

Mean theoretic approach to a further extension of grand Furuta inequality

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Abstract

Very recently, Furuta has shown a further extension of grand Furuta inequality. In this report, we obtain a more precise and clear expression of Furuta’s extension by considering a mean theoretic proof of grand Furuta inequality.

1 Introduction

In what follows, A and B are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

Löwner-Heinz theorem “ $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$ ” is very famous as an order preserving operator inequality. As an extension of Löwner-Heinz theorem, Furuta [8] established the following result called Furuta inequality (see also [2, 3, 9, 12, 18, 20]).

Theorem 1.A (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

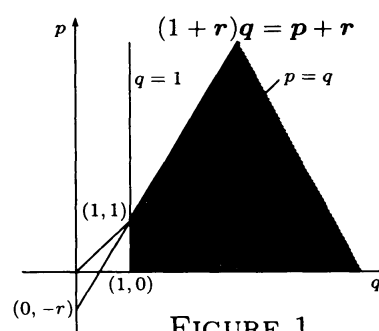


FIGURE 1

Theorem 1.B ([3]). Let $A \geq B \geq 0$ with $A > 0$. Then

$$f(p, r) = A^{-\frac{r}{2}} (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} A^{-\frac{r}{2}} \quad (1.1)$$

is decreasing for $p \geq 1$ and $r \geq 0$.

In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

Theorem 1.C (Grand Furuta inequality [10]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$F(r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

For $A > 0$ and $B \geq 0$, α -power mean $\sharp_\alpha$ for $\alpha \in [0, 1]$ is defined by $A \sharp_\alpha B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$. In this report, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that α -power mean is very useful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geq 0$, Theorem 1.A can be arranged in terms of α -power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

We can also rewrite (1.1) in Theorem 1.B by

$$f(p, r) = A^{-r} \sharp_{\frac{1+r}{p+r}} B^p. \quad (1.1')$$

Similarly, by putting $\beta = (p-t)s + t$ and $\gamma = r - t$, we can arrange Theorem 1.C in terms of α -power mean as follows [5]: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$ with $p \neq t$,

$$\hat{F}(\beta, \gamma) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{is decreasing for } \beta \geq p \text{ and } \gamma \geq 0,$$

and

$$A \geq B \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0, \quad (1.2)$$

where $A \natural_s B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}}$ for a real number s . (If $s \in [0, 1]$, then $\natural_s = \sharp_s$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem 1.D. We call this “FGF inequality” here.

Theorem 1.D (FGF inequality [14, 15]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n-1} \geq 1$ for natural number n . Then*

$$G(r, p_{2n}) = A^{\frac{-r}{2}} \left[A^{\frac{r}{2}} \left(A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \dots (A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \right. \right. \\ \left. \left. \times (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}})^{p_4} \dots A^{\frac{t}{2}} \}^{p_{2n-1}} A^{\frac{-t}{2}} \right)^{p_{2n}} A^{\frac{r}{2}} \right]^{\frac{1-t+r}{q[2n]-t+r}} A^{\frac{-r}{2}} \quad (1.3)$$

is decreasing for $r \geq t$ and $p_{2n} \geq 1$, and

$$A^{1-t+r} \geq \left[A^{\frac{r}{2}} \left(A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \dots (A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \right. \right. \\ \left. \left. \times (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}})^{p_4} \dots A^{\frac{t}{2}} \}^{p_{2n-1}} A^{\frac{-t}{2}} \right)^{p_{2n}} A^{\frac{r}{2}} \right]^{\frac{1-t+r}{q[2n]-t+r}} \quad (1.4)$$

holds for $r \geq t$ and $p_{2n} \geq 1$, where

$$q[2n] = (\{ \dots (\{ (p_1 - t)p_2 + t \} p_3 - t) p_4 + \dots + t \} p_{2n-1} - t) p_{2n} + t.$$

In this report, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2 FGF inequality

Firstly, we show that a sequence $\{B_i\}$ such that $B_i = (A^t \natural_{\alpha_i-t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$ is decreasing. Theorem 2.1 is a key result in the proof of FGF inequality.

Theorem 2.1. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \dots, n$,*

$$A \geq B \geq B_1 \geq \dots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\alpha_i-t} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Lemma 2.A ([5]). *Let $A \geq B \geq 0$ with $A > 0$. Then*

$$A \geq B \geq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0, 1]$, $\beta \geq p \geq 1$ and $p \neq t$.

We remark that Lemma 2.A plays an important role in the proof of grand Furuta inequality (1.2).

Proof of Theorem 2.1. By applying Lemma 2.A to that $A \geq B \geq 0$ with $A > 0$, we have

$$A \geq B \geq (A^t \natural_{\frac{\beta_1-t}{\alpha_1-t}} B^{\alpha_1})^{\frac{1}{\beta_1}} = B_1$$

for $t \in [0, 1]$, $\beta_1 \geq \alpha_1 \geq 1$ and $\alpha_1 \neq t$, and also by applying Lemma 2.A repeatedly to that $A \geq B_{i-1} \geq 0$ with $A > 0$ for $i = 1, 2, \dots, n$, we have

$$B_{i-1} \geq (A^{t_i} \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}} = B_i$$

for $t \in [0, 1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$, so that

$$A \geq B \geq B_1 \geq \dots \geq B_{n-1} \geq B_n.$$

Hence the proof is complete. $\square$

Furuta [15] has given an extension of Lemma 2.A as an application of Theorem 1.D.

Theorem 2.B ([15]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n-1}, p_{2n} \geq 1$ for natural number n . Then*

$$A \geq B \geq \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{\frac{1}{q[2n]}} \geq \dots \geq [A^{\frac{t}{2}} (A^{\frac{-t}{2}} \{A^{\frac{t}{2}} \dots (A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}})^{p_4} \dots A^{\frac{t}{2}}\}^{p_{2n-1}} A^{\frac{-t}{2}})^{p_{2n}} A^{\frac{t}{2}}]^{\frac{1}{q[2n]}},$$

where

$$q[2n] = (\{ \dots (\{(p_1 - t)p_2 + t\} p_3 - t) p_4 + \dots + t\} p_{2n-1} - t) p_{2n} + t.$$

We can rewrite Theorem 2.B by putting

$$\beta_0 = 1, \alpha_i = \beta_{i-1} p_{2i-1}, \beta_i = (\alpha_i - t) p_{2i} + t \text{ and } \gamma = r - t \quad (2.1)$$

as follows:

Theorem 2.B'. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \dots \geq \beta_1 \geq \alpha_1 \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \dots, n$,*

$$A \geq B \geq B_1 \geq \dots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Therefore we recognize that Theorem 2.1 is a fine extension of Theorem 2.B. More precisely, $\beta_i \geq \alpha_i \geq 1$ in Theorem 2.1 is looser than $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$ in Theorem 2.B.

By using Theorem 2.1, we obtain an improvement of (1.4) in Theorem 1.D and Theorem 2.B. Theorem 2.2 is a satellite form of Theorem 1.D in our sense. Theorem 2.2 leads (1.4) in Theorem 1.D by the same replacement to (2.1).

Theorem 2.2. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$, $\gamma \geq 0$ and $\alpha_1 \neq t$,*

$$\begin{aligned} A \geq B &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_1+\gamma}} B^{\alpha_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_2+\gamma}} B_1^{\alpha_2} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geq \cdots \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_n+\gamma}} B_{n-1}^{\alpha_n} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \sharp_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. Let $\beta_0 = 1$. By Theorem 2.1, $A \geq B_{i-1}$ holds for $i = 1, 2, \dots, n$, so that we have

$$\begin{aligned} &A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} \\ &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_i+\gamma}} B_{i-1}^{\alpha_i} \quad \text{by Theorem 1.B} \\ &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} (A^t \sharp_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i}) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i} \quad \text{by Theorem 1.C} \end{aligned}$$

since $\beta_i \geq \alpha_i \geq \beta_{i-1} \geq 1$. Hence the proof is complete. $\square$

3 Variant of FGF inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2, and also we have a result on a FGF-type operator function. We omit their proofs here.

Theorem 3.1. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\alpha_i \geq 1$, $1 \leq \frac{\beta_i-t}{\alpha_i-t} \leq 2$ and $\alpha_i \neq t$ for $i = 1, 2, \dots, n$,*

$$B_{i-1}^{\beta_i} \geq B_i^{\beta_i},$$

where $B_0 = B$ and $B_i = (A^t \sharp_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Theorem 3.2. Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\alpha_i \geq 1$, $\beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq 1$, $1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \dots, n$,

$$\begin{aligned} A \geq B \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_1+\gamma}} B^{\beta_1} &\geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_2+\gamma}} B_1^{\beta_2} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geq \cdots \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_n+\gamma}} B_{n-1}^{\beta_n} \geq A^{-\gamma} \#_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Theorem 3.3. Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0, 1]$, $\beta_i \geq \alpha_i \geq 1$ for $i = 1, 2, \dots, n-1$, $\alpha_n \geq 1$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \dots, n$,

$$\hat{G}(\beta_n) = A^{-\gamma} \#_{\frac{1+\gamma}{\beta_n+\gamma}} (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n}) \quad (3.1)$$

is decreasing for $\beta_n \geq \alpha_n$, where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Remark. (3.1) is also decreasing for $\gamma \geq 0$ by Theorem 1.B since $A \geq B \geq 0$ with $A > 0$ ensures $A \geq B_n = (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n})^{\frac{1}{\beta_n}}$ by Theorem 2.1. Therefore, similarly to Theorem 2.1, we recognize that Theorem 3.3 is a slight extension of (1.3) in Theorem 1.D.

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