

Two remarks on the subadditivity inequalities in von Neumann algebras¹

by O. E. TIKHONOV

Taking into account a developed theory of operator monotone and operator convex functions (see e.g. [1–3]) it appears interesting to study operator subadditive functions. We do this within the context of von Neumann algebras though the main result is essentially a statement on 2×2 -matrices.

In what follows we suppose that M is a von Neumann algebra and $\phi : [0, \infty) \rightarrow \mathbb{R}$ is a Borel measurable function bounded on bounded subsets of $[0, \infty)$. We say that ϕ is *operator subadditive with respect to M* or briefly *M -subadditive* if $\phi(a + b) \leq \phi(a) + \phi(b)$ for every pair a, b of positive operators from M .

EXAMPLES. It is easy to see that the following functions on $[0, \infty)$ are M -subadditive for any M :

- 1) $\phi(t) = \alpha t + \beta$ ($\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$);
- 2) $\phi(t) = 1/(\alpha t + 1)$ ($\alpha \in \mathbb{R}^+$);
- 3) an arbitrary function ϕ satisfying $\alpha \leq \phi(t) \leq 2\alpha$ for some $\alpha \in \mathbb{R}^+$.

Subadditive real functions used in analysis often satisfy $\phi(0) = 0$. The following theorem shows that the class of operator subadditive functions satisfying this condition uses to be very small.

THEOREM 1. *Let M be a von Neumann algebra and let there exist a function $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\phi(0) = 0$, ϕ is M -subadditive, and ϕ is not of the form $\phi(t) = \alpha t$ with $\alpha \in \mathbb{R}$. Then M is commutative.*

Proof. Let M be noncommutative. Suppose ϕ is M -subadditive and $\phi(0) = 0$. We will show that ϕ has to be of the form $\phi(t) = \alpha t$ for some $\alpha \in \mathbb{R}$.

Since M is noncommutative, it is easy to check that there exist two equivalent and mutually orthogonal nonzero projections in M , i.e., there exists a nonzero partial isometry $v \in M$ such that the projections $p = v^*v$ and $q = vv^*$ are mutually orthogonal (see e.g. [4]). Take positive reals ϵ, δ such that $\epsilon \leq \delta$ and consider the pair of operators:

$$\begin{aligned} a &= \epsilon p + \sqrt{\epsilon(\delta - \epsilon)} v + \sqrt{\epsilon(\delta - \epsilon)} v^* + (\delta - \epsilon) q, \\ b &= \epsilon p - \sqrt{\epsilon(\delta - \epsilon)} v - \sqrt{\epsilon(\delta - \epsilon)} v^* + (\delta - \epsilon) q. \end{aligned}$$

Observe that a and b are positive scalar multiples of projections and a straightforward computation shows that $\phi(a + b) = \phi(2\epsilon)p + \phi(2(\delta - \epsilon))q$, $\phi(a) = (\phi(\delta)/\delta)a$, $\phi(b) = (\phi(\delta)/\delta)b$. Whence, after multiplying the inequality $\phi(a + b) \leq \phi(a) + \phi(b)$ by p from the left and the right we obtain $\phi(2\epsilon)p \leq (2\epsilon\phi(\delta)/\delta)p$. Hence, $\phi(2\epsilon)/2\epsilon \leq \phi(\delta)/\delta$. As the only restriction imposed on ϵ and δ is $0 < \epsilon \leq \delta$, it follows that $\phi(t)/t$ is a constant, say α , on $(0, \infty)$. Thus, $\phi(t) = \alpha t$ on $[0, \infty)$.

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By similar arguments, we can also prove the following.

THEOREM 2. *Let τ be a semifinite normal faithful trace on a noncommutative von Neumann algebra M . Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ satisfy $\phi(0) = 0$ and*

$$\tau(\phi(a + b)) \leq \tau(\phi(a)) + \tau(\phi(b)) \quad (*)$$

for every pair a, b of positive operators from M such that both sides of the inequality are well-defined. Then ϕ is concave on $[0, \infty)$.

This theorem may be viewed as a converse to the well-known assertion that, under certain restrictions, the inequality $(*)$ holds if ϕ is supposed to be a concave function on $[0, \infty)$ with $\phi(0) = 0$ (see [5], [6], [7]).

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Research Institute of Mathematics and Mechanics
Kazan University,
Universitetskaya 17,
Kazan, Tatarstan,
420008, Russia.
E-mail: tikhonov@niimm.kazan.su