{i-1}(X_0, \dots, X_{i-2}) + \lambda_i X_{i-1}}],$$

where $D_k(X_0, \dots, X_{k-1}) : \mathcal{W}_{k,A/\lambda_{k+1}} \rightarrow \mathbb{G}_{m,A/\lambda_{k+1}}$ is a homomorphism for $k = 1, \dots, i-1$. Here we understand that $D_0 = 1$. Now we assume that for $1 \leq i \leq n-1$, each $D_i(X_0, X_1, \dots, X_{i-1})$ is given by

$$\begin{aligned} D_i(X_0, X_1, \dots, X_{i-1}) &= E_p \left(\begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_i^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1} \right) \\ &:= \prod_{\ell=1}^i E_p(a_\ell^i, \lambda_\ell; \frac{X_{\ell-1}}{D_{\ell-1}(X_0, \dots, X_{\ell-2})}), \end{aligned}$$

and

$$\begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_i^i \end{pmatrix} \in \text{Ker} \left(U_i := \begin{pmatrix} F^{(\lambda_1)} & -T_{a_1^1} & -T_{a_1^2} & \cdots & -T_{a_1^{i-1}} \\ 0 & F^{(\lambda_2)} & -T_{a_2^2} & \cdots & -T_{a_2^{i-1}} \\ 0 & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & -T_{a_{i-1}^{i-1}} \\ bm0 & 0 & 0 & \ddots & F^{(\lambda_{i-1})} \end{pmatrix} : \widehat{W}(A/\lambda_i)^i \rightarrow \widehat{W}(A/\lambda_i)^i \right).$$

Here the ℓ -th components of $\mathbf{a}_j^i$'s are defined inductively by

$$\lambda_i(\mathbf{a}_j^i)_\ell = \left(F^{(\lambda_j)} \mathbf{a}_j^i - \sum_{\ell=j}^{i-1} T_{\mathbf{a}_j^\ell} \mathbf{a}_\ell^i \right)_\ell.$$

We put

$$H_i := \frac{1}{\lambda_{i+1}} \left(\frac{D_i(X_0, \dots, X_{i-1}) D_i(Y_0, \dots, Y_{i-1})}{D_i((X_0, \dots, X_{i-1}) + (Y_0, \dots, Y_{i-1}))} - 1 \right).$$

Furthermore we define a formal power series by

$$\begin{aligned} & F_p \left(\begin{pmatrix} \mathbf{b}_1^i \\ \mathbf{b}_2^i \\ \vdots \\ \mathbf{b}_i^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1} \right) \\ &:= \prod_{j=1}^i F_p(\mathbf{b}_j^i, \lambda_j; \frac{X_{j-1}}{D_{j-1}(X_0, \dots, X_{j-2})}, \frac{Y_{j-1}}{D_{j-1}(Y_0, \dots, Y_{j-2})}) \times \\ & \quad \prod_{k=2}^i F_p(\mathbf{b}_k^i, \lambda_k; H_{k-1}, \frac{X_{k-1}}{D_{k-1}(X_0, \dots, X_{k-2})} + \frac{Y_{k-1}}{D_{k-1}(Y_0, \dots, Y_{k-2})}) \times \\ & \quad G_p(-\mathbf{b}_k^i, \lambda_k; \lambda_k H_{k-1} + 1). \end{aligned}$$

Then we can show the following theorem inductively.

Theorem 4.4. *The homomorphisms*

$$\begin{aligned} \xi_0^i : \text{Ker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) &\rightarrow \text{Hom}(\mathcal{W}_i, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^i \end{pmatrix} &\mapsto E_p \left(\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1} \right) \end{aligned}$$

and

$$\begin{aligned} \xi_1^i : \text{Coker}(\widehat{W}(A)^i \xrightarrow{U_i} \widehat{W}(A)^i) &\rightarrow H_0^2(\mathcal{W}_i, \mathbb{G}_{m,A}); \\ \begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix} &\mapsto F_p \left(\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^i \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{pmatrix}; X_0, X_1, \dots, X_{i-1}, Y_0, Y_1, \dots, Y_{i-1} \right) \end{aligned}$$

are bijective, for each i .

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(Tsutomu Sekiguchi) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, CHUO UNIVERSITY 1-13-27 KASUGA BUNKYO-KU TOKYO 112-8551, JAPAN

E-mail address: sekiguti@math.chuo-u.ac.jp

(Noriyuki Suwa) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, CHUO UNIVERSITY 1-13-27 KASUGA BUNKYO-KU TOKYO 112-8551, JAPAN

E-mail address: suwa@math.chuo-u.ac.jp