

THE EXISTENCE AND THE CONTINUATION OF HOLOMORPHIC SOLUTIONS FOR CONVOLUTION EQUATIONS IN A HALF-SPACE IN $\mathbb{C}^n$

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ABSTRACT. — We study holomorphic solutions for convolution equations (E) $T * f = g$ in a half-space in $\mathbb{C}^n$. Under a natural condition (the condition (S)), we will prove the existence of solutions of (E) and the analytic continuation of homogeneous equation (E') $T * f = 0$.

1. INTRODUCTION

Let Ω be a convex domain and let K be a compact convex set in $\mathbb{C}^n$. We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω provided with the topology of uniform convergence on compact subsets of Ω , and by $\mathcal{O}(K)$ the space of germs of functions holomorphic on

K , endowed with the usual topology of the inductive limit. Then each nonzero analytic functional $T \in \mathcal{O}'(\mathbb{C}^n)$ carried by K (or equivalently, $T \in \mathcal{O}'(K)$) defines a continuous linear convolution operator

$$T* : \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

which is given by

$$(T * f)(z) = T_{\zeta}(f(z + \zeta)), \quad z \in \Omega.$$

If $K = \{0\}$, the convolution operator $T*$ is a linear partial differential operator of infinite order with constant coefficients on $\mathcal{O}(\Omega)$. The convolution equation has been historically studied by many authors, especially the equation in the category of holomorphic functions defined on a complex domain. For example, using the notion of an entire function of completely regular growth on a fixed ray, Morzhakov [1] established sufficient condition for $T*$ to be surjective in the general case, and gave a criterion for the solvability for three classes of domains: smooth domains, products of smooth planar domains, and domains whose boundaries consist of smooth points and vertices. On the other hand, under the condition (S) due to Kawai [2], Ishimura - Y. Okada [3] studied the existence and the continuation problem of holomorphic solutions for convolution equations of hyperfunction kernels in the tube domain. In [4], Ishimura and the author proved that the property of completely regular growth is equivalent to the condition (S) for entire functions, in more general case, *i.e.* for sub-harmonic functions.

In this paper, we consider the convolution equation in the case where Ω is a half-space, and under the condition $(S)_{\zeta_0}$, we will prove the existence of solutions of (E) and the analytic continuation of homogeneous equation (E').

Most of results is based on the paper [3], and refer to it for the details of proofs.

2. PRELIMINARIES

Let

$$|z|^2 = z_1 \overline{z_1} + \cdots + z_n \overline{z_n}, \quad \langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

for

$$z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

For $\zeta_0 \in \mathbb{C}^n$ and $|\zeta_0| = 1$, we put

$$\Omega = \{ \zeta \in \mathbb{C}^n \mid \operatorname{Re} \langle \zeta_0, \zeta \rangle < 0 \}$$

and take a compact convex set K as $K \subset \Omega$, i.e. $\Omega + K = \Omega$. As it is well-known, the properties of convolution operator are reflected in the properties of the Fourier-Borel transform of T , namely

$$\hat{T}(\zeta) = T_z(\exp\langle z, \zeta \rangle),$$

which is an entire function of exponential type satisfying the following estimate:

Theorem 2.1. (Polyà-Ehrenpreis-Martineau) If $T \in \mathcal{O}'(\mathbf{C}^n)$ and T is carried by a compact set $K \subset \mathbf{C}^n$, then $\hat{T}(\zeta)$ is an entire function and for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$(2.1) \quad |\hat{T}(\zeta)| \leq C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|), \quad \zeta \in \mathbf{C}^n$$

where $H_K(\zeta) = \sup_{z \in K} \operatorname{Re}\langle z, \zeta \rangle$ is the supporting function of K .

Conversely, if K is a compact convex set and $f(\zeta)$ an entire function satisfying (2.1) for every $\varepsilon > 0$, there exists an analytic functional $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K such that $\hat{T}(\zeta) = f(\zeta)$.

In this paper, we suppose the following condition for the entire function $\hat{T}(\zeta)$.

Definition 2.2. We say that $\hat{T}(\zeta)$ satisfies the condition (S) to the direction ζ_0 or simply it satisfies the condition $(S)_{\zeta_0}$ if we have

$$\left\{ \begin{array}{l} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } r \in \mathbf{R} \text{ with } r > N, \\ \text{we can find } \zeta \in \mathbf{C}^n, \text{ which satisfies} \\ |\zeta - \zeta_0| < \varepsilon r, \\ |\hat{T}(\zeta)| \geq \exp(-\varepsilon r). \end{array} \right.$$

3. THE EXISTENCE OF HOLOMORPHIC SOLUTIONS

We will prove the surjectivity theorem under the condition $(S)_{\zeta_0}$.

Theorem 3.1. Let $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K . Assume that $\hat{T}(\zeta)$ satisfies the condition $(S)_{\zeta_0}$. Then the convolution operator

$$T* : \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

is surjective.

Proof. The transpose of

$$P = T* : \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)$$

is

$${}^tP = \check{T}* : \mathcal{O}'(\Omega) \longrightarrow \mathcal{O}'(\Omega + K)$$

with $\hat{\check{T}}(\zeta) = \hat{T}(-\zeta)$. By the standard argument, it is enough to prove that tP is injective and that the image of tP is weakly closed. In fact, the injectivity of tP shows that the image of P is dense, and the closedness of the image of tP shows the closedness of the image of P . Because T is not 0, the injectivity of tP is clear. Then we will show that the image of tP is weakly closed. To do this, we use the following division lemma, which we can prove in an analogous way to the proof of Lemma 2.1 in [3].

Lemma 3.2. Let f, g and h be entire functions satisfying $fg = h$, and K and L be two compact convex sets in $\mathbf{C}^n$ with $K, L \subset \Omega$. We suppose that for every $\varepsilon > 0$, f and h satisfy the following estimates with constants $A_\varepsilon > 0$ and $B_\varepsilon > 0$,

$$\begin{cases} \log |f(\zeta)| \leq A_\varepsilon + H_K(\zeta) + \varepsilon|\zeta|, \\ \log |h(\zeta)| \leq B_\varepsilon + H_L(\zeta) + \varepsilon|\zeta|, \end{cases}$$

for any $\zeta \in \mathbf{C}^n$. We also assume that f satisfies the condition $(S)_{\zeta_0}$. Then for any $\varepsilon > 0$, there exists a compact convex set $M = M_\varepsilon \subset \mathbf{C}^n$ and $C_\varepsilon > 0$ such that

$$\begin{cases} M \subset \Omega \\ \log |g(\zeta)| \leq C_\varepsilon + H_M(\zeta). \end{cases}$$

End of the proof of the theorem. — Let $\{T_\nu\}$ be a sequence in $\mathcal{O}'(\Omega)$ and assume that $\{{}^tPT_\nu\}$ converges to $S \in \mathcal{O}'(\Omega + K)$ in $\mathcal{O}'(\Omega + K)$. By taking the Fourier-Borel transform,

$\hat{T}(-\zeta)\hat{T}_\nu(\zeta)$ converges to $\hat{S}(\zeta)$. Then it is well-known that $G(\zeta) = \frac{\hat{S}(\zeta)}{\hat{T}(-\zeta)}$ becomes an

entire function. By Lemma 3.2 and Theorem 2.1, there exists a compact convex set M and $\mu \in \mathcal{O}'(\mathbf{C}^n)$ carried by M such that $\hat{\mu}(\zeta) = G(\zeta)$ and ${}^tP\mu = \check{T} * \mu = S$, i.e. $S \in \text{Im } {}^tP$. $\square$

4. THE CHARACTERISTIC SET AND THE CONTINUATION OF HOMOGENEOUS SOLUTIONS

Under the condition $(S)_{\zeta_0}$, we shall now solve the problem of continuation for the solutions of the homogeneous equation (E') . For any open set $U \subset \mathbf{C}^n$, we set:

$$\mathcal{N}(U) = \{ f \in \mathcal{O}(U) \mid T * f = 0 \}.$$

For an open set $V \subset \mathbf{C}^n$ with $U \subset V$, the problem is formulated as to get the condition so that the restriction map

$$r : \mathcal{N}(V) \longrightarrow \mathcal{N}(U)$$

is surjective.

In order to describe the theorem of continuation, we will prepare the notion of characteristics which is a natural generalization of the case of usual differential operators of finite order with constant coefficients. We define the sphere at infinity S_∞^{2n-1} by $(\mathbf{C}^n \setminus \{0\})/\mathbf{R}_+$ and consider the compactification with directions $\mathbf{D}^{2n} = \mathbf{C}^n \sqcup S_\infty^{2n-1}$ of $\mathbf{C}^n$. For $\zeta \in \mathbf{C}^n \setminus \{0\}$, we denote by $\zeta_\infty \in S_\infty^{2n-1}$ the equivalence class of ζ , *i.e.*

$$\{\zeta_\infty\} = (\text{the closure of } \{ t\zeta \mid t > 0 \} \text{ in } \mathbf{D}^{2n}) \cap S_\infty^{2n-1}.$$

For $\varepsilon > 0$, we put:

$$\begin{cases} V_{\hat{T}}(\varepsilon) = \{ \zeta \in \mathbf{C}^n \mid \exp(\varepsilon|\zeta|)|\hat{T}(\zeta)| < 1 \}, \\ W_{\hat{T}}(\varepsilon) = (\text{the closure of } V_{\hat{T}}(\varepsilon) \text{ in } \mathbf{D}^{2n}) \cap S_\infty^{2n-1}. \end{cases}$$

Now we define the characteristic set of $T*$.

Definition 4.1. With the above notation, we define the characteristics of $T*$ (at infinity)

$$\text{Char}_\infty(T*) = \text{the closure of } \bigcup_{\varepsilon > 0} W_{\hat{T}}(\varepsilon).$$

Under the above situation, we can state the theorem of the continuation without proof.

Theorem 4.2. Let $T \in \mathcal{O}'(\mathbf{C}^n)$ carried by K and $f \in \mathcal{O}(\Omega + K)$ be a solution of $T * f = 0$. Assume that $\hat{T}(\zeta)$ satisfies the condition $(S)_{\zeta_0}$. If $\zeta_0 \notin \text{Char}_\infty(T*)$, then the restriction map

$$r : \mathcal{N}(\mathbf{C}^n) \longrightarrow \mathcal{N}(\Omega + K)$$

is surjective, that is, f can be analytically continued to the whole $\mathbf{C}^n$.

References

- [1] V. V. Morzhakov : *Convolution equations in convex domains of $\mathbb{C}^n$* , Compl. Anal. and Appl. '87, Sofia, pp.360–364 (1989).
- [2] T. Kawai : *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **17**, pp.467–517 (1970).
- [3] R. Ishimura and Y. Okada : *The existence and the continuation of holomorphic solutions for convolution equations in tube domains*, Bull. Soc. Math. France, **122**, pp.413–433 (1994).
- [4] R. Ishimura et J. Okada : *Sur la condition (S) de kawai et la propriété de croissance régulière d'une fonction sous-harmonique et d'une fonction entière*, Kyushu J. Math. **48**, pp.257–263 (1994).