

Some remarks on subvarieties of Hopf manifolds

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§ 1. Introduction

A holomorphic automorphism g of a complex space $\mathbb{X}$ is called a contraction to a point $0 \in \mathbb{X}$ if g satisfies the following three conditions:

$$(i) \quad g(0) = 0,$$

$$(ii) \quad \lim_{\nu \rightarrow +\infty} g^\nu(x) = 0 \quad \text{for any point } x \in \mathbb{X},$$

(iii) for any small neighborhood U of 0 in $\mathbb{X}$, there exists an integer ν_0 such that $g^\nu(U) \subset U$ for all $\nu \geq \nu_0$,

where g^ν is the ν -times composite of g . By [2]* the complex space $\mathbb{X}$ which admits a contracting automorphism is holomorphically isomorphic to an algebraic subset of $\mathbb{C}^N$ for some N . We identify $\mathbb{X}$ to the algebraic subset of $\mathbb{C}^N$. Then there exists a contracting automorphism $\tilde{g}$ of $\mathbb{C}^N$ to the origin 0 such that $\tilde{g}|_{\mathbb{X}} = g$ ([2], [3]). Obviously the action of $\tilde{g}$ on $\mathbb{C}^N - \{0\}$ is free and properly discontinuous. Hence the quotient space $H = \mathbb{C}^N - \{0\} / \langle \tilde{g} \rangle$ is a compact complex manifold which is called a primary Hopf manifold. Sometimes we indicate by H^N a N -dimensional primary Hopf manifold. The compact complex space $\mathbb{X} - \{0\} / \langle g \rangle$ is clearly an analytic subset of a primary Hopf manifold. A compact complex manifold X of dimension n ($n \geq 2$) is called a Hopf manifold if its universal covering is holomorphically isomorphic to $\mathbb{C}^n - \{0\}$ (Kodaira[4]).

The purpose of this paper is to show several properties of subvarieties of Hopf manifolds.

* In [2], the condition (iii) is forgotten.

§ 2. Hopf manifolds

The following proposition shows that it is sufficient to consider only subvarieties of primary Hopf manifolds.

Proposition 1. Any Hopf manifold is a submanifold of a (higher dimensional) primary Hopf manifold.

Proof. Let X be any Hopf manifold. Then, by definition, there exists a group G of holomorphic transformations of $\mathbb{C}^n - \{0\}$ such that $X = \mathbb{C}^n - \{0\} / G$ ($n = \dim X$)^(≥2). It follows from a theorem of Hartogs that any element of G can be extended to a holomorphic transformation of $\mathbb{C}^n$. Hence we may assume that each element of G is a holomorphic transformation of $\mathbb{C}^n$ which fixes the origin $0 \in \mathbb{C}^n$. By the same argument as in [4] pp 694-695, G contains a contraction.

For each element $x \in G$, we denote by $dx(0)$ the jacobian matrix at the origin $0 \in \mathbb{C}^n$.

Lemma 1. An element $x \in G$ is a contraction if and only if $|\det(dx(0))| < 1$.

Proof. If $x \in G$ is a contraction, then any eigenvalue α of $dx(0)$ satisfies $|\alpha| < 1$ (see [3] for the detail). Hence $|\det(dx(0))| < 1$.

Conversely, let x be an element of G satisfying $|\det(dx(0))| < 1$.

Let g be a contraction contained in G . Since $\mathbb{C}^n - \{0\} / \langle g \rangle$ is compact, the index of the infinite cyclic subgroup $\{g\}$ generated by g is finite in G . Now assume that x is not a contraction. Then x^n is not a contraction for any integers n . Hence $x^n \neq g^m$ for any pair of integers n and m except $n = m = 0$. This implies that $\{x\} \cap \{g\} = \{1\}$.

This contradicts the fact that $\{g\}$ is of the finite index in G , q.e.d.

Let U be a subgroup of G defined by

$$U = \{x \in G : |\det(dx(0))| = 1\}.$$

Obviously U is a normal subgroup of G .

Lemma 2. There exists an infinite cyclic subgroup Z of G such that G is the semi-direct product of Z and U ; $G = Z \cdot U$.

Proof. Define a group homomorphism $\mathfrak{l} : G \rightarrow \mathbb{R}$ by $\mathfrak{l}(x) = -\log |\det(dx(0))|$ ($x \in G$). Let $g_1 \in G$ be a contraction. Then the index d of the infinite cyclic group $\{\mathfrak{l}(g_1)\}$ generated by $\mathfrak{l}(g_1)$ in $\mathfrak{l}(G)$ is finite. Hence $d^{-1} \mathfrak{l}(g_1)$ is a minimum positive element of $\mathfrak{l}(G)$. Let g be an element of G such that $\mathfrak{l}(g) = d^{-1} \mathfrak{l}(g_1)$. We put $Z = \{g\}$. Then it is clear that $G = Z \cdot U$, q.e.d.

Lemma 3. U is a finite normal subgroup of G .

Proof. Clear by Lemma 2.

Now continue the proof of Proposition 1. It is easy to see that any holomorphic transformation u of $\mathbb{C}^n$ which fixes the origin is linear, if u is of the finite order. Hence U is a finite subgroup of $GL(n, \mathbb{C})$. Hence, by H. Cartan [1], $\mathcal{X} = \mathbb{C}^n/U$ is a complex space with unique possible singularity at $\bar{0}$, where $\bar{0}$ is the corresponding point to the origin $0 \in \mathbb{C}^n$. The generator g of Z induces a contracting automorphism $\bar{g}$ of $\mathcal{X}$ such that $\bar{g}(\bar{0}) = \bar{0}$. Hence $X = \mathcal{X} - \{\bar{0}\}/\langle \bar{g} \rangle$ is a submanifold of a primary Hopf manifold as we have seen in the introduction. Q.E.D.

§ 3. Line bundles defined by divisors

Let M be an arbitrary compact complex manifold and N be a divisor of M . The line bundle $[N]$ defined by N is an element of $H^1(M, \mathcal{O}^*)$. There is a natural homomorphism $i : H^1(M, \mathbb{C}^*) \longrightarrow H^1(M, \mathcal{O}^*)$ induced by the natural injection $\mathbb{C}^* \longrightarrow \mathcal{O}^*$. If $[N]$ is in the image of i , then $[N]$ is called a locally flat line bundle. In other words, $[N]$ is locally flat if and only if its transition functions can be written by constant functions.

Now let $\tilde{g}$ be any contracting automorphism of $\mathbb{C}^N$ which fixes the origin $0 \in \mathbb{C}^N$. Then, by L. Reich ([6], [7]), we can choose a system of coordinates of $\mathbb{C}^N$ such that $\tilde{g}$ can be written in the following form:

$$\begin{aligned}
 (1) \quad & z'_1 = \alpha_1 z_1 \\
 & z'_2 = z_1 + \alpha_2 z_2 \\
 & \vdots \\
 & z'_{r_1} = z_{r_1-1} + \alpha_{r_1} z_{r_1} \\
 & z'_{r_1+1} = \alpha_{r_1+1} z_{r_1+1} + P_{r_1+1}(z_1, \dots, z_{r_1}) \\
 & \vdots \\
 & z'_{r_1+r_2} = z_{r_1+r_2-1} + \alpha_{r_1+r_2} z_{r_1+r_2} + P_{r_1+r_2}(z_1, \dots, z_{r_1}) \\
 & z'_{r_1+r_2+1} = \alpha_{r_1+r_2+1} z_{r_1+r_2+1} + P_{r_1+r_2+1}(z_1, \dots, z_{r_1+r_2}) \\
 & \vdots \\
 & z'_N = z_{N-1} + \alpha_N z_N + P_N(z_1, \dots, z_{r_1+r_2+\dots+r_{\mu-1}}),
 \end{aligned}$$

where $1 > |\alpha_1| \geq \dots \geq |\alpha_N| > 0$, μ is the number of Jordan blocks of the linear part, P_j ($r_1 + \dots + r_s < j \leq r_1 + \dots + r_{s+1}$) are finite sums of monomials $z_1^{m_1} \dots z_s^{m_s}$ which satisfy

$$(2) \quad \alpha_j = \alpha_1^{m_1} \dots \alpha_s^{m_s},$$

$$m_1 + \dots + m_s \geq 2 \quad (\text{all } m_i > 0).$$

Let $\tilde{\omega}: \mathbb{C}^N - \{0\} \rightarrow H = \mathbb{C}^N - \{0\} / \langle \tilde{g} \rangle$ be the covering projection. For any analytic subset X in H , the set $\tilde{\omega}^{-1}(X)$ is an analytic subset in $\mathbb{C}^N - \{0\}$. If $\dim X \geq 1$, then by a theorem of Remmert-Stein, $\mathcal{X} = \tilde{\omega}^{-1}(X) \cup \{0\}$ is an analytic subset of $\mathbb{C}^N$. In what follows, we indicate by the script letters the analytic subsets in $\mathbb{C}^N$ corresponding in the above manner to the analytic subsets of H written by the Roman letters. An analytic subset is called a variety if it is irreducible.

Assume that X is an analytic subvariety in H of $\dim X \geq 2$ and that D is an analytic subvariety of codimension 1 in X . It is clear that $\mathcal{X}$ and $\mathcal{D}$ are both $\tilde{g}$ -invariant in $\mathbb{C}^N$, i.e. $g(\mathcal{X}) = \mathcal{X}$ and $g(\mathcal{D}) = \mathcal{D}$.

Lemma 4 ([2]). There exists a non-constant holomorphic function f on $\mathcal{X}$ such that $g^*f = \alpha f$ for some constant α ($0 < |\alpha| < 1$) and that $f|_{\mathcal{D}} = 0$.

Remark 1. In [2], the word "variety" is used as "analytic set".

Let X be a non-singular manifold. Consider f of Lemma 4 as a

multiplicative multi-valued holomorphic function on X (K. Kodaira [4] pp 701). The divisor $D_1 = (f)$ is well-defined. The equation $g^*f = \alpha f$ implies that the line bundle $[D_1]$ is locally flat of which the transition functions are some powers of α . We summarize these facts as follows.

Theorem 1. Let X be a submanifold of H and D an effective divisor on X . Assume that $\dim X \geq 2$. Then there exists an effective divisor E on X such that the line bundle $[D + E]$ is locally flat of which the transition functions are some powers of a certain constant $\alpha \in \mathbb{C}^* (0 < |\alpha| < 1)$.

Remark 2. The following example shows that there are cases such that the "additional" effective divisor E of Theorem 1 is indispensable.

Let (x_0, x_1, x_2, x_3) be a standard system of coordinates of $\mathbb{C}^4$. Fix a complex number α such that $0 < |\alpha| < 1$. Let $\tilde{g}$ be a contracting holomorphic automorphism of $\mathbb{C}^4$ defined by

$$\tilde{g} : (x_0, x_1, x_2, x_3) \longmapsto (\alpha x_0, \alpha x_1, \alpha x_2, \alpha x_3).$$

Define $\tilde{g}$ -invariant subvarieties of $\mathbb{C}^4$ by

$$\mathcal{X} : x_0 x_1 = x_2 x_3$$

and

$$\mathcal{A} : x_3 = 0$$

Denote the intersection $\mathcal{X} \cap \mathcal{A}$ by $\mathcal{S}$. Then $\mathcal{S} = \{x_0 = x_3 = 0\} \cup \{x_1 = x_3 = 0\}$. We put

$$\mathcal{S}_1 = \{x_0 = x_3 = 0\}$$

and

$$\mathcal{S}_2 = \{x_1 = x_3 = 0\}.$$

Then $S = \mathcal{S} - \{0\}/\langle \tilde{g} \rangle$, $S_1 = \mathcal{S}_1 - \{0\}/\langle \tilde{g} \rangle$ and $S_2 = \mathcal{S}_2 - \{0\}/\langle \tilde{g} \rangle$ are subvarieties of a compact complex manifold $X = \mathcal{X} - \{0\}/\langle \tilde{g} \rangle$. It is clear that $[S_1 + S_2] = [S]$ is locally flat. We shall prove that either $[S_1]$ or $[S_2]$ is not locally flat. Assume that both $[S_1]$ and $[S_2]$ are locally flat. Let $\mathcal{U} = \{U_\lambda\}$ be a sufficiently fine finite open covering of X . We represent $[S_1]$ as a 1-cocycle $\{c_{1\lambda\mu}\} \in Z^1(\mathcal{U}, \mathbb{C}^*)$. Since $\dim H^0(X, \mathcal{O}[S_1]) > 0$, there exists a non-zero section φ_1 which vanishes exactly on S_1 . Let $\varphi_{1\lambda} = c_{1\lambda\mu} \varphi_{1\mu}$ on $U_\lambda \cap U_\mu$. As we can easily see,

$$\eta_1 = \frac{d\varphi_{1\lambda}}{\varphi_{1\lambda}} = \frac{d\varphi_{1\mu}}{\varphi_{1\mu}} = \dots$$

is a meromorphic 1-form on X . Since $\mathcal{X} - \{0\}$ is simply connected,

$$f_1(x) = \exp \int^x \eta_1$$

is a holomorphic function on $\mathcal{X} - \{0\}$ such that $\tilde{g}^* f_1 = \beta_1 f_1$

($\beta_1 \in \mathbb{C}^*$, $0 < |\beta_1| < 1$) which vanishes exactly on $\mathcal{S}_1 - \{0\}$ with multiplicity 1. Since $\mathcal{X}$ is normal at 0, f_1 uniquely extends to a holomorphic function on $\mathcal{X}$. Comparing the initial terms of $\tilde{g}^* f_1$ and f_1 at 0, we see that β_1 is some power of α , i.e. $\beta_1 = \alpha^{m_1}$ ($m_1 \geq 1$).

By the same manner, we construct f_2 for a non-zero section $\varphi_2 \in H^0(X, \mathcal{O}[S_2])$ such that $\tilde{g}^* f_2 = \alpha^{m_2} f_2$ ($m_2 \geq 1$). Let f_0 be a restriction of a holomorphic function x_3 to $\mathcal{X} - \{0\}$. Then $\tilde{g}^* f_0 = \alpha f_0$. It is easy to see that $f = f_1 \cdot f_2 \cdot f_0^{-1}$ is a non-vanishing holomorphic function on $\mathcal{X} - \{0\}$ such that $\tilde{g}^* f = \alpha^{m_1+m_2-1} f$ ($m_1+m_2-1 \geq 1$). But this does not occur if $\dim X > 1$. In fact, using the non-vanishing

holomorphic function f , we get the following commutative diagram:

$$\begin{array}{ccc} X - \{0\} & \xrightarrow{\tilde{g}} & X - \{0\}, \\ \downarrow f & & \downarrow f \\ \mathbb{C}^* & \xrightarrow{\times \alpha^{m_1+m_2-1}} & \mathbb{C}^*. \end{array}$$

Then f induces a proper surjective holomorphic mapping $\bar{f} : X \longrightarrow \mathbb{C}^* / \langle \alpha^{m_1+m_2-1} \rangle$. For any point $\tau \in \mathbb{C}^* / \langle \alpha^{m_1+m_2-1} \rangle$, $\bar{f}^{-1}(\tau) = X_\tau$ is a compact subvariety in X . Hence $\tilde{\omega}^{-1}(X_\tau)$ is a complex analytic subset in $\mathbb{C}^4 - \{0\}$ whose connected components are compact, where $\tilde{\omega}$ is the covering map $\mathbb{C}^4 - \{0\} \longrightarrow \mathbb{C}^4 - \{0\} / \langle \tilde{g} \rangle$. This implies that $\tilde{\omega}^{-1}(X_\tau)$ is a countable union of points. Hence $\dim X_\tau = 0$. This contradicts $\dim X > 1$. This implies that either $[S_1]$ or $[S_2]$ is not locally flat.

Remark 3. If $\dim X = 2$, then $[D]$ is always locally flat ([3]).

§ 4. Some properties of subvarieties

By Lemma 5 in [2], we have easily

Proposition 2. Let Y_1 and Y_2 be subvarieties of a (primary) Hopf manifold (H) such that $Y_1 \subset Y_2$ and $0 < n_1 = \dim Y_1 < n_2 = \dim Y_2$. Then there exists a sequence of subvarieties $W_0, W_1, \dots, W_p$ ($p = n_2 - n_1$) in H with following properties:

- (i) $W_0 = Y_1, \quad W_p = Y_2,$
- (ii) $W_i \subset W_{i+1} \quad (i = 0, \dots, p-1), \quad \dim W_i + 1 = \dim W_{i+1}.$

Proposition 3. Let $H^N = \mathbb{C}^N - \{0\} / \langle \tilde{g} \rangle$ be a primary Hopf manifold.

Then

- (a) any positive dimensional subvariety in H^N contains a curve,
- (b) any irreducible curve in H^N is non-singular elliptic,
- (c) for any elliptic curve C in H^N , there exist an eigenvalue α of $\tilde{g}$, a constant β and certain positive integers m, n with $\alpha^m = \beta^n$ such that C is isomorphic to $\mathbb{C}^* / \langle \beta \rangle$.

Proof. (a) Let Y be a n -dimensional subvariety in H^N ($n \geq 1$). For any integer k ($1 \leq k \leq N$), the $(N-k)$ -dimensional subspace $\mathbb{C}^{N-k}$ defined by $z_1 = \dots = z_k = 0$ is $\tilde{g}$ -invariant. There exists an integer k such that $\dim(\mathbb{C}^{N-(k-1)} \cap Y) = 1$. Then $\tilde{\omega}((\mathbb{C}^{N-(k-1)} \cap Y) - \{0\})$ is a 1-dimensional analytic subset of Y .

(b) Let C be any irreducible curve in H^N . Then C is a 1-dimensional analytic subset of $\mathbb{C}^N$. Let C_0 be one of the irreducible components of C . Then, for some positive integer n_0 , g^{n_0} acts on C_0 as a contracting automorphism of C_0 . Let $\lambda: C_0^* \rightarrow C_0$ be the normalization of C_0 . Then g^{n_0} naturally induces a contracting automorphism of C_0^* . By [2], $C_0^* \simeq \mathbb{C}$. It is clear that $\lambda^{-1}(0)$ consists of one point 0^* . Hence $C_0 - \{0\} \simeq C_0^* - \{0^*\} \simeq \mathbb{C}^*$. Thus $\mathbb{C}^*$ is an infinite cyclic unramified covering of C . Therefore C is a non-singular elliptic curve.

(c) Consider the $\tilde{g}$ -invariant subspaces $\mathbb{C}^{N-k}$ defined in (a). For $k = 0$, $\mathbb{C}^{N-k}$ is the total space. Fix the integer k ($0 \leq k \leq N-1$)

such that $C \subset e^{N-k}$ and $C \not\subset e^{N-k-1}$. If $C \cap e^{N-k-1}$ contains a point p other than 0, then $C \cap e^{N-k-1}$ contains an infinite sequence of points $\tilde{g}^n(p) \rightarrow 0$ ($n = 1, 2, \dots$). Hence one of the irreducible components of C is contained in e^{N-k-1} . Since $\tilde{g}$ is transitive over all the irreducible components of C , this implies that $C \subset e^{N-k-1}$. Therefore $C \cap e^{N-k-1} = \{0\}$. Hence $f = z_{k+1}|_{C-N-k}$, the restriction of z_{k+1} to e^{N-k} , vanishes nowhere on $C - \{0\}$. Moreover f satisfies the equation $g^*f = \alpha'_{k+1}f$. Hence we get the following commutative diagram:

$$\begin{array}{ccc} C - \{0\} & \xrightarrow{g} & C - \{0\} \\ \downarrow f & & \downarrow f \\ \mathbb{C}^* & \xrightarrow{\alpha'_{k+1}} & \mathbb{C}^*. \end{array}$$

This induces a covering $\bar{f} : C \rightarrow \mathbb{C}^* / \langle \alpha'_{k+1} \rangle$. Since both C and $\mathbb{C}^* / \langle \alpha'_{k+1} \rangle$ are non-singular elliptic curves, $\bar{f}$ has no branch points by the Hurwitz's formula. Hence there exist $\beta \in \mathbb{C}^*$ and positive integers m, n such that $C \simeq \mathbb{C}^* / \langle \beta \rangle$ and $\alpha'_{k+1} = \beta^n$. Q.E.D.

Remark 4. By Propositions 2 and 3 (a), it follows that any n -dimensional subvariety of a Hopf manifold contains subvarieties of arbitrary dimensions less than n .

§ 5. Subvarieties of algebraic dimension 0

In general, let M be a compact complex analytic subvariety. Then the field $\mathcal{H}(M)$ of all meromorphic functions on M has the finite transcendental degree $a(M)$ over $\mathbb{C}$. We call $a(M)$ the algebraic dimension of M . It is well known that $a(M) \leq \dim M$. The number $\dim M - a(M)$ is called the algebraic codimension of M .

Theorem 2. Let Y be a subvariety of dimension k in N -dimensional primary Hopf manifold H^N . Assume that $a(Y) = 0$. Then the number of $(k-1)$ -dimensional subvarieties in Y is at most N .

Before proving the theorem, we shall make some preparations.

Let $\alpha_1, \dots, \alpha_N$ be the eigenvalues of $\tilde{g}((1))$. Put $\theta_j = \log \alpha_j$ ($0 \leq \arg \theta_j < 2\pi$, $j = 1, 2, \dots, N$). Let K be a vector space over the field of rational numbers $\mathbb{Q}$ generated by the elements $2\pi\sqrt{-1}$, $\theta_1, \dots, \theta_N$. Choose a basis $\tau_0, \tau_1, \dots, \tau_\lambda$ of K so that the following conditions may be satisfied:

- (i) $\tau_0 = 2\pi\sqrt{-1}$,
- (ii) $\{\tau_1, \dots, \tau_\lambda\}$ is a subset of $\{\theta_1, \dots, \theta_N\}$,
- (iii) for any $\nu \geq 1$, τ_ν is linearly independent to $\mathbb{Q}\tau_0 + \mathbb{Q}\tau_1 + \dots + \mathbb{Q}\tau_{\nu-1}$,
- (iv) if $\tau_\nu = \theta_j$, $\tau_\mu = \theta_k$ and $\nu < \mu$, then $j < k$.

It is easy to check that we can choose such a basis. We denote

by α_{i_ν} the element of $\{\alpha_1, \dots, \alpha_N\}$ corresponding to τ_ν . Note that $\tau_\nu = \theta_{i_\nu} = \log \alpha_{i_\nu}$ ($\nu = 1, 2, \dots, \lambda$). If the equation

$$\alpha_{i_\nu} = \alpha_1^{a_1} \dots \alpha_\lambda^{a_\lambda} \quad (1 \leq i_\nu)$$

holds for some integers $a_1, \dots, a_\lambda$, then

$$\tau_\nu = \theta_{i_\nu} = \sum_{j=1}^l a_j \theta_j + p \tau_0 \quad (p \in \mathbb{Z}).$$

Since $\sum_{j=1}^l a_j \theta_j$ is written by a linear combination of $\tau_0, \tau_1, \dots, \tau_{l-1}$,

this is absurd. Therefore α_{i_ν} has no such relations. Hence by (1),

$$z_i' = \alpha_{i_\nu} z_i \quad (\nu = 1, 2, \dots, \lambda).$$

Proof of Theorem 2. We may assume that Y can't be contained any primary Hopf manifold of dimension less than N . Let D be a subvariety of codimension 1 in Y . By Lemma 4, $\mathcal{D}$ is contained in the zero locus of a non-constant holomorphic function f on $\mathcal{Y}$ such that $\tilde{g}^* f = \alpha f$ ($0 < |\alpha| < 1$). There exist some integers $m, m_1, \dots, m_\lambda$ such that

$$\alpha^m = \alpha_{i_1}^{m_1} \dots \alpha_{i_\lambda}^{m_\lambda}.$$

Put

$$h = z_{i_1}^{m_1} \dots z_{i_\lambda}^{m_\lambda}.$$

Since Y is not contained in any lower dimensional primary Hopf manifold, h is not equal to zero on $\mathcal{Y}$. Hence both f^m and h are eigenfunctions of $\tilde{g}^*$ of which the eigenvalues are the same α^m .

Then h/f^m defines a non-zero meromorphic function on Y . By the assumption $a(Y) = 0$, $h/f^m = \text{constant} = c \neq 0$. Hence we get

$$(3) \quad h = c f^m.$$

Let Z_{i_ν} ($\nu = 1, \dots, \lambda$) be analytic subsets of Y corresponding to $\{z_{i_\nu} = 0\} \cap \mathcal{Y}$. The equation (3) implies that D is contained in

$\bigcup_{\nu=1}^{\lambda} z_{i_{\nu}}$. Since $\lambda \leq N$, this proves the theorem.

Q.E.D.

§ 6. $\mathbb{C}^*$ -actions

Proposition 4. There exists a holomorphic mapping

$$\begin{aligned} \tilde{\varphi}: \mathbb{C} \times \mathbb{C}^N &\longrightarrow \mathbb{C}^N \\ (t, z) &\longrightarrow \tilde{\varphi}_t(z) \end{aligned}$$

which satisfies the following properties:

- (i) for every $t \in \mathbb{C}$, $\tilde{\varphi}_t$ is a holomorphic automorphism of $\mathbb{C}^N$ which fixes the origin,
- (ii) $\tilde{\varphi}_{t+s} = \tilde{\varphi}_t \circ \tilde{\varphi}_s$,
- (iii) there exists an integer n_0 such that $\tilde{\varphi}_1 = \tilde{g}^{n_0}$,
- (iv) every $\tilde{g}$ -invariant subvarieties in $\mathbb{C}^N$ is $\tilde{\varphi}_t$ -invariant for all $t \in \mathbb{C}$.

We say that an analytic subset of $\mathbb{C}^N$ is $\tilde{\varphi}$ -invariant, if it is $\tilde{\varphi}_t$ -invariant for all $t \in \mathbb{C}$.

Proof. Let $\alpha_{i_1}, \dots, \alpha_{i_{\lambda}}$ be the eigenvalues of $\tilde{g}$ considered in § 5. For any eigenvalue α_j of $\tilde{g}$, there exist some integers $m_j, m_{j_1}, \dots, m_{j_{\lambda}}$ such that

$$\alpha_j^{m_j} = \alpha_{i_1}^{m_{j_1}} \dots \alpha_{i_{\lambda}}^{m_{j_{\lambda}}} \quad (j = 1, 2, \dots, N).$$

Put $n_0 = m_1 \dots m_N$ and $g_0 = g^{n_0}$. We define

$$(4) \quad \alpha_{i_{\nu}}^t = \exp t \tau_{\nu} \quad (t \in \mathbb{C}, \nu = 1, 2, \dots, \lambda),$$

and

$$(5) \quad \alpha_j^{not} = \exp \left(t n_j \sum_{\nu=1}^{\lambda} m_{j_{\nu}} \tau_{\nu} \right) \quad (n_j = n_0 m_j^{-1}, j = 1, 2, \dots, N).$$

Let $R(\alpha_1^{n_0}, \dots, \alpha_N^{n_0}) = 1$ be any relation among the eigenvalues of g_0 , where $R(u_1, \dots, u_N)$ is a product of some (possibly negative) powers of u_j ($j = 1, 2, \dots, N$), u_j being indeterminates. Now let

$R(u_1, \dots, u_N) = u_1^{a_1} \dots u_N^{a_N}$ ($a_j \in \mathbb{Z}$). Then, for $t \in \mathbb{C}$,

$$\begin{aligned}
 (6) \quad R(\alpha_1^{n_0 t}, \dots, \alpha_N^{n_0 t}) &= \alpha_1^{a_1 n_0 t} \dots \alpha_N^{a_N n_0 t} \\
 &= \exp \left(t \sum_{j=1}^N a_j n_j \sum_{\nu=1}^{\lambda} m_{j\nu} \tau_\nu \right) \\
 &= \exp \left(t \sum_{\nu=1}^{\lambda} \left(\sum_{j=1}^N a_j n_j m_{j\nu} \right) \tau_\nu \right).
 \end{aligned}$$

Put $t = 1$ in (6). Then we get

$$\sum_{\nu=1}^{\lambda} \left(\sum_{j=1}^N a_j n_j m_{j\nu} \right) \tau_\nu = p \tau_0 \quad (p \in \mathbb{Z}).$$

Hence we get $p = 0$ and $\sum_{j=1}^N a_j n_j m_{j\nu} = 0$ ($\nu = 1, 2, \dots, \lambda$). Therefore

$$(7) \quad R(\alpha_1^{n_0 t}, \dots, \alpha_N^{n_0 t}) = 1$$

for all $t \in \mathbb{C}$. Put $\beta_j = \alpha_j^{n_0}$. By (1), the j -th coordinate of the point

$g_0^n(z)$ is given by

$$(8) \quad (g_0^n(z))_j = \beta_j^n \{ z_j + Q_j(n, z_1, \dots, z_{j-1}) \},$$

where Q_j is a polynomial of $n, z_1, \dots, z_{j-1}$. Replace n and β_j^n of (8) by t and $\alpha_j^{n_0 t}$, respectively. Then we get a holomorphic automorphism

$\tilde{\varphi}_t$ of $\mathbb{C}^N$ defined by

$$(\tilde{\varphi}_t(z))_j = \beta_j^t \{ z_j + Q_j(t, z_1, \dots, z_{j-1}) \}.$$

We shall prove that $\tilde{\varphi} = \{ \tilde{\varphi}_t \}_{t \in \mathbb{C}}$ satisfies the desired conditions.

The condition (i) and (iii) are clearly satisfied. To prove the condition (ii) is satisfied we put

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad Q(t, z) = \begin{pmatrix} Q_1(t, z) \\ \vdots \\ Q_N(t, z) \end{pmatrix} \quad \text{and} \quad A^t = \begin{pmatrix} \beta_1^t & & 0 \\ & \ddots & \\ 0 & & \beta_N^t \end{pmatrix}.$$

We write $\tilde{\varphi}_t(z)$ as

$$(9) \quad \tilde{\varphi}_t(z) = A^t(z + Q(t, z)).$$

Again we put

$$(10) \quad d(t, s, z) = \tilde{\varphi}_{t+s}(z) - \tilde{\varphi}_t \circ \tilde{\varphi}_s(z).$$

It is sufficient to prove that $d(t, s, z)$ vanishes identically. By (9),

$$(11) \quad \begin{aligned} d(t, s, z) &= A^{t+s}(z + Q(t+s, z)) - A^t(A^s(z + Q(s, z)) + Q(t, A^s(z + Q(s, z)))) \\ &= A^{t+s}Q(t+s, z) - A^{t+s}Q(s, z) - A^tQ(t, A^s(z + Q(s, z))). \end{aligned}$$

Let $Q_j(s, z) = \sum q_{i_1 \dots i_{j-1}}(s) z_1^{i_1} \dots z_{j-1}^{i_{j-1}}$ be the j -th component of $Q(s, z)$, where $i_1, \dots, i_{j-1}$ satisfy $\beta_1^{i_1} \dots \beta_{j-1}^{i_{j-1}} = \beta_j$ and $i_l > 0$.

Then, by (7),

$$\begin{aligned} &Q_j(t, A^s(z + Q(s, z))) \\ &= \sum q_{i_1 \dots i_{j-1}}(t) \{\beta_1^s(z_1 + Q_1(s, z))\}^{i_1} \dots \{\beta_{j-1}^s(z_{j-1} + Q_{j-1}(s, z))\}^{i_{j-1}} \\ &= \beta_j^s \sum q_{i_1 \dots i_{j-1}}(t) (z_1 + Q_1(s, z))^{i_1} \dots (z_{j-1} + Q_{j-1}(s, z))^{i_{j-1}}. \end{aligned}$$

Hence we get

$$(12) \quad A^tQ(t, A^s(z + Q(s, z))) = A^{t+s}Q(t, z + Q(s, z)).$$

Combining (11) with (12), we obtain

$$d(t,s,z) = A^{t+s}(Q(t+s,z) - Q(s,z) - Q(t, z+Q(s,z))).$$

Hence it is sufficient to show that

$$d_1(t,s,z) = Q(t+s,z) - Q(s,z) - Q(t, z+Q(s,z))$$

vanishes identically. Note that every component of $d_1(t,s,z)$ is a polynomial of t , s and z .

Fix any integer $t = m$. Since $d_1(m,n,z)$ vanishes identically for any $n \in \mathbb{Z}$, the algebraic subset in $\mathbb{C}^{N+1}$ defined by

$$\{(s,z) \in \mathbb{C}^{N+1} : d_1(m, s, z) = 0\}$$

contains infinitely many N -dimensional subspaces of $\mathbb{C}^{N+1}$. Hence we infer that $d_1(m,s,z)$ vanishes identically for any integer m . Again, since $d_1(m,s,z) = 0$ for any $m \in \mathbb{Z}$, the algebraic subset in $\mathbb{C}^{N+2}$ defined by $d_1(t,s,z) = 0$ contains infinitely many $(N+1)$ -dimensional subspaces of $\mathbb{C}^{N+2}$. Hence we conclude that d_1 vanishes identically on $\mathbb{C}^{N+2}$. Therefore the condition (ii) is satisfied.

Next we prove that the condition (iv) is satisfied. We need the following

Lemma 5. Let $\mathcal{U}$ be a $(\tilde{g}$ - and $\tilde{\varphi}$ -invariant analytic subvariety in $\mathbb{C}^N$.

Let $\mathcal{Z}$ be a pure 1-codimensional $\tilde{g}$ -invariant analytic subset of $\mathcal{U}$.

Then each irreducible component of $\mathcal{Z}$ is $\tilde{\varphi}$ -invariant.

Proof. By Lemma 4, there exists a holomorphic function f on

$\mathcal{U}$ such that $\tilde{g}^*f = \alpha f$ ($0 < |\alpha| < 1$) and that $f|_{\mathcal{Z}} = 0$. Here we shall prove the following equation:

$$(13) \quad \tilde{\varphi}_t^* f = \alpha^t f.$$

Once the equation (13) is proved, the lemma is clear. In fact, each irreducible component of $\mathcal{Z}$ is an irreducible component of the zero locus of f . Since everything continuously varies depending on t , (13) implies that the irreducible components of $\mathcal{Z}$ is $\tilde{\varphi}$ -invariant.

We put

$$M(\alpha) = \{ h \in \mathcal{O}_{\mathcal{Y}} : \tilde{g}^*h = \alpha h \}.$$

Then $M(\alpha)$ is a finite dimensional vector space over $\mathbb{C}$ (cf. [2]). Let $\sigma_1, \dots, \sigma_s$ be a basis of $M(\alpha)$. Put $\sigma_i^t(z) = \sigma_i(\tilde{\varphi}_t(z))$ ($i = 1, 2, \dots, s$). Since $\mathcal{Y}$ is $\tilde{\varphi}_t$ -invariant, the elements $\sigma_1^t, \dots, \sigma_s^t$ form another basis of $M(\alpha)$. Hence there exist some constants $c_{ij}(t)$ depending on t such that

$$\sigma_i^t = \sum_{j=1}^s c_{ij}(t) \sigma_j.$$

We claim that $C(t) = (c_{ij}(t))$ is holomorphically dependent on t .

In fact, we can choose points $z_1, \dots, z_s \in \mathcal{Y}$ such that

$$S = \begin{pmatrix} \sigma_1(z_1) & \dots & \sigma_1(z_s) \\ \vdots & & \vdots \\ \sigma_s(z_1) & \dots & \sigma_s(z_s) \end{pmatrix}$$

is a non-singular matrix. Then,

$$(14) \quad \begin{pmatrix} \sigma_1^t(z_1) & \dots & \sigma_1^t(z_s) \\ \vdots & & \vdots \\ \sigma_s^t(z_1) & \dots & \sigma_s^t(z_s) \end{pmatrix} S^{-1} = C(t).$$

Since the left hand side of (14) is holomorphically dependent on t ,

$C(t)$ is holomorphic.

It is easy to see that $\{C(t)\}_{t \in \mathbb{C}}$ is a 1-parameter subgroup of $GL(s, \mathbb{C})$, satisfying the equality,

$$(15) \quad C(n) = \alpha^n I \quad (n \in \mathbb{Z}).$$

Hence there exist a matrix A which has ^(the) Jordan canonical form and a non-singular matrix P such that

$$C(t) = P^{-1} \exp(tA) P.$$

By (15), A is a diagonal matrix. Put $P^{-1} \sigma_j = \tau_j$ ($j = 1, 2, \dots, s$).

Then,

$$(16) \quad \tau_j^t = (\exp ta_j) \tau_j \quad (j = 1, 2, \dots, s),$$

where $a_1, \dots, a_s$ are the diagonal components of A . Comparing the initial term^s of the both sides of (16), we get

$$(17) \quad \exp ta_j = \exp \sum_{\nu=1}^{\lambda} t n_{j\nu} \tau_{\nu} \quad (j = 1, 2, \dots, s),$$

for some integers $n_{j\nu}$. Letting $t = 1$, we get

$$\alpha = \exp a_j = \exp \sum_{\nu=1}^{\lambda} n_{j\nu} \tau_{\nu} \quad (j = 1, 2, \dots, s).$$

Hence for any i and j ,

$$\sum_{\nu=1}^{\lambda} (n_{j\nu} - n_{i\nu}) \tau_{\nu} = p_{ij} \tau_0,$$

choosing some integers p_{ij} . Since $\tau_0, \tau_1, \dots, \tau_{\lambda}$ are linearly independent over $\mathbb{Q}$, this implies that $n_{j\nu} = n_{i\nu}$ and $p_{ij} = 0$. Hence $\exp ta_j = \exp ta_i$ for any i and j . Therefore $C(t)$ is a scalar matrix:

$$C(t) = \alpha^t I \quad (\alpha^t = \exp ta_j).$$

Since $f \in M(\alpha)$, f can be expressed as

$$f = c_1 \tau_1 + \dots + c_s \tau_s \quad (c_j \in \mathbb{C}).$$

$$\text{Then } \tilde{\varphi}_t^* f = \sum_j c_j \tilde{\varphi}_t^* \tau_j = \alpha^t \sum_j c_j \tau_j = \alpha^t f, \quad \text{q.e.d.}$$

Proof of (iv). By Lemma 5 [2], there exists a sequence

$\{\mathcal{W}_j : j = 0, 1, \dots, p\}$ of $\tilde{g}$ -invariant subvarieties of $\mathbb{C}^N$ such that $\mathcal{W}_0 =$ a given $\tilde{g}$ -invariant subvariety, $\mathcal{W}_j \subset \mathcal{W}_{j+1}$, $\dim \mathcal{W}_j + 1 = \dim \mathcal{W}_{j+1}$ and $\mathcal{W}_p = \mathbb{C}^N$ ($p = N - \dim \mathcal{W}_0$). Since $\mathbb{C}^N$ is obviously $\tilde{g}$ - and $\tilde{\varphi}$ -invariant, we infer that $\mathcal{W}$ is $\tilde{\varphi}$ -invariant by the previous lemma. Q.E.D.

As a corollary, we obtain

Theorem 3. For any primary Hopf manifold H^N , there exists another primary Hopf manifold H'^N with following properties:

- (i) H'^N is a finite cyclic unramified covering of H^N ,
- (ii) H'^N has a free $\mathbb{C}^*$ -action $\varphi = \{\varphi_\tau\}_{\tau \in \mathbb{C}^*}$ such that every positive dimensional subvariety in H'^N is φ -invariant.

Proof. Let $H' = \mathbb{C}^N - \{0\} / \langle \tilde{g}^{n_0} \rangle$. Then everything is clear from Proposition 4.

Corollary. The Euler number of a submanifold of a Hopf manifold is equal to 0.

Proof. By Theorem 3, every submanifold of a Hopf manifold has a finite unramified covering which admits a free S^1 -action. Hence the Euler number vanishes. Q.E.D.

§ 7. Subvarieties of algebraic codimension 1

Let Y be a n -dimensional ($n \geq 2$) subvariety of a primary Hopf manifold H^N . Take another primary Hopf manifold H'^N of Theorem 3. Let $\omega : H'^N \rightarrow H^N$ be the covering map. We denote by Y' a connected component of $\omega^{-1}(Y)$.

Theorem 4. The algebraic dimension of Y is $n-1$ if and only if the $\mathbb{C}^*$ -action φ on Y' reduces to a complex torus action.

Proof. Assume that $a(Y) = n-1$. Since $a(Y') = a(Y) = n-1$, Y' has an $(n-1)$ -dimensional algebraic family of elliptic curves.

The moduli of curves depends continuously on the parameters. Hence, by Proposition 3, the moduli are constant. Since every curve in Y is φ -invariant, the $\mathbb{C}^*$ -action reduces to a complex torus action on the open dense subset of Y' and therefore on the whole Y' .

Conversely, assume that φ reduces to a complex torus action ψ on Y' . Then $\mathcal{Y}'$ is an affine variety in $\mathbb{C}^N$ with the $\mathbb{C}^*$ -action $\tilde{\psi}$ induced by $\tilde{\varphi}$. Moreover the action $\tilde{\psi}$ is compatible with g' , where g' is a contracting automorphism to 0 of $\mathbb{C}^N$ defining H'^N . It is not difficult to check that the $\mathbb{C}^*$ -action $\tilde{\psi}$ on $\mathcal{Y}'$ is algebraic. (Construct a contracting automorphism on $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$ which leaves invariant the closure $\overline{\Gamma}$ of the graph Γ of $\tilde{\psi}$, where $\overline{\Gamma}$ is an analytic subset of $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$. Use the result of [2] to show that $\overline{\Gamma}$ is an algebraic subset of $\mathbb{C} \times \mathcal{Y}' \times \mathcal{Y}'$.) Hence, by Proposition (1.1.3) in Orlik-Wagreich [5], there is an embedding $j : \mathcal{Y}' \rightarrow \mathbb{C}^{N'}$ for some N' and a $\mathbb{C}^*$ -action $\tilde{\psi}'$ on $\mathbb{C}^{N'}$ such that $j(\mathcal{Y}')$ is $\tilde{\psi}'$ -invariant and that $\tilde{\psi}'$ induces $\tilde{\psi}$ on $\mathcal{Y}'$. Moreover, by a suitable choice of coordinates $(z_1, \dots, z_{N'})$ on $\mathbb{C}^{N'}$, the action $\tilde{\psi}'$ on $\mathbb{C}^{N'}$ can be written

as

$$\tilde{\Psi}'(\varrho, (z_1, \dots, z_N)) = (\varrho^{q_1} z_1, \dots, \varrho^{q_{N'}} z_{N'}),$$

where the q_i 's are positive integers. There exists a constant α such that $\tilde{\Psi}'_\alpha$ induces g' on $\mathcal{U}'$. Then $Y' = \mathcal{U}' - \{0\} / \langle g' \rangle$ can be considered as a submanifold of $\mathbb{C}^{N'} - \{0\} / \langle \tilde{\Psi}'_\alpha \rangle$.

The following theorem is known.

Theorem (Ueno [8]). Let M_1 be a subvariety of a compact complex variety M_0 . Then

$$(18) \quad \dim M_1 - a(M_1) \leq \dim M_0 - a(M_0).$$

Now it is clear that $a(\mathbb{C}^{N'} - \{0\} / \langle \tilde{\Psi}'_\alpha \rangle) = N' - 1$. Hence, by (18), we get $a(Y') \geq \dim Y' - 1$. Since $a(Y') < \dim Y'$, we obtain $a(Y') = a(Y) = n - 1$. Q.E.D.

Remark 5. Topologically, any submanifold of a Hopf manifold is diffeomorphic to a fibre bundle over a 1-dimensional circle of which the transition function has a finite order as an element of the diffeomorphism group of the fibre. This can be seen without difficulty from Theorem 3.

Remark 6. A compact complex surface S is a submanifold of a Hopf manifold if and only if S is a relatively minimal surface of class VI_0 , VII_0 -elliptic or a Hopf surface. (See [3] for the proof of the "if" part.) Let S be a submanifold of a Hopf manifold. It is clear by Proposition 3 that S is relatively minimal. By Theorem 1, S is not algebraic. Hence $a(S) \leq 1$. Assume that $a(S) = 1$. Then, by Theorem 1, there exists a locally flat line bundle L on S such that the mapping $\Phi_L : S \rightarrow \mathbb{P}^n$ defined by the linear system $|L|$ gives an algebraic reduction of S which is defined everywhere. Put $\Delta = \Phi_L(S)$. Let η be the line bundle on Δ associated to a hyperplane section of Δ . Then we have $\Phi_L^* \eta = L$. We note that every fibre of $\Phi_L : S \rightarrow \Delta$ is a non-singular elliptic curve (Proposition 3). We indicate by $b_i(M)$ the i -th Betti number of a manifold M . It is clear that $b_1(\Delta) \leq b_1(S) \leq b_1(\Delta) + 2$. Assume first that $b_1(\Delta) = b_1(S)$. Since L is a locally flat line bundle on S , L is raised from a group representation ρ of $H_1(S, \mathbb{Z})$ into $\mathbb{C}^*$. Let m be a certain positive integer such that ρ^m is trivial on the torsion part of $H_1(S, \mathbb{Z})$. Then, in view of $b_1(\Delta) = b_1(S)$, there exists a locally flat line bundle ξ on Δ such that $\Phi_L^* \xi = L^m$. Hence we get $\Phi_L^* \xi = \Phi_L^* \eta^m$.

Since $\bar{\Phi}_1^* : H^1(\Delta, 0^*) \longrightarrow H^1(S, 0^*)$ is ^(an) injection, this implies that the ample line bundle γ on Δ is locally flat. This is absurd. Hence we get $b_1(\Delta) < b_1(S)$. Next assume that $b_1(S) = b_1(\Delta) + 2$. By Corollary to Theorem 3, we get $b_2(S) = 2b_1(\Delta) + 2$. This implies that the dual of the homology class represented by a general fibre is a Betti base of $H^2(S, \mathbb{Z})$. This contradicts Theorem 1. Hence we conclude that $b_1(S) = b_1(\Delta) + 1$. Therefore $b_1(S)$ is odd. Hence S is either a surface of VI_0 or VII_0 -elliptic. Consider the case $a(S) = 0$. By the classification theory of surfaces [4], a relatively minimal surface with no non-constant meromorphic functions and vanishing Euler number is either a complex torus or a surface of VII_0 . A complex torus has a positive algebraic dimension if it contains a divisor. Hence by Proposition 3 we infer that S is of VII_0 -class. Moreover $b_1(S) = 1$ and $b_2(S) = 0$. Hence, by Theorem 34 [4], S is a Hopf surface.

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