

On v -adic periods of t -motives: a resume

By

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Abstract

This is a resume of our results ([7]) on v -adic periods of t -motives, where v is a “finite” place of the rational function field over a finite field. For a t -motive M , we define v -adic periods of M and the fundamental group of the Tannakian category generated by M . Our main result is the transcendental degree of the extension generated by the v -adic periods is equal to the dimension of the fundamental group.

§ 1. Introduction

The special values $\zeta(n)$ of the Riemann zeta function are important objects in number theory. However we do not know how many algebraic relations are there among them over $\mathbb{Q}$. Euler proved that if $n \geq 2$ is an even integer, then we have $\zeta(n)/\pi^n \in \mathbb{Q}$. The odd integer points are more mysterious and we have the following conjecture:

Conjecture 1.1. *For each integer $n \geq 2$, we have the equality*

$$\mathrm{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\pi, \zeta(2), \dots, \zeta(n)) = n - \lfloor n/2 \rfloor.$$

Here, for a real number x , we denote by $\lfloor x \rfloor$ the largest integer not greater than x .

To prove Conjecture 1.1 seems to be very difficult, but the function field analogue of this conjecture was proved by Chang and Yu ([4]). We explain this briefly.

Let $\mathbb{F}_q$ be the finite field with q elements, p the characteristic of $\mathbb{F}_q$, $K := \mathbb{F}_q(\theta)$ the rational function field over $\mathbb{F}_q$, and $K_{\infty} := \mathbb{F}_q((\theta^{-1}))$ the ∞ -adic completion of K . For each integer $n \geq 1$, the Carlitz zeta value is defined by

$$\zeta_C(n) := \sum_{a \in \mathbb{F}_q[\theta], \text{monic}} a^{-n} \in K_{\infty}.$$

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This is the function field analogue of the Riemann zeta values. Fix a $(q-1)$ st root $(-\theta)^{\frac{1}{q-1}}$ of $-\theta$ and set

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_{\infty}((-\theta)^{\frac{1}{q-1}}).$$

This is an analogue of $2\pi\sqrt{-1}$. By the definition of ζ_C , we have the equalities $\zeta_C(np^m) = \zeta_C(n)p^m$ for all integers $m, n \geq 1$. Carlitz proved that if n is divisible by $q-1$, then we have $\zeta_C(n)/\tilde{\pi}^n \in K$. This is an analogue of the corresponding fact about the Riemann zeta values at positive even integers (note that $q-1$ and 2 are respectively the cardinalities of $\mathbb{F}_q[\theta]^{\times}$ and $\mathbb{Z}^{\times}$). Chang and Yu proved that these are essentially the only relations among the special values.

Theorem 1.2 ([4, Corollary 4.6]). *For each integer $n \geq 1$, we have the equality*

$$\text{tr.deg}_K K(\tilde{\pi}, \zeta_C(1), \dots, \zeta_C(n)) = n + 1 - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/p(q-1) \rfloor.$$

The proof of Theorem 1.2 uses Papanikolas' result on periods of t -motives. We explain this result. Let t be a variable independent of θ . A t -motive over K is a free $K[t]$ -module M of finite rank equipped with a “Frobenius action” satisfying certain conditions. Let $\mathbb{C}_{\infty}$ be the ∞ -adic completion of an algebraic closure of K_{∞} and $|\cdot|_{\infty}$ its valuation. Set $\mathbb{T} := \{f \in \mathbb{C}_{\infty}[[t]] \mid f \text{ converges on } |t|_{\infty} \leq 1\}$ and $\mathbb{L} := \text{Frac } \mathbb{T}$ the fraction field of $\mathbb{T}$. For a t -motive M , a Betti realization $H_B(M) \subset \mathbb{L} \otimes_{K[t]} M$ is defined. This is an $\mathbb{F}_q(t)$ -vector space and we have $\dim_{\mathbb{F}_q(t)} H_B(M) \leq \text{rank}_{K[t]} M$. Assume that the equality holds. Such t -motives are called *rigid analytically trivial*. Fix bases $\mathbf{x}$ of $H_B(M)$ and $\mathbf{m}$ of M . We obtain the matrix $\Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(\mathbb{L})$ such that $\mathbf{m} = \Psi \mathbf{x}$ in $\mathbb{L} \otimes M$, where r is the rank of M . We can construct a “good” category of rigid analytically trivial t -motives and this category forms a neutral Tannakian category with fiber functor H_B . Thus we obtain an algebraic group $\Gamma \subset \text{GL}_r$ over $\mathbb{F}_q(t)$ which corresponds to the Tannakian subcategory generated by M via the Tannakian duality. In this situation, Papanikolas proved the following theorem:

Theorem 1.3 ([8, Theorem 4.3.1, 4.5.10]). *Let M , Ψ and Γ be as above. Then we have*

$$\text{tr.deg}_{\bar{K}(t)} \bar{K}(t)(\Psi_{11}, \Psi_{12}, \dots, \Psi_{rr}) = \dim \Gamma.$$

Note that, each component of Ψ converges at $t = \theta$, and moreover Papanikolas proved the equality

$$\text{tr.deg}_{\bar{K}} \bar{K}(\Psi_{11}|_{t=\theta}, \Psi_{12}|_{t=\theta}, \dots, \Psi_{rr}|_{t=\theta}) = \dim \Gamma$$

by using the “ABP-criterion” ([2]). Anderson and Thakur showed ([3]) that the Carlitz zeta values are described by linear combinations of entries of $\Psi|_{t=\theta}$ over K for certain t -motives. Hence Theorem 1.2 is proved by calculations of algebraic groups.

For a finite place v , there exist v -adic zeta values and v -adic realizations of t -motives. In [7] we proved a v -adic analogue of Theorem 1.3. However we do not know whether we can apply this result to the v -adic zeta values.

§ 2. t -motives

Before we state our results, we review t -motives. The notion of t -motive was introduced by Anderson in [1]. Let K and t be as in Section 1. We define an endomorphism σ on $K[t]$ by

$$\sigma: K[t] \rightarrow K[t]; \quad \sum_i a_i t^i \mapsto \sum_i a_i^q t^i.$$

Definition 2.1. A t -motive over K is a free $K[t]$ -module M of finite rank equipped with a σ -semilinear map $\varphi: M \rightarrow M$ such that

- $\det \varphi = c(t - \theta)^n$ ($c \in K^\times, n \geq 0$),
- M is finitely generated over $K[\varphi]$.

Note that $\det \varphi$ in the first condition is the determinant of the matrix $A \in \mathrm{GL}_r(K[t])$ which satisfies $\varphi \mathbf{m} = A \mathbf{m}$ for a fixed basis $\mathbf{m}$ of M , where r is the $K[t]$ -rank of M . Since $K[t]^\times = K^\times$, the validity of the first condition is independent of the choice of $\mathbf{m}$.

Remark. There exists an anti-equivalence of categories between the category of t -motives over K and the category of “abelian t -modules” over K . Roughly speaking, an abelian t -module is an algebraic group $\mathbb{G}_a^d$ over K for some $d \geq 0$ equipped with an $\mathbb{F}_q[t]$ -action which satisfies certain conditions.

Example 2.2 (Carlitz t -motive). As a $K[t]$ -module, set $M := K[t]$. We define a φ -action on M by

$$\varphi(a) := (t - \theta)\sigma(a)$$

for each $a \in M$. This forms a t -motive. We call this t -motive the *Carlitz t -motive*. The Carlitz t -motive corresponds to an abelian t -module $\mathbb{G}_a$ equipped with an $\mathbb{F}_q[t]$ -action defined by $\mathbb{F}_q[t] \rightarrow \mathrm{End}(\mathbb{G}_a); t \mapsto (x \mapsto \theta x + x^q)$.

§ 3. v -adic case

Let $v \in \mathbb{F}_q[t]$ be an irreducible monic polynomial of degree d . Set $K^{\mathrm{sep}}(t)_v := K^{\mathrm{sep}}(t) \otimes_{K^{\mathrm{sep}}[t]} \varprojlim_n K^{\mathrm{sep}}[t]/v^n$, the v -adic completion of $K^{\mathrm{sep}}(t)$, where K^{sep} is a separable closure of K . We define $K(t)_v$ and $\mathbb{F}_q(t)_v$ similarly. The endomorphism σ on $K[t]$

naturally extends to an endomorphism on $K^{\text{sep}}(t)_v$, and we have $(K^{\text{sep}}(t)_v)^\sigma = \mathbb{F}_q(t)_v$, where $(-)^\sigma$ is the σ -fixed part. For a t -motive M over K , we set

$$V(M) := (K^{\text{sep}}(t)_v \otimes_{K[t]} M)^{\sigma \otimes \varphi}.$$

We call $V(M)$ the v -adic realization of M . This is an $\mathbb{F}_q(t)_v$ -vector space and the absolute Galois group $G_K := \text{Gal}(K^{\text{sep}}/K)$ of K acts on $V(M)$ naturally. For any t -motive M , we can prove that $\dim_{\mathbb{F}_q(t)_v} V(M) = \text{rank}_{K[t]} M$. Thus if we fix bases $\mathbf{x}$ of $V(M)$ and $\mathbf{m}$ of M , we obtain the matrix $\Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(K^{\text{sep}}(t)_v)$ such that $\mathbf{m} = \Psi \mathbf{x}$ in $K^{\text{sep}}(t)_v \otimes M$. Each component of the matrix Ψ is called a v -adic period of M . If we factorize $v = \prod_{l \in \mathbb{Z}/d} (t - \lambda_l)$ in $\overline{\mathbb{F}_q}$, we have the decomposition $K^{\text{sep}}(t)_v = \prod_l K^{\text{sep}}((t - \lambda_l))$. Thus we can write $\Psi_{ij} = (\Psi_{ijl})_{l \in \mathbb{Z}/d}$ where $\Psi_{ijl} \in K^{\text{sep}}((t - \lambda_l))$ for each i, j and l .

To construct the v -adic analogue of the algebraic group Γ , we consider a certain subcategory of the category of φ -modules over $K(t)_v$. A φ -module over $K(t)_v$ is a finite-dimensional $K(t)_v$ -vector space N equipped with a σ -semilinear map $\varphi: N \rightarrow N$. A morphism between φ -modules is a $K(t)_v$ -linear map which commutes with φ 's. For a φ -module N , we set

$$V(N) := (K^{\text{sep}}(t)_v \otimes_{K(t)_v} N)^{\sigma \otimes \varphi}.$$

We have a natural injection

$$K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V(N) \rightarrow K^{\text{sep}}(t)_v \otimes_{K(t)_v} N.$$

Let $\mathcal{C}$ be the full subcategory of the category of φ -modules over $K(t)_v$ whose objects are the φ -modules such that the above map is an isomorphism. We can prove that $K(t)_v \otimes_{K[t]} M$ is an object of $\mathcal{C}$ for each t -motive M . Recall that a neutral Tannakian category over a field k is a rigid abelian k -linear tensor category $\mathcal{A}$ for which $k \xrightarrow{\sim} \text{End}(\mathbf{1})$ and there exists an exact faithful k -linear tensor functor $\omega: \mathcal{A} \rightarrow \mathbf{Vec}(k)$, where $\mathbf{1}$ is the unit object of $\mathcal{A}$ and $\mathbf{Vec}(k)$ is the category of finite-dimensional k -vector spaces (cf. [5, Definition 2.19]). Any such functor ω is said to be a fiber functor for $\mathcal{A}$. We can prove that the category $\mathcal{C}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)_v$, and the functor V is a fiber functor for $\mathcal{C}$. For a t -motive M over K , we set $\mathcal{C}_M$ to be the Tannakian subcategory of $\mathcal{C}$ generated by $K(t)_v \otimes M$. Let Γ_v be the algebraic group over $\mathbb{F}_q(t)_v$ corresponding to $\mathcal{C}_M$ via the Tannakian duality. Thus we obtain the matrix Ψ and the algebraic group Γ_v from a t -motive M . We have the following theorem, which is a v -adic analogue of Theorem 1.3:

Theorem 3.1 ([7, Theorem 4.14, 5.15]). *Let M , Ψ and Γ_v be as above. Then we have*

$$\text{tr.deg}_{K(t)_v} K(t)_v(\Psi_{11l}, \Psi_{12l}, \dots, \Psi_{rrl}) = \dim \Gamma_v$$

for all $l \in \mathbb{Z}/d$.

Example 3.2. Let M be the Carlitz t -motive defined in Example 2.2. Then we have $\Gamma_v = \mathbb{G}_m$, the multiplicative group over $\mathbb{F}_q(t)_v$, and the transcendental degree is one.

By using Theorem 3.1, we can prove the following proposition:

Proposition 3.3 ([7, Proposition 6.4, Corollary 7.4]). *Fix an integer $n \geq 1$.*

(1) *For each $\alpha \in K$, there exists an element $L_{\alpha,n} = (L_{\alpha,n,l})_l \in K^{\text{sep}}[[t]]_v = \prod_l K^{\text{sep}}[[t - \lambda_l]]$ such that $\sigma(L_{\alpha,n}) = \sigma(\alpha) + L_{\alpha,n}/(t - \theta)^n$.*

(2) *Fix elements $\alpha_1, \dots, \alpha_r \in K$. If $L_{\alpha_1,n,l}, \dots, L_{\alpha_r,n,l}, 1$ are linearly independent over $K(t)_v$ for some $l \in \mathbb{Z}/d$, then $L_{\alpha_1,n,l'}, \dots, L_{\alpha_r,n,l'}$ are algebraically independent over $K(t)_v$ for each $l' \in \mathbb{Z}/d$.*

Remark. In the ∞ -adic case, an analogous element of $L_{\alpha,n}$ is constructed explicitly, and its value at $t = \theta$ is the n -th Carlitz polylogarithm of α . Thus we consider $L_{\alpha,n}$ as a v -adic formal polylogarithm.

§ 4. Outline of the proof of Theorem 3.1

In this section, we will sketch the proof of Theorem 3.1. We will construct an algebraic group Γ' defined over $\mathbb{F}_q(t)_v$ such that the dimension of Γ' is equal to the transcendental degree in Theorem 3.1, and there exists an isomorphism $\Gamma' \xrightarrow{\sim} \Gamma$. We continue to use the notations of the previous sections, and factorize $v = \prod_{l \in \mathbb{Z}/d} (t - \lambda_l)$ in $\overline{\mathbb{F}_q}$ so that $\lambda_l^q = \lambda_{l+1}$ for each l . Set $F := \mathbb{F}_q(t)_v$, $E := K(t)_v$, $L := K^{\text{sep}}(t)_v$ and $L_l := K^{\text{sep}}((t - \lambda_l))$ for each l .

Let $X := (X_{ij})$ be an $r \times r$ matrix of independent variables X_{ij} , and set $\Delta := \det(X)$. We set $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \dots, X_{rr}, \Delta^{-1}]$. We define E -algebra homomorphisms $\nu : E[X, \Delta^{-1}] \rightarrow L$; $X_{ij} \mapsto \Psi_{ij}$ and $\nu_l : E[X, \Delta^{-1}] \rightarrow L_l$; $X_{ij} \mapsto \Psi_{ijl}$, and set

$$Z := \text{Spec } E[X, \Delta^{-1}] / \text{Ker } \nu$$

and

$$Z_l := \text{Spec } E[X, \Delta^{-1}] / \text{Ker } \nu_l$$

for each l . It is clear that the dimension of Z_l is equal to the transcendental degree which we want to calculate. To construct Γ' , we define matrices $\tilde{\Psi} = (\tilde{\Psi}_{ij})_{i,j} := (\Psi_{ij} \otimes 1)_{i,j}^{-1} (1 \otimes \Psi_{ij})_{i,j} \in \text{GL}_r(L \otimes_E L)$ and $\tilde{\Psi}_{lm} = (\tilde{\Psi}_{ijlm})_{i,j} := (\Psi_{ijl} \otimes 1)_{i,j}^{-1} (1 \otimes \Psi_{ijm})_{i,j} \in \text{GL}_r(L_l \otimes_E L_m)$ for each $l, m \in \mathbb{Z}/d$. We define F -algebra homomorphisms

$\mu : F[X, \Delta^{-1}] \rightarrow L \otimes_E L$; $X_{ij} \mapsto \tilde{\Psi}_{ij}$ and $\mu_{lm} : F[X, \Delta^{-1}] \rightarrow L_l \otimes_E L_m$; $X_{ij} \mapsto \tilde{\Psi}_{ijlm}$ for each l and m . Set

$$\Gamma' := \operatorname{Spec} F[X, \Delta^{-1}] / \operatorname{Ker} \mu$$

and

$$\Gamma'_{lm} := \operatorname{Spec} F[X, \Delta^{-1}] / \operatorname{Ker} \mu_{lm}.$$

By a simple calculation, we have $\operatorname{Ker} \mu_{0,m} = \operatorname{Ker} \mu_{1,m+1} = \dots$. Thus we can set $\Gamma'_m := \Gamma'_{0,m} = \Gamma'_{1,m+1} = \dots$. For each scheme Y over F , we set $Y_E := Y \times_{\operatorname{Spec} F} \operatorname{Spec} E$.

Proposition 4.1 ([7, Proposition 4.11]). (1) *Let $\psi : Z \times_E Z \rightarrow Z \times_E \operatorname{GL}_{r/E}$ be the morphism of affine E -schemes defined by $(u, v) \mapsto (u, u^{-1}v)$. Then ψ factors through an isomorphism $\psi' : Z \times_E Z \xrightarrow{\sim} Z \times_E \Gamma'_E$ of affine E -schemes.*

(2) *For any l and m , let $\psi_{lm} : Z_l \times_E Z_{l+m} \rightarrow Z_l \times_E \operatorname{GL}_{r/E}$ be the morphism of affine E -schemes defined by $(u, v) \mapsto (u, u^{-1}v)$. Then ψ factors through an isomorphism $\psi'_{lm} : Z_l \times_E Z_{l+m} \xrightarrow{\sim} Z_l \times_E \Gamma'_{m,E}$ of affine E -schemes.*

$$\begin{array}{ccc} Z \times Z & \xrightarrow{\psi: (u,v) \mapsto (u, u^{-1}v)} & Z \times \operatorname{GL}_{r/E} \\ \psi' \searrow & & \nearrow \text{natural} \\ & Z \times \Gamma'_E & \end{array} \quad \begin{array}{ccc} Z_l \times Z_{l+m} & \xrightarrow{\psi_{lm}: (u,v) \mapsto (u, u^{-1}v)} & Z_l \times \operatorname{GL}_{r/E} \\ \psi'_{lm} \searrow & & \nearrow \text{natural} \\ & Z_l \times \Gamma'_{m,E} & \end{array}$$

By Proposition 4.1, we conclude that

- Γ' is a closed subgroup scheme of $\operatorname{GL}_{r/F}$ and Z is a Γ'_E -torsor under right multiplication,
- Γ'_0 is a closed subgroup scheme of $\operatorname{GL}_{r/F}$ and Z_l is a $\Gamma'_{0,E}$ -torsor under right multiplication for each l ,
- Γ'_m is a Γ'_0 -torsor under right and left multiplications for each m .

In particular, we have $\dim \Gamma' = \dim \Gamma'_m = \dim Z_l = \operatorname{tr.deg}_E E(\Psi_{11l}, \Psi_{12l}, \dots, \Psi_{rrl})$ for each l and m .

For any object N in $\mathcal{C}_M$, we can define a Γ' -action on $V(N)$. This is functorial in N and we have a functor

$$\xi : \mathcal{C}_M \rightarrow \mathbf{Rep}_F(\Gamma')$$

which is compatible with tensor products, where $\mathbf{Rep}_F(\Gamma')$ is the category of finite-dimensional F -representations of Γ' . Thus we obtain the morphism $f : \Gamma' \rightarrow \Gamma_v$ of algebraic groups over F which corresponds to the functor ξ via the Tannakian duality. We can check easily that the morphism f is a closed immersion. To prove that f is an

isomorphism, we consider the Galois representation $G_K \rightarrow \mathrm{GL}(V(M))$ which is obtained naturally by the definition of V . This representation factors as follows:

$$G_K \rightarrow \Gamma'(F) \hookrightarrow \Gamma_v(F) \hookrightarrow \mathrm{GL}(V(M)).$$

Since the functor V induces an equivalence of categories $V: \mathcal{C} \xrightarrow{\sim} \mathbf{Rep}_F(G_K)$, where $\mathbf{Rep}_F(G_K)$ is the category of finite-dimensional continuous F -representations of G_K (cf. [6, Appendix]), the set of F -valued points $\Gamma_v(F)$ is dense in Γ_v . Therefore we conclude that the immersion $f: \Gamma' \hookrightarrow \Gamma_v$ is an isomorphism. This is an essentially different point from Papanikolas' proof for the ∞ -adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism.

References

- [1] G. W. Anderson, t -motives, *Duke Math. J.*, **53** (1986), 457–502.
- [2] G. W. Anderson, W. D. Brownawell, M. A. Papanikolas, Determination of the algebraic relations among special Γ -values in positive characteristic, *Ann. of Math.*, **160** (2004), 237–313.
- [3] G. W. Anderson, D. S. Thakur, Tensor powers of the Carlitz module and zeta values, *Ann. of Math.*, **132** (1990), 159–191.
- [4] C.-Y. Chang, J. Yu, Determination of algebraic relations among special zeta values in positive characteristic, *Adv. Math.*, **216** (2007), 321–345.
- [5] P. Deligne, J. S. Milne, Tannakian categories, *Hodge Cycles, Motives and Shimura Varieties*, *Lectures Notes in Math.*, **900**, Springer, Berlin (1982), 101–228.
- [6] D. Goss, The adjoint of the Carlitz module and Fermat's Last Theorem, *J. Finite Fields*, **1** (1995), 165–188. (Appendix: Y. Taguchi, φ -modules and adjoint operators)
- [7] Y. Mishiba, On v -adic periods of t -motives, *J. Number Theory*, **132** (2012), 2132–2165.
- [8] M. A. Papanikolas, Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms, *Invent. Math.*, **171** (2008), 123–174.