

Alexander duality theorem and Stanley-Reisner rings

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Introduction

In this article we survey [Te] and [Fr-Te].

Alexander duality theorem plays an important role in the study on a minimal free resolution of Stanley-Reisner rings. (See [Br-He₂], [Te-Hi₁], [Te-Hi₂], for example.) In particular, Eagon and Reiner used Alexander dual complexes and proved the following interesting theorem:

THEOREM 0.1 ([Ea-Re, Theorem 3]). *Let k be a field. and let Δ be a simplicial complex and Δ^* its Alexander dual complex. Then $k[\Delta]$ has a linear resolution if and only if $k[\Delta^*]$ is Cohen-Macaulay.*

The above result is a starting point of this article. We generalize it in the following way.

THEOREM 0.2. *Let k be a field. Let Δ be a $(d-1)$ -dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\operatorname{reg} I_{\Delta} - \operatorname{indeg} I_{\Delta} = \dim k[\Delta^*] - \operatorname{depth} k[\Delta^*].$$

Note that Theorem 0.2 corresponds to Theorem 0.1 in the case that either side of the equality is 0.

Using the Auslander-Buchsbaum formula, we have the following corollary:

COROLLARY 0.3. *Let k be a field. Let Δ be a $(d-1)$ -dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\operatorname{reg} I_{\Delta} = \operatorname{pd} k[\Delta^*].$$

Here, we use $\text{indeg } I_\Delta = \text{embdim } k[\Delta^*] - \dim k[\Delta^*]$.

It is an interesting problem to estimate regularity of homogeneous ideals. Upper bounds of regularity are studied very actively in algebraic geometry and commutative algebra, that seems to be motivated by Eisenbud-Goto Conjecture. See, for example, [Kw] and [Mi-Vo]. Here we focus on monomial ideals. We give two kind of inequalities as an application of Alexander duality.

THEOREM 0.4 ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). *Let I be a monomial ideal in the polynomial $A = k[x_1, x_2, \dots, x_n]$ over a field k . Assume $\text{codim } A/I \geq 2$. Then we have*

$$\text{reg } I \leq \text{arith-deg } I.$$

Theorem 0.4 was first proved by Hoa and Trung. After that, Frübis-Krüger and the author proved it independently using Alexander duality.

THEOREM 0.5 (Monomial version of Eisenbud-Goto Conjecture). *Let k be a field. and let Δ be a pure simplicial complex connected in codimension 1. Then we have*

$$\text{reg } I_\Delta \leq \text{deg } I_\Delta - \text{codim } k[\Delta] + 1.$$

As another application, we give some upper bound for the multiplicities of homogeneous k -algebras. In [Ba-Mu] and [He-Sr], among other things, the following inequality is proved :

THEOREM 0.6 ([Ba-Mu, Proposition 3.6], [He-Sr, Corollary 3.8]). *Let k be a field and let $R = k[x_1, x_2, \dots, x_n]/I$ be a homogeneous k -algebra of codimension h_1 . Then*

$$e(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1}.$$

We improve it as follows:

THEOREM 0.7. *Let k be a field and let $R = k[x_1, x_2, \dots, x_n]/I$ be a homogeneous k -algebra of codimension $h_1 \geq 2$. Then*

$$e(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1} - \binom{\text{reg } I - \text{indeg } I + h_1 - 1}{h_1}.$$

§1. Preliminaries

We first fix notation. Let $\mathbf{N}$ (resp. $\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $|S|$ denote the cardinality of a set S .

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [St]. We refer the reader to, e.g., [Br-He], [Hi], [Ho] and [St] for the detailed information about combinatorial and algebraic background.

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ such that (i) $\{i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. Each element F of Δ is called a *face* of Δ . We call $F \in \Delta$ an *i-face* if $|F| = i + 1$. We set $d = \max\{|F| \mid F \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$. We call a maximal face a *facet*. We say that Δ is *pure* if every facet has the same cardinality. When Δ is pure, we call Δ *connected in codimension 1*, if for every two facets F and G , there is a sequence of facets $F = F_0, F_1, \dots, F_p = G$ such that $|F_i \cap F_{i+1}| = |F_i| - 1$ for $0 \leq i \leq p - 1$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d - 1$, denote the number of i -faces in Δ . We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ the *f-vector* of Δ . Define the *h-vector* $h(\Delta) = (h_0, h_1, \dots, h_d)$ of Δ by

$$\sum_{i=0}^d f_{i-1}(t-1)^{d-i} = \sum_{i=0}^d h_i t^{d-i}.$$

If F is a face of Δ , then we define a subcomplex $\text{link}_\Delta F$ as follows:

$$\text{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i -th *reduced simplicial homology group* of Δ with the coefficient field k .

Let $A = k[x_1, x_2, \dots, x_n]$ be the polynomial ring in n -variables over a field k . Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \dots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k .

THEOREM 1.1 (Hochster's formula on the local cohomology modules

(cf. [St, Theorem 4.1])).

$$F(H_{\mathfrak{m}}^i(k[\Delta]), t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\text{link}_{\Delta} F; k) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{|F|}.$$

where $H_{\mathfrak{m}}^i(k[\Delta])$ denote the i -th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\mathfrak{m}$.

Let A be the polynomial ring $k[x_1, x_2, \dots, x_n]$ for a field k . Let M be a finitely generated graded A -module and let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A . We call $\beta_i(M) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M)$ the i -th Betti number of M over A . We sometimes denote $\beta_i^A(M)$ for $\beta_i(M)$ to emphasize the base ring A . We define a *Castelnuovo-Mumford regularity* $\text{reg } M$ of M by

$$\text{reg } M = \max \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

We define an *initial degree* $\text{indeg } M$ of M by

$$\text{indeg } M = \min \{i \mid M_i \neq 0\} = \min \{j \mid \beta_{0,j}(M) \neq 0\}.$$

THEOREM 1.2 (Hochster's formula on the Betti numbers [Ho, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F \subset [n], |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

Finally we quote some result on Gröbner bases we use later. See [Ei, Chapter 15] for complete explanation.

Let A be the polynomial ring $k[x_1, x_2, \dots, x_n]$ for a infinite field k . Let I be a homogeneous ideal in A . We denote $\text{Gin } (I)$ to be a *generic initial ideal* of I with respect to the reverse lexicographic order. It is well known that $e(A/\text{Gin } (I)) = e(A/I)$.

Further we have:

THEOREM 1.3 ([Ba-St]).

- (1) $\text{depth } A/\text{Gin } (I) = \text{depth } A/I$.
- (2) $\text{reg } \text{Gin } (I) = \text{reg } I$.

§2. Alexander duality and some generalization of the Eagon-Reiner theorem

First we recall the definition of Alexander dual complexes.

Definition (cf. [Ea-Re]). For a simplicial complex Δ on the vertex set $[n]$, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

If $\dim \Delta \leq n - 3$, then Δ^* is also a simplicial complex on the vertex set $[n]$.

In the rest of the paper we always assume $\dim k[\Delta] = d$ and $\dim k[\Delta^*] = d^*$ for a fixed field k .

Now we give some generalization of the Eagon-Reiner theorem.

THEOREM 2.1. *Let Δ be a $(d - 1)$ -dimensional complex on the vertex set $[n]$. Suppose $d \leq n - 2$. Then*

$$\operatorname{reg} I_{\Delta} - \operatorname{indeg} I_{\Delta} = \dim k[\Delta^*] - \operatorname{depth} k[\Delta^*].$$

Proof. Put $\operatorname{depth} k[\Delta^*] = p^*$. By Hochster's formula on the local cohomology modules, we have

$$F(H_{\mathbf{m}}^l(k[\Delta^*]), t) = \sum_{F \in \Delta^*} \dim_k \tilde{H}_{l-|F|-1}(\operatorname{link}_{\Delta^*} F; k) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{|F|}.$$

Hence if $l < p^*$, then $\tilde{H}_{l-|F|-1}(\operatorname{link}_{\Delta^*} F; k) = (0)$ for all $F \in \Delta^*$. By the proof in [Ea-Re, Proposition 1], we have $\tilde{H}_{n-l-2}(\Delta_F; k) = (0)$ for all $F \subset [n]$. By Hochster's formula on the Betti numbers this means that $\beta_{i,i+n-l-1}(k[\Delta]) = 0$ for $i \geq 1$. Hence

$$\beta_{i,i+n}(I_{\Delta}) = \beta_{i,i+n-1}(I_{\Delta}) = \cdots = \beta_{i,i+n-p^*+1}(I_{\Delta}) = 0$$

for $i \geq 0$. Similarly, since $\tilde{H}_{n-p^*-2}(\Delta_{[n] \setminus F}; k) \cong \tilde{H}_{p^*-|F|-1}(\operatorname{link}_{\Delta^*} F; k) \neq (0)$ for some $F \in \Delta$, we have $\beta_{i,i+n-p^*}(I_{\Delta}) \neq 0$ for some $i \geq 0$. Hence $\operatorname{reg} I_{\Delta} = n - p^*$. By the definition of the Alexander dual complex we have $\operatorname{indeg} I_{\Delta} = n - d^*$. Therefore, we have $\operatorname{reg} I_{\Delta} - \operatorname{indeg} I_{\Delta} = d^* - p^*$. Q.E.D.

§3. On upper bounds for regularity of monomial ideals

In this section we give some upper bounds for regularity of monomial ideals.

THEOREM 3.1 ([Fr-Te, Theorem 3.1]). *Let k be a field. and let Δ be a simplicial complex. Assume $\text{codim } k[\Delta] \geq 2$. Then we have*

$$\text{reg } I_{\Delta} \leq \text{arith-deg } I_{\Delta}.$$

See, for example, [Ba-Mu] for the definition of arithmetic degree of an ideal I . Here we just remark that arithmetic degree $\text{arith-deg } I_{\Delta}$ of a square-free monomial ideal I_{Δ} is the number of the facets in Δ .

Proof. Taylor resolution guarantees $\text{pd } k[\Delta^*] \leq \beta_0(I_{\Delta^*})$. Then we have

$$\text{reg } I_{\Delta} = \text{pd } k[\Delta^*] \leq \beta_0(I_{\Delta^*}) = \text{arith-deg } I_{\Delta}$$

by Corollary 0.3.

Q.E.D.

By combinatorial argument on standard pairs, which are introduced by [St-Tr-Vo], we can show:

THEOREM 3.2 ([Fr-Te, Corollary 3.6]). *Let I be a monomial ideal of a polynomial ring. Put I^{pol} be the polarization of I . Then we have*

$$\text{reg } I = \text{reg } I^{\text{pol}}.$$

See, for example, [St-Vo] for the definition and basic properties of the polarization of monomial ideals.

Combining Theorem 3.1 and 3.2, we have:

THEOREM 3.3 ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). *Let I be a monomial ideal in the polynomial $A = k[x_1, x_2, \dots, x_n]$ over a field k . Assume $\text{codim } A/I \geq 2$. Then we have*

$$\text{reg } I \leq \text{arith-deg } I.$$

Next, we will prove a certain conjecture of Eisenbud (see [Ei-Po].), which is a monomial version of Eisenbud-Goto Conjecture (see [Ei-Go]).

THEOREM 3.4. Let k be a field and let Δ be a pure simplicial complex connected in codimension 1. Then we have

$$\operatorname{reg} I_{\Delta} \leq \deg k[\Delta] - \operatorname{codim} k[\Delta] + 1.$$

We give a sketch of a proof, which is simplified by suggestions of Eisenbud.

Sketch of proof. Let V be the vertex set of Δ . Put $\sharp(V) = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number f_{d-1} of facets in Δ .

First if $\operatorname{codim} k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose $\operatorname{codim} k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. Then there exists a facet $\sigma \in \Delta$ such that

$$\Delta' := \Delta \setminus \{\tau \in \Delta \mid \text{For any facet } \rho (\neq \sigma) \in \Delta; \tau \not\subset \rho\}$$

is pure and connected in codimension 1. Denote by V' the vertex set of Δ' and by f'_{d-1} the number of facets in Δ' . There are two cases.

Case(i) $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \notin W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, We have $\tilde{H}_i(\Delta_W; k) \cong \tilde{H}_i(\Delta'_{W \setminus \{v\}}; k)$ for $i \geq 1$. Since

$$\operatorname{reg} I_{\Delta} = \max\{i + 2 \mid \tilde{H}_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\begin{aligned} \operatorname{reg} I_{\Delta} &= \operatorname{reg} I_{\Delta'} \\ &\leq f'_{d-1} - (n - 1 - d) + 1 \\ &= f_{d-1} - (n - d) + 1. \end{aligned}$$

Case(ii) $V = V'$. We have $\operatorname{reg} I_{\Delta} = \operatorname{pd} k[\Delta^*]$ by Corollary 0.3. If we prove $\operatorname{pd} k[\Delta^*] \leq \operatorname{pd} k[(\Delta')^*] + 1$, we have

$$\begin{aligned} \operatorname{reg} I_{\Delta} &\leq \operatorname{reg} I_{\Delta'} + 1 \\ &\leq f'_{d-1} - (n - d) + 2 \\ &= f_{d-1} - (n - d) + 1. \end{aligned}$$

Then we have only to prove

$$\text{pd } k[\Delta^*] \leq \text{pd } k[(\Delta')^*] + 1.$$

Put $k[\Delta^*] = k[(\Delta')^*]/(m)$, where $m = \prod_{x_i \in V \setminus \sigma} x_i$. If we show that

$$\text{pd } k[(\Delta')^*] \geq \text{pd } (I_{(\Delta')^*} + (m))/I_{(\Delta')^*},$$

then the mapping cone guarantees that

$$\text{pd } k[\Delta^*] \leq \text{pd } k[(\Delta')^*] + 1$$

by [E, Exercise A.3.30]. But now we have

$$\begin{aligned} (I_{(\Delta')^*} + (m))/I_{(\Delta')^*} &\cong (m)/((m) \cap I_{(\Delta')^*}) \\ &\cong (m)/((m) \cap (m_1, \dots, m_t)) \\ &\cong (m)/(\text{lcm}(m, m_1) \dots, \text{lcm}(m, m_t)) \\ &\cong A/(m'_1, \dots, m'_t) \otimes_A (m), \end{aligned}$$

where $I_{(\Delta')^*} = (m_1, \dots, m_t)$, $m'_i = \frac{\text{lcm}(m, m_i)}{m}$, and $A = k[x_i \mid x_i \in V]$. Hence, we have only to show

$$\text{pd } k[(\Delta')^*] \geq \text{pd } A/(m'_1, \dots, m'_t).$$

Now we have $k[(\Delta')^*]_m \cong A_m/(m'_1, \dots, m'_t)A_m$. Hence we have

$$\text{pd } k[(\Delta')^*] \geq \text{pd } k[(\Delta')^*]_m = \text{pd } A_m/(m'_1, \dots, m'_t)A_m = \text{pd } A/(m'_1, \dots, m'_t).$$

Q.E.D.

§4. On upper bounds for multiplicities

In this section we give some upper bound for the multiplicities of homogeneous k -algebras.

First we prove the following lemma:

LEMMA 4.1.

$$e(k[\Delta]) = \beta_{1, h_1}(k[\Delta^*]).$$

Proof. We have

$$h_0(\Delta) + h_1(\Delta)(1-t) + \dots + h_d(\Delta)(1-t)^d \quad (1)$$

$$= \frac{1 - (1-t)^{n-d^*} (h_0(\Delta^*) + h_1(\Delta^*)t + \dots + h_{d^*}(\Delta^*)t^{d^*})}{t^{n-d}}. \quad (2)$$

Since $\text{indeg } I_{\Delta^*} = n - d = h_1$, we have

$$\begin{aligned}
 & \beta_{1,n-d}(k[\Delta^*]) \\
 &= (\text{the coefficient of } t^{n-d} \text{ in } -(1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})) \\
 &= (\text{the coefficient of } t^{n-d} \text{ in the numerator in (2)}) \\
 &= \lim_{t \rightarrow 0} (h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d) \\
 &= e(k[\Delta]).
 \end{aligned}$$

Q.E.D.

THEOREM 4.2. *Let $R = A/I$ be a homogeneous k -algebra of codimension $h_1 \geq 2$. Then*

$$e(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1} - \binom{\text{reg } I - \text{indeg } I + h_1 - 1}{h_1}.$$

Proof. We may assume $|k| = \infty$. By Theorem 1.3, we have $\text{reg } \text{Gin}(I) = \text{reg } I$ and $h(A/I) = h(A/\text{Gin}(I))$. Considering the polarization, we obtain a Stanley-Reisner ring $k[\Delta] = B/I_{\Delta}$ with $e(A/I) = e(k[\Delta])$ and $\text{reg } I = \text{reg } I_{\Delta}$. Put $p^* = \text{depth } k[\Delta^*]$. By Theorem 2.1, we have $d^* - p^* = \text{reg } I - (n - d^*)$, where $n = \text{embdim } k[\Delta^*]$. Hence $\text{reg } I = n - p^*$.

Let $y_1, y_2, \dots, y_{p^*}$ be a regular sequence in $k[\Delta^*]_1$, and let $z_1, z_2, \dots, z_{d^*-p^*} \in (k[\Delta^*]/(y_1, y_2, \dots, y_{p^*}))_1$ be a system of parameters of $k[\Delta^*]/(y_1, y_2, \dots, y_{p^*})$. We have $k[z_1, z_2, \dots, z_{d^*-p^*}] \subset k[\Delta^*]/(y_1, y_2, \dots, y_{p^*})$. Since $k[z_1, z_2, \dots, z_{d^*-p^*}]$ is isomorphic to the polynomial ring with $d^* - p^*$ variables, we have $\dim_k(k[\Delta^*]/(y_1, y_2, \dots, y_{p^*}))_{h_1} \geq \binom{d^*-p^*+h_1-1}{h_1}$. By Lemma 4.1, we have

$$\begin{aligned}
 e(k[\Delta]) &= \beta_{1,h_1}(k[\Delta^*]) \\
 &= \beta_{1,h_1}^{B/(y_1, y_2, \dots, y_{p^*})}(k[\Delta^*]/(y_1, y_2, \dots, y_{p^*})) \\
 &= \dim_k(B/(y_1, y_2, \dots, y_{p^*}))_{h_1} - \dim_k(k[\Delta^*]/(y_1, y_2, \dots, y_{p^*}))_{h_1} \\
 &\leq \binom{n - p^* + h_1 - 1}{h_1} - \binom{d^* - p^* + h_1 - 1}{h_1}.
 \end{aligned}$$

Q.E.D.

References

- [Ba-Mu] D. Bayer and D. Mumford, *What can be computed in algebraic geometry*, in "Computational Algebraic Geometry and Commutative

- Algebra (D. Eisenbud and L. Robbiano, eds.)," Cambridge University Press, 1993, pp.1 – 48.
- [Ba-St] D. Bayer and M. Stillman, *A theorem on refining division orders by the reverse lexicographic orders*, Duke J. Math. **55** (1987) 321–328.
- [Br-He₁] W. Bruns and J. Herzog, "Cohen-Macaulay Rings," Cambridge University Press, Cambridge / New York / Sydney, 1993.
- [Br-He₂] W. Bruns and J. Herzog, *Semigroup rings and simplicial complexes*, Journal of Pure and Applied Algebra **122** (1997) 185–208.
- [Ea-Re] J. A. Eagon and V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, Preprint.
- [Ei] D. Eisenbud, "Commutative Algebra with a view toward Algebraic Geometry," Springer-Verlag, New York, 1995.
- [Ei-Go] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicities*, J. of Algebra **88** (1984) 89–133.
- [Ei-Po] D. Eisenbud and S. Popescu, Research Problems: PRAGMATIC 1997.
- [Fr-Te] A. Fröbis-Krüger and N. Terai, *Bounds for the regularity of monomial ideals*, Preprint.
- [He-Sr] J. Herzog and H. Srinivasan, *Bounds for multiplicities*, Transaction of American Math. Society **350** (1998) 2879–2902.
- [Hi] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe, N.S.W., Australia, 1992.
- [Ho] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in "Ring Theory II (B. R. McDonald and R. Morris, eds.)," Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, 1977, pp.171 – 223.
- [Ho-Tr] L. T. Hoa and N. V. Trung, *On the Castelnuovo-Mumford regularity and the arithmetic degree of monomial ideals*, Preprint.
- [Kw] S. Kwak, *Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4*, J. of Algebraic Geometry **7** (1998) 195–206.

- [Mi-Vo₁] C. Miyazaki and W. Vogel, *Bounds on cohomology and Castelnuovo-Mumford regularity*, J. of Algebra **185** (1996) 626–642.
- [Mi-Vo₂] C. Miyazaki and W. Vogel, *Towards a theory of arithmetic degrees*, manuscripta math. **89** (1996), 427–438.
- [St] R. P. Stanley, “Combinatorics and Commutative Algebra, Second Edition,” Birkhäuser, Boston / Basel / Stuttgart, 1996.
- [St-Vo] J. Stückrad and W. Vogel, “Buchsbaum Rings and Applications,” Springer-Verlag, Berlin / Heidelberg / New York, 1986.
- [St-Tr-Vo] B. Sturmfels, N. V. Trung, and W. Vogel, *Bounds on degrees of projective schemes*, Math. Ann. **302** (1995), 417–432.
- [Te] N. Terai, *Generalization of Eagon-Reiner theorem and h-vectors of graded rings*, Preprint.
- [Te-Hi₁] N. Terai and T. Hibi, *Some results on Betti numbers of Stanley-Reisner rings*, Discrete Math. **157** (1996), 311–320.
- [Te-Hi₂] N. Terai and T. Hibi, *Alexander duality theorem and second Betti numbers of Stanley-Reisner rings*, Advances in Math. **124** (1996), 332–333.

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