

N-Fractional Calculus of Some Algebraic Functions

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Abstract

In this article, N-fractional calculus for the functions

$$\frac{1}{z^2 + (a+b)z + ab} \quad \text{and} \quad \frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab}$$

where $z^2 + (a+b)z + ab \neq 0$ are discussed.

1 Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$, C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = C(f)_\nu \\ &= \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in Z^+), \quad (2)$$

where

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \text{ Gamma function,}$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}} \right) \quad (\nu \notin Z^-), \quad (\text{Refer to [1]}) \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in Z^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in R), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii),

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

2 Preliminary

The following theorem has been reported by K. Nishimoto [12].

Theorem D. We have

(i)

$$(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta}\right)^k \quad (1)$$

$$(|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta}\right)^k \quad (2)$$

$$(n \in \mathbb{Z}_0^+, \quad |\frac{c}{(z-b)^\beta}| < 1),$$

where

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with } [\lambda]_0 = 1,$$

(Pochhammer's Notation).

We apply this theorem to have some theorems for some algebraic functions.

3 Some Theorems and Identities

We describe some results for the functions $\frac{1}{z^2+(a+b)z+ab}$ and $\frac{z+\frac{a+b}{2}}{z^2+(a+b)z+ab}$ by applying the theorem in §2.

Theorem 1. We have

(i)

$$\begin{aligned} \left(\frac{1}{z^2 + (a+b)z + ab} \right)_\gamma &= e^{-i\pi\gamma} \left(\frac{2z + a + b}{2} \right)^{-2-\gamma} \\ &\times \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+2+\gamma)}{k! \Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \end{aligned} \quad (1)$$

$$(|\Gamma(2k+2+\gamma)| < \infty)$$

and

(ii)

$$\begin{aligned} \left(\frac{1}{z^2 + (a+b)z + ab} \right)_n &= (-1)^n \left(\frac{2z + a + b}{2} \right)^{-2-n} \\ &\times \sum_{k=0}^{\infty} [2k+2]_n \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k, \end{aligned} \quad (n \in \mathbb{Z}_0^+) \quad (2)$$

where $z^2 + (a+b)z + ab \neq 0$, $|(a-b)/(2z+a+b)| < 1$.

Proof of (i). Setting $p = -(a+b)/2$, $q = (\frac{a+b}{2})^2 - ab$, we have

$$z^2 + (a+b)z + ab = (z-p)^2 - q, \quad (3)$$

hence

$$\begin{aligned} \left(\frac{1}{z^2 + (a+b)z + ab} \right)_\gamma &= (((z-p)^2 - q)^{-1})_\gamma \\ &= e^{-i\pi\gamma} (z-p)^{-2-\gamma} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+2+\gamma)}{k! \Gamma(2k+2)} \left(\frac{q}{(z-p)^2} \right)^k \end{aligned} \quad (4)$$

by Theorem D, (i), and we can rewrite this equation immediately to (1).

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (1).

Theorem 2. We have

(i)

$$\left(\frac{1}{z^2 + (a+b)z + ab}\right)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma+1)}{b-a} \left(\frac{1}{(z+a)^{1+\gamma}} - \frac{1}{(z+b)^{1+\gamma}}\right) \quad (5)$$

$$(|\Gamma(1+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{1}{z^2 + (a+b)z + ab}\right)_n = (-1)^n \frac{n!}{b-a} \left(\frac{1}{(z+a)^{1+n}} - \frac{1}{(z+b)^{1+n}}\right) \quad (6)$$

$$(n \in Z_0^+)$$

where $b-a \neq 0$, $z^2 + (a+b)z + ab \neq 0$.

Proof of (i). According to Lemma (i), we have

$$\left(\frac{1}{z^2 + (a+b)z + ab}\right)_\gamma = \frac{1}{b-a} \left(\frac{1}{z+a} - \frac{1}{z+b}\right)_\gamma \quad (7)$$

$$= \frac{1}{b-a} \{((z+a)^{-1})_\gamma - ((z+b)^{-1})_\gamma\} \quad (8)$$

$$= e^{-i\pi\gamma} \frac{\Gamma(\gamma+1)}{b-a} \{(z+a)^{-1-\gamma} - (z+b)^{-1-\gamma}\}, \quad (9)$$

This is the result.

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (5).

{Note} When $a+b=0$, we obtain the following case from this theorem,

$$\left(\frac{1}{z^2 - a^2}\right)_\gamma = \frac{1}{2a} \left(\frac{1}{z-a} + \frac{1}{z+a}\right)_\gamma \quad (z^2 - a^2 \neq 0).$$

Theorem 3. We have the identities

(i)

$$\left(\frac{2z+a+b}{2}\right)^{-2-\gamma} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k+2+\gamma)}{k! \Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$

$$= \frac{\Gamma(\gamma+1)}{b-a} \left(\frac{1}{(z+a)^{1+\gamma}} - \frac{1}{(z+b)^{1+\gamma}}\right) \quad (10)$$

$$(|\Gamma(2k+2+\gamma)| < \infty, \quad |\Gamma(\gamma+1)| < \infty)$$

and

(ii)

$$\begin{aligned}
& \left(\frac{2z + a + b}{2} \right)^{-2-n} \sum_{k=0}^{\infty} [2k+2]_n \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \\
&= \frac{n!}{b-a} \left(\frac{1}{(z+a)^{1+n}} - \frac{1}{(z+b)^{1+n}} \right) \quad (11) \\
& \quad (n \in Z_0^+)
\end{aligned}$$

where $|\frac{a-b}{2z+a+b}| < 1$, $b-a \neq 0$, $z^2 + (a+b)z + ab \neq 0$.

Proof. It is valid from Theorem 1 and Theorem 2.

Theorem 4. We have

(i)

$$\begin{aligned}
& \left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab} \right)_\gamma = e^{-i\pi\gamma} \left(\frac{2}{2z+a+b} \right)^{1+\gamma} \\
& \times \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^m \quad (12) \\
& \quad (|\Gamma(2m+1+\gamma)| < \infty)
\end{aligned}$$

and

(ii)

$$\begin{aligned}
& \left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab} \right)_n = (-1)^n \left(\frac{2}{2z+a+b} \right)^{1+n} \\
& \times \sum_{m=0}^{\infty} [2m+1]_n \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^m \quad (13) \\
& \quad (n \in Z_0^+)
\end{aligned}$$

where $z^2 + (a+b)z + ab \neq 0$, $|(a-b)/(2z+a+b)| < 1$.

Proof of (i). Setting $p = -(a+b)/2$, $q = (\frac{a+b}{2})^2 - ab$, we have

$$z^2 + (a+b)z + ab = (z-p)^2 - q, \quad (14)$$

hence we obtain

$$\left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab} \right)_\gamma = \left(((z-p)^2 - q)^{-1} \cdot (z-p) \right)_\gamma \quad (15)$$

$$= \sum_{k=0}^1 \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} (((z-p)^2 - q)^{-1})_{\gamma-k} (z-p)_k \quad (16)$$

by Lemma (iv). According to Theorem D, (i) and Lemma (i), we have

$$\begin{aligned} (((z-p)^2 - q)^{-1})_{\gamma-k} &= e^{-i\pi(\gamma-k)} (z-p)^{k-2-\gamma} \\ &\times \sum_{m=0}^{\infty} \frac{[1]_m}{m!} \frac{\Gamma(2m+2+\gamma-k)}{\Gamma(2m+2)} \left(\frac{q}{(z-p)^2} \right)^m, \end{aligned} \quad (17)$$

$$(|q/(z-p)^2| < 1, \quad |\Gamma(2m+2+\gamma-k)| < \infty)$$

and

$$(z-p)_k = e^{-i\pi k} \frac{\Gamma(k-1)}{\Gamma(-1)} (z-p)^{1-k}. \quad (18)$$

So we have

$$\begin{aligned} (((z-p)^2 - q)^{-1}) \cdot (z-p)_\gamma &= e^{-i\pi\gamma} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(2m+2+\gamma)}{\Gamma(2m+2)} (z-p)^{-1-\gamma} \left(\frac{q}{(z-p)^2} \right)^m \right. \\ &\quad \left. - \gamma \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+2)} (z-p)^{-1-\gamma} \left(\frac{q}{(z-p)^2} \right)^m \right\} \end{aligned} \quad (19)$$

$$= e^{-i\pi\gamma} (z-p)^{-1-\gamma} \sum_{m=0}^{\infty} \frac{(2m+1)\Gamma(2m+1+\gamma)}{\Gamma(2m+2)} \left(\frac{q}{(z-p)^2} \right)^m \quad (20)$$

$$= e^{-i\pi\gamma} \left(z + \frac{a+b}{2} \right)^{-1-\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)} \left(\frac{(\frac{a+b}{2})^2 - ab}{(z + \frac{a+b}{2})^2} \right)^m. \quad (21)$$

Therefore we have (12) from (21) clearly under the conditions stated before.

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (12).

Corollary 1. We have

(i)

$$\left(\frac{z}{z^2 - a^2} \right)_\gamma = e^{-i\pi\gamma} \left(\frac{1}{z} \right)^{1+\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)} \left(\left(\frac{a}{z} \right)^2 \right)^m \quad (22)$$

$$(|\Gamma(2m+1+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{z}{z^2 - a^2}\right)_n = (-1)^n \left(\frac{1}{z}\right)^{1+n} \sum_{m=0}^{\infty} [2m+1]_n \left(\left(\frac{a}{z}\right)^2\right)^m \quad (23)$$

$$(n \in Z_0^+),$$

$$\text{where } z^2 - a^2 \neq 0, \quad \left|\frac{a}{z}\right| < 1.$$

Proof. It is clear from Theorem 4, by setting $a + b = 0$.

Theorem 5. We have

(i)

$$\left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(1+\gamma)}{2} \left\{ \frac{1}{(z+a)^{1+\gamma}} + \frac{1}{(z+b)^{1+\gamma}} \right\} \quad (24)$$

$$(|\Gamma(1+\gamma)| < \infty)$$

and

(ii)

$$\left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_n = (-1)^n \frac{n!}{2} \left\{ \frac{1}{(z+a)^{1+n}} + \frac{1}{(z+b)^{1+n}} \right\} \quad (25)$$

$$(n \in Z_0^+)$$

$$\text{where } z^2 + (a+b)z + ab \neq 0, \quad z + \frac{a+b}{2} \neq 0.$$

Proof of (i). We have

$$\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab} = \frac{1}{2} \left(\frac{1}{z+a} + \frac{1}{z+b} \right) \quad (26)$$

hence, based on Lemma (i) we have the followings under the conditions,

$$LHS \text{ of (24)} = \frac{1}{2} \left(\frac{1}{z+a} + \frac{1}{z+b} \right)_\gamma \quad (27)$$

$$= \frac{1}{2} \left(\left(\frac{1}{z+a} \right)_\gamma + \left(\frac{1}{z+b} \right)_\gamma \right) \quad (28)$$

$$= \frac{1}{2} e^{-i\pi\gamma} \Gamma(\gamma+1) \left(\frac{1}{(z+a)^{1+\gamma}} + \frac{1}{(z+b)^{1+\gamma}} \right). \quad (29)$$

Proof of (ii). We have the result by setting $\gamma = n$ in (i).

Theorem 6. We have the identities

(i)

$$\begin{aligned} \left(\frac{2}{2z+a+b}\right)^{1+\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(2m+1+\gamma)}{\Gamma(2m+1)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m \\ = \frac{\Gamma(\gamma+1)}{2} \left(\frac{1}{(z+a)^{1+\gamma}} + \frac{1}{(z+b)^{1+\gamma}}\right) \quad (30) \\ (|\Gamma(2m+1+\gamma)| < \infty) \end{aligned}$$

and

(ii)

$$\begin{aligned} \left(\frac{2}{2z+a+b}\right)^{1+n} \sum_{m=0}^{\infty} [2m+1]_n \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^m \\ = \frac{n!}{2} \left(\frac{1}{(z+a)^{1+n}} + \frac{1}{(z+b)^{1+n}}\right) \quad (31) \\ (n \in Z_0^+) \end{aligned}$$

where $|\frac{a-b}{2z+a+b}| < 1$, $z + (a+b)/2 \neq 0$, $z^2 + (a+b)z + ab \neq 0$.

Proof. It is valid from Theorem 4 and Theorem 5.

4 Semi Derivatives and Integrals

In this section we give semi derivatives and integrals from Theorems 1, 4, 5 and Corollary.

(I) From Theorem 1, (i), we have

(i)

$$\begin{aligned} \left(\frac{1}{z^2 + (a+b)z + ab}\right)_{\frac{1}{2}} = -i \left(\frac{2z+a+b}{2}\right)^{-\frac{5}{2}} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k + \frac{5}{2})}{k! \Gamma(2k+2)} \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k, \\ (1) \\ (\text{semi derivative}) \end{aligned}$$

(ii)

$$\left(\frac{1}{z^2 + (a+b)z + ab}\right)_{-\frac{1}{2}} = i \left(\frac{2z + a + b}{2}\right)^{-\frac{3}{2}} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k + \frac{3}{2})}{k! \Gamma(2k + 2)} \left(\left(\frac{a-b}{2z + a + b}\right)^2\right)^k \quad (2)$$

(semi integral)

where

$$z^2 + (a+b)z + ab \neq 0 \quad \text{and} \quad |(a-b)/(2z + a + b)| < 1.$$

(II) From Theorem 4, (i), we have

(i)

$$\left(\frac{1 + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_{\frac{1}{2}} = -i \left(\frac{2}{2z + a + b}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m + \frac{3}{2})}{\Gamma(2m + 1)} \left(\left(\frac{a-b}{2z + a + b}\right)^2\right)^m \quad (3)$$

(semi derivative)

and

(ii)

$$\left(\frac{1 + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_{-\frac{1}{2}} = i \left(\left(\frac{2}{2z + a + b}\right)^2\right)^m \sum_{m=0}^{\infty} \frac{\Gamma(2m + \frac{1}{2})}{\Gamma(2m + 1)} \left(\left(\frac{a-b}{2z + a + b}\right)^2\right)^m \quad (4)$$

(semi integral)

where

$$z^2 + (a+b)z + ab \neq 0 \quad \text{and} \quad |(a-b)/(2z + a + b)| < 1.$$

(III) From Corollary 1, (i), we have

(i)

$$\left(\frac{z}{z^2 - a^2}\right)_{\frac{1}{2}} = -i \left(\frac{1}{z}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m + \frac{3}{2})}{\Gamma(2m + 1)} \left(\frac{a}{z}\right)^{2m}, \quad (5)$$

(semi derivative)

(ii)

$$\left(\frac{z}{z^2 - a^2}\right)_{-\frac{1}{2}} = i \left(\frac{1}{z}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(2m + \frac{1}{2})}{\Gamma(2m + 1)} \left(\frac{a}{z}\right)^{2m} \quad (6)$$

(semi integral)

where

$$z^2 - a^2 \neq 0 \quad \text{and} \quad |a/z| < 1.$$

(IV) From Theorem 5, (i), we have

(i)

$$\left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_{\frac{1}{2}} = -i \frac{\sqrt{\pi}}{4} \left\{ \frac{1}{(z+a)^{(3/2)}} + \frac{1}{(z+b)^{(3/2)}} \right\} \quad (7)$$

(semi derivative)

and

(ii)

$$\left(\frac{z + \frac{a+b}{2}}{z^2 + (a+b)z + ab}\right)_{-\frac{1}{2}} = i \frac{\sqrt{\pi}}{2} \left\{ \frac{1}{\sqrt{z+a}} + \frac{1}{\sqrt{z+b}} \right\} \quad (8)$$

(semi integral)

where

$$z^2 + (a+b)z + ab \neq 0 \quad \text{and} \quad |z + (a+b)/2| \neq 0.$$

5 Examples of Theorem 3, (ii)

We illustrate some examples of Theorem 3 in cases of which the order of derivative are integers $n = 0, 1, 2$.

(I) When $n = 0$, from Theorem 3 the left-hand side of (11) in §3 is derived as follows,

$$\left(\frac{2z + a + b}{2}\right)^{-2} \sum_{k=0}^{\infty} [2k + 2]_0 \left(\left(\frac{a-b}{2z+a+b}\right)^2\right)^k$$

$$= \left(\frac{2}{2z+a+b} \right)^2 \sum_{k=0}^{\infty} \frac{[1]_k}{k!} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \quad (1)$$

$$= \left(\frac{2}{2z+a+b} \right)^2 \left(1 - \left(\frac{a-b}{2z+a+b} \right)^2 \right)^{-1} \quad (2)$$

$$= \frac{1}{(z+a)(z+b)}. \quad (3)$$

And we derive the followings from the right-hand side of (11),

$$\frac{1}{b-a} \left(\frac{1}{z+a} - \frac{1}{z+b} \right) = \frac{1}{(z+a)(z+b)}. \quad (4)$$

Then (11) holds true for $n = 0$.

(II) When $n = 1$, the LHS of (11) is

$$\begin{aligned} & \left(\frac{2z+a+b}{2} \right)^{-3} \sum_{k=0}^{\infty} [2k+2]_1 \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \\ &= \left(\frac{2}{2z+a+b} \right)^3 \sum_{k=0}^{\infty} \frac{[1]_k}{k!} (2k+2) \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \end{aligned} \quad (5)$$

$$= \left(\frac{2}{2z+a+b} \right)^3 \left(\frac{(a-b)^2(2z+a+b)^2}{8(z+a)^2(z+b)^2} + \frac{(2z+a+b)^2}{2(z+a)(z+b)} \right) \quad (6)$$

$$= \frac{2z+a+b}{(z+a)^2(z+b)^2}, \quad (7)$$

since

$$\sum_{k=0}^{\infty} \frac{[1]_k}{k!} k \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k = \sum_{k=1}^{\infty} \frac{[1]_k}{(k-1)!} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \quad (8)$$

$$= \left(\frac{a-b}{2z+a+b} \right)^2 \sum_{k=0}^{\infty} \frac{[2]_k}{k!} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k \quad (9)$$

$$= \left(\frac{a-b}{2z+a+b} \right)^2 \left(1 - \left(\frac{a-b}{2z+a+b} \right)^2 \right)^{-2} \quad (10)$$

$$= \frac{(a-b)^2(2z+a+b)^2}{4^2(z+a)^2(z+b)^2} \quad (11)$$

and

$$\sum_{k=0}^{\infty} \frac{[1]_k}{k!} \left(\left(\frac{a-b}{2z+a+b} \right)^2 \right)^k = \left(1 - \left(\frac{a-b}{2z+a+b} \right)^2 \right)^{-1} \quad (12)$$

$$= \frac{(2z+a+b)^2}{4(z+a)(z+b)}. \quad (13)$$

Furthermore from RHS of (11) in §3, we have

$$\frac{1}{b-a} \left(\frac{1}{(z+a)^2} - \frac{1}{(z+b)^2} \right) = \frac{2z+a+b}{(z+a)^2(z+b)^2}. \quad (14)$$

Then for $n = 1$, section 3.(11) holds.

(III) When $n = 2$, setting

$$T = \left(\frac{a-b}{2z+a+b} \right)^2, \quad (15)$$

we have

$$LHS \text{ of section 3.(11)} = \left(\frac{2}{2z+a+b} \right)^4 \sum_{k=0}^{\infty} (2k+2)[2k+3]_1 T^k \quad (16)$$

$$= \left(\frac{2}{2z+a+b} \right)^4 \left\{ 2 \sum_{k=0}^{\infty} k[2k+3]_1 T^k + 2 \sum_{k=0}^{\infty} [2k+3]_1 T^k \right\} \quad (17)$$

$$= \left(\frac{2}{2z+a+b} \right)^4 2 \left(\frac{5T - T^2}{(1-T)^3} + \frac{3-T}{(1-T)^2} \right) \quad (18)$$

$$= \left(\frac{2}{2z+a+b} \right)^4 \left(\frac{6+2T}{(1-T)^3} \right) \quad (19)$$

$$= \frac{6z^2 + 6z(a+b) + 2(a^2 + ab + b^2)}{(z+a)^3(z+b)^3}. \quad (20)$$

On the other hand we obtain the following from the right side of (11),

$$\begin{aligned} RHS \text{ of section 3.(11)} &= \frac{2}{2z+a+b} \left(\frac{1}{(z+a)^3} - \frac{1}{(z+b)^3} \right) \\ &= \frac{6z^2(b-a) + 6z(b^2 - a^2) + 2(b^3 - a^3)}{(b-a)(z+a)^3(z+b)^3} \end{aligned} \quad (22)$$

$$= \frac{6z^2 + 6z(a+b) + 2(a^2 + ab + b^2)}{(z+a)^3(z+b)^3}. \quad (23)$$

Therefore (11) in §3 holds for the case of $n = 2$.

And so on.

{ Note } Examples for Theorem 6 (ii) shall be shown in another paper.

6 Appendix

In this section, we give some remarks on the infinite series. We have used these results in the previous section.

We have

$$\sum_{k=0}^{\infty} k[2k+3]_1 T^k = \sum_{k=1}^{\infty} \frac{[1]_k}{(k-1)!} [2k+3]_1 T^k \quad (1)$$

$$= T \sum_{k=0}^{\infty} \frac{[2]_k}{k!} (2(k+1)+3) T^k \quad (2)$$

$$= 2T \sum_{k=0}^{\infty} \frac{[2]_k}{k!} k T^k + 5T \sum_{k=0}^{\infty} \frac{[2]_k}{k!} T^k \quad (3)$$

$$= 2T^2 \sum_{k=0}^{\infty} \frac{[2]_{k+1}}{k!} T^k + 5T(1-T)^{-2} \quad (4)$$

$$= 2^2 T^2 \sum_{k=0}^{\infty} \frac{[3]_k}{k!} T^k + 5T(1-T)^{-2} \quad (5)$$

$$= 4T^2(1-T)^{-3} + 5T(1-T)^{-2} \quad (6)$$

$$= \frac{5T - T^2}{(1-T)^3}, \quad (7)$$

and

$$\sum_{k=0}^{\infty} [2k+3]_1 T^k = \sum_{k=0}^{\infty} \frac{[1]_k}{k!} (2k+3) T^k \quad (8)$$

$$= 2 \sum_{k=0}^{\infty} \frac{[1]_k}{k!} k T^k + 3 \sum_{k=0}^{\infty} \frac{[1]_k}{k!} T^k \quad (9)$$

$$= 2T \sum_{k=0}^{\infty} \frac{[2]_k}{k!} T^k + 3(1-T)^{-1} \quad (10)$$

$$= 2T(1-T)^{-2} + 3(1-T)^{-1} \quad (11)$$

$$= \frac{3-T}{(1-T)^2}. \quad (12)$$

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