

# APPROXIMATION OF FIXED POINTS AND PROXIMAL POINT ALGORITHMS

WATARU TAKAHASHI

Department of Mathematical and Computing Sciences  
Tokyo Institute of Technology  
2-12-1, Ohokayama, Meguro-ku, Tokyo 152-8552, Japan

**ABSTRACT.** In this article, we give three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space. Then we discuss weak and strong convergence theorems for nonlinear operators of accretive and monotone type in a Hilbert space or a Banach space. In particular, we state weak and strong convergence theorems for resolvents of  $m$ -accretive operators and maximal monotone operators in a Banach space. Using these results, we also consider the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

## 1. INTRODUCTION

We consider the following problem: Let  $f_0, f_1, f_2, \dots, f_m$  be convex continuous functions of a Hilbert space  $H$  into  $\mathbb{R}$ . Then, the problem is to find a  $z \in C$  such that

$$f_0(z) = \min\{f_0(x) : x \in C\}, \quad (1)$$

where  $C = \{x \in H : f_1(x) \leq 0, f_2(x) \leq 0, \dots, f_m(x) \leq 0\}$ . Such a problem is called the convex minimization problem. Let us define a function  $g : H \rightarrow (-\infty, \infty]$  as follows:

$$g(x) = \begin{cases} f_0(x), & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then,  $g$  is a proper lower semicontinuous convex function and a minimizer  $z \in H$  of  $g$  is a solution of the convex minimization problem (1). So, let  $g : H \rightarrow (-\infty, \infty]$  be a proper convex lower semicontinuous function. Consider a convex minimization problem:

$$\min\{g(x) : x \in H\}. \quad (2)$$

For such a  $g$ , we can define a multivalued operator  $\partial g$  on  $H$  by

$$\partial g(x) = \{x^* \in H : g(y) \geq g(x) + \langle x^*, y - x \rangle, y \in H\}$$

for all  $x \in H$ . Such a  $\partial g$  is said to be the subdifferential of  $g$ . A monotone operator  $A \subset H \times H$  is called maximal if its graph

$$G(A) = \{(x, y) : y \in Ax\}$$

is not properly contained in the graph of any other monotone operator. We know that if  $A$  is a maximal monotone operator, then  $R(I + \lambda A) = H$  for all  $\lambda > 0$ . A monotone operator  $A$  is also called  $m$ -accretive if  $R(I + \lambda A) = H$  for all  $\lambda > 0$ .

So, we can define, for each positive  $\lambda$ , the resolvent  $J_\lambda : R(I + \lambda A) \rightarrow D(A)$  by  $J_\lambda = (I + \lambda A)^{-1}$ . We know that  $J_\lambda$  is a nonexpansive mapping. If  $g : H \rightarrow (-\infty, \infty]$  is a proper lower semicontinuous convex function, then  $\partial g$  is a maximal monotone operator.

We know that one method for solving (2) is the proximal point algorithm first introduced by Martinet [16]. The proximal point algorithm is based on the notion of resolvent  $J_\lambda$ , i.e.,

$$J_\lambda x = \arg \min \left\{ g(z) + \frac{1}{2\lambda} \|z - x\|^2 : z \in H \right\}.$$

The proximal point algorithm is an iterative procedure, which starts at a point  $x_1 \in H$ , and generates recursively a sequence  $\{x_n\}$  of points  $x_{n+1} = J_{\lambda_n} x_n$ , where  $\{\lambda_n\}$  is a sequence of positive numbers; see, for instance, Rockafellar [26].

On the other hand, Halpern [6] and Mann [15] introduced the following iterative schemes to approximate a fixed point of a nonexpansive mapping  $T$  of  $H$  into itself:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad n = 1, 2, \dots$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n = 1, 2, \dots,$$

respectively, where  $x_1 = x \in H$  and  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ . Recently, Nakajo and Takahashi [18] also introduced an iterative scheme of finding a fixed point of a nonexpansive mapping in a Hilbert space by using an idea of the hybrid method in mathematical programming.

In this article, we first state three convergence theorems for nonexpansive mappings in a Hilbert space. They are convergence theorems of Halpern's type, Mann's type and Nakajo-Takahashi's type. Then, we prove a strong convergence theorem of Halpern's type and a weak convergence theorem of Mann's type for inverse-strongly-monotone mappings in a Hilbert space. In Section 6, we prove weak and strong convergence theorems for resolvents of accretive operators in a Banach space. In Section 7, we consider the strong convergence of a sequence defined by resolvents of maximal monotone operators in a Banach space. Using these results, we also discuss the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

## 2. PRELIMINARIES

Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and let  $E^*$  denote the dual of  $E$ . We denote the value of  $y^* \in E^*$  at  $x \in E$  by  $\langle x, y^* \rangle$ . When  $\{x_n\}$  is a sequence in  $E$ , we denote the strong convergence of  $\{x_n\}$  to  $x \in E$  by  $x_n \rightarrow x$  and the weak convergence by  $x_n \rightharpoonup x$ . The modulus of convexity of  $E$  is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every  $\epsilon$  with  $0 \leq \epsilon \leq 2$ . A Banach space  $E$  is said to be uniformly convex if  $\delta(\epsilon) > 0$  for every  $\epsilon > 0$ . If  $E$  is uniformly convex, then  $\delta$  satisfies that  $\delta(\epsilon/r) > 0$  and

$$\left\| \frac{x + y}{2} \right\| \leq r \left( 1 - \delta \left( \frac{\epsilon}{r} \right) \right)$$

for every  $x, y \in E$  with  $\|x\| \leq r$ ,  $\|y\| \leq r$  and  $\|x - y\| \geq \epsilon$ . Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Then we know that

for any  $x \in E$ , there exists a unique element  $z \in C$  such that  $\|x - z\| \leq \|x - y\|$  for all  $y \in C$ . Putting  $z = P_C(x)$ , we call  $P_C$  the metric projection of  $E$  onto  $C$ . The duality mapping  $J$  from  $E$  into  $2^{E^*}$  is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every  $x \in E$ . Let  $U = \{x \in E : \|x\| = 1\}$ . The norm of  $E$  is said to be Gâteaux differentiable if for each  $x, y \in U$ , the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (3)$$

exists. In the case,  $E$  is called smooth. The norm of  $E$  is said to be uniformly Gâteaux differentiable if for each  $y \in U$ , the limit (3) is attained uniformly for  $x \in U$ . It is also said to be Fréchet differentiable if for each  $x \in U$ , the limit (3) is attained uniformly for  $y \in U$ . It is known that if the norm of  $E$  is uniformly Gâteaux differentiable, then the duality mapping  $J$  is single valued and uniformly norm to weak\* continuous on each bounded subset of  $E$ . A Banach space  $E$  is said to satisfy Opial's condition [20] if for any sequence  $\{x_n\} \subset E$ ,  $x_n \rightharpoonup y$  implies

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|$$

for all  $z \in E$  with  $z \neq y$ . A Hilbert space satisfies Opial's condition.

Let  $C$  be a closed convex subset of  $E$ . A mapping  $T: C \rightarrow C$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . We denote the set of all fixed points of  $T$  by  $F(T)$ . A closed convex subset  $C$  of  $E$  is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset  $D$  of  $C$  into itself has a fixed point in  $D$ . Let  $D$  be a subset of  $E$ . We denote the closure of the convex hull of  $D$  by  $\overline{\text{co}}D$ .

Let  $I$  denote the identity operator on  $E$ . An operator  $A \subset E \times E$  with domain  $D(A) = \{z \in E : Az \neq \emptyset\}$  and range  $R(A) = \bigcup \{Az : z \in D(A)\}$  is said to be accretive if for each  $x_i \in D(A)$  and  $y_i \in Ax_i$ ,  $i = 1, 2$ , there exists  $j \in J(x_1 - x_2)$  such that  $\langle y_1 - y_2, j \rangle \geq 0$ . If  $A$  is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all  $r > 0$ . An accretive operator  $A$  is said to satisfy the range condition if  $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$ . If  $A$  is accretive, then we can define, for each  $r > 0$ , a nonexpansive single valued mapping  $J_r: R(I + rA) \rightarrow D(A)$  by  $J_r = (I + rA)^{-1}$ . It is called the resolvent of  $A$ . We also define the Yosida approximation  $A_r$  by  $A_r = (I - J_r)/r$ . We know that  $A_r x \in AJ_r x$  for all  $x \in R(I + rA)$  and  $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$  for all  $x \in D(A) \cap R(I + rA)$ . We also know that for an accretive operator  $A$  satisfying the range condition,  $A^{-1}0 = F(J_r)$  for all  $r > 0$ . An accretive operator  $A$  is said to be  $m$ -accretive if  $R(I + rA) = E$  for all  $r > 0$ . A multi-valued operator  $A: E \rightarrow 2^{E^*}$  with domain  $D(A) = \{z \in E : Az \neq \emptyset\}$  and range  $R(A) = \bigcup \{Az : z \in D(A)\}$  is said to be monotone if  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$  for each  $x_i \in D(A)$  and  $y_i \in Ax_i$ ,  $i = 1, 2$ . A monotone operator  $A$  is said to be maximal if its graph  $G(A) = \{(x, y) : y \in Ax\}$  is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance [32].

**Theorem 1.** *Let  $E$  be a reflexive, strictly convex and smooth Banach space and let  $A: E \rightarrow 2^{E^*}$  be a monotone operator. Then  $A$  is maximal if and only if  $R(J + rA) = E^*$  for all  $r > 0$ .*

**Theorem 2.** *Let  $E$  be a strictly convex and smooth Banach space and let  $x, y \in E$ . If  $\langle x - y, Jx - Jy \rangle = 0$ , then  $x = y$ .*

By Theorem 1, a monotone operator  $A$  in a Hilbert space  $H$  is maximal if and only if  $A$  is  $m$ -accretive.

### 3. APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

There are three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space which are related to the problem of finding a minimizer of a convex function.

Halpern [6] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space. For the proof, see Wittmann [36] and Takahashi [32].

**Theorem 3** ([36]). *Let  $C$  be a closed convex subset of a Hilbert space  $H$  and let  $T$  be a nonexpansive mapping of  $C$  into itself that  $F(T)$  is nonempty. Let  $P$  be the metric projection of  $H$  onto  $F(T)$ . Let  $x \in C$  and let  $\{x_n\}$  be a sequence defined by  $x_1 = x$  and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  satisfies

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then,  $\{x_n\}$  converges strongly to  $Px \in F(T)$ .

Mann [15] also introduced the iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [32].

**Theorem 4** ([15]). *Let  $C$  be a closed convex subset of a Hilbert space  $H$  and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T)$  is nonempty. Let  $P$  be the metric projection of  $H$  onto  $F(T)$ . Let  $x \in C$  and let  $\{x_n\}$  be a sequence defined by  $x_1 = x$  and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  satisfies

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then,  $\{x_n\}$  converges weakly to  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Px_n$ .

Recently, Nakajo and Takahashi [18] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

**Theorem 5** ([18]). *Let  $C$  be a closed convex subset of a Hilbert space  $H$  and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T)$  is nonempty. Let  $P$  be the metric projection of  $H$  onto  $F(T)$ . Let  $x_1 = x \in C$  and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \dots, \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$  satisfies  $\liminf_{n \rightarrow \infty} \alpha_n < 1$  and  $P_{C_n \cap Q_n}$  is the metric projection of  $H$  onto  $C_n \cap Q_n$ . Then,  $\{x_n\}$  converges strongly to  $Px_1 \in F(T)$ .

Shioji and Takahashi [27] extended Theorem 3 to that of a Banach space whose norm is uniformly Gâteaux differentiable. Let  $C$  and  $D$  be closed convex subsets of a Banach space  $E$  and let  $D$  be a subset of  $C$ . Then, a mapping  $P$  of  $C$  onto  $D$  is called sunny if

$$P(Px + t(x - Px)) = Px$$

whenever  $Px + t(x - Px) \in C$  for  $x \in C$  and  $t \geq 0$ .

**Theorem 6** ([27]). *Let  $E$  be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T)$  is nonempty. Let  $\{\alpha_n\}$  be a sequence of real numbers such that*

$$0 \leq \alpha_n \leq 1, \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Suppose  $x_1 = x \in C$  and  $\{x_n\}$  is given by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots$$

Then,  $\{x_n\}$  converges strongly to  $Px \in F(T)$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(T)$ .

Reich [22] extended also Mann's result to that of a Banach space whose norm is Fréchet differentiable.

**Theorem 7** ([22]). *Let  $C$  be a closed convex subset of a uniformly convex Banach space  $E$  with a Fréchet differentiable norm, let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T)$  is nonempty, and let  $\{\alpha_n\}$  be a real sequence such that  $0 \leq \alpha_n \leq 1$  and  $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ . If  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad n = 1, 2, \dots,$$

then  $\{x_n\}$  converges weakly to a fixed point of  $T$ .

**Problem.** Is a Hilbert space in Theorem 5 replaced by a uniformly convex and smooth Banach space?

#### 4. APPROXIMATING SOLUTIONS OF VALIATIONAL INEQUALITIES

Let  $C$  be a closed convex subset of a Hilbert space  $H$ . Then, a mapping  $A$  of  $C$  into  $H$  is called inverse-strongly-monotone if there exists a positive real number  $\alpha$  such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all  $x, y \in C$ ; see [4] and [14]. For such a case,  $A$  is called  $\alpha$ -inverse-strongly-monotone. If a mapping  $T$  of  $C$  into itself is nonexpansive, then  $A = I - T$  is  $\frac{1}{2}$ -inverse-strongly-monotone and  $F(T) = \text{VI}(C, A)$ ; for example, see [8]. A mapping  $A$  of  $C$  into  $H$  is called strongly monotone if there exists a positive number  $\eta$  such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2$$

for all  $x, y \in C$ . In such a case, we say that  $A$  is  $\eta$ -strongly monotone. If  $A$  is  $\eta$ -strongly monotone and  $k$ -Lipschitz continuous, i.e.,  $\|Ax - Ay\| \leq k\|x - y\|$  for all  $x, y \in C$ , then  $A$  is  $\frac{\eta}{k^2}$ -inverse-strongly-monotone; see [14]. Let  $f$  be a continuously Fréchet differentiable convex function  $H$  and let  $\nabla f$  be the gradient of  $f$ . If  $\nabla f$  is

$\frac{1}{\alpha}$ -Lipschitz continuous, then  $\nabla f$  is an  $\alpha$ -inverse-strongly-monotone mapping of  $C$  into  $H$ ; see [1]. We also have that for all  $x, y \in C$  and  $\lambda > 0$ ,

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2.\end{aligned}$$

So, if  $\lambda \leq 2\alpha$ , then  $I - \lambda A$  is a nonexpansive mapping of  $C$  into  $H$ .

**Theorem 8** ([7]). *Let  $C$  be a closed convex subset of a Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly-monotone mapping of  $C$  into  $H$  and let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then,  $\{x_n\}$  converges strongly to  $z = P_{F(S) \cap VI(C, A)} x$ .

**Theorem 9** ([34]). *Let  $C$  be a closed convex subset of a Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly-monotone mapping of  $C$  into  $H$  and let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\}$  and  $\{\lambda_n\}$  satisfy

$$0 < c \leq \alpha_n \leq d < 1 \quad \text{and} \quad 0 < a \leq \lambda_n \leq b < 2\alpha.$$

Then,  $\{x_n\}$  converges weakly to  $z \in F(S) \cap VI(C, A)$ .

## 5. PROXIMAL POINT ALGORITHMS IN HILBERT SPACES

We consider two proximal point algorithms for solving (2) in Section 1, with parameters  $\{r_n\}$ , starting at an initial point  $x_1$  in a Hilbert space  $H$ .

**Theorem 10** ([9]). *Let  $H$  be a Hilbert space and let  $A \subset H \times H$  be a maximal monotone operator. Let  $x_1 = x \in H$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges strongly to  $Px \in A^{-1}0$ , where  $P$  is the metric projection of  $H$  onto  $A^{-1}0$ .

**Theorem 11** ([9]). *Let  $H$  be a Hilbert space and let  $A \subset H \times H$  be a maximal monotone operator. Let  $x_1 = x \in H$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy  $\alpha_n \in [0, k]$  for some  $k$  with  $0 < k < 1$  and  $\lim_{n \rightarrow \infty} r_n = \infty$ . If  $A^{-1}0 \neq \phi$ , then  $\{x_n\}$  converges weakly to  $v \in A^{-1}0$ , where  $v = \lim_{n \rightarrow \infty} Px_n$  and  $P$  is the metric projection of  $H$  onto  $A^{-1}0$ .

Using Theorems 10 and 11, we obtain the following theorems.

**Theorem 12** ([9]). *Let  $H$  be a Hilbert space and let  $f : H \rightarrow (-\infty, \infty]$  be a lower semicontinuous proper convex function. Let  $x_1 = x \in H$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

$$J_{r_n} x_n = \arg \min \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 : z \in H \right\},$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

If  $(\partial f)^{-1}0 \neq \phi$ , then  $\{x_n\}$  converges strongly to  $v \in H$ , which is the minimizer of  $f$  nearest to  $x$ . Further

$$f(x_{n+1}) - f(v) \leq \alpha_n (f(x) - f(v)) + \frac{1 - \alpha_n}{r_n} \|J_{r_n} x_n - v\| \|J_{r_n} x_n - x_n\|.$$

**Theorem 13** ([9]). *Let  $H$  be a Hilbert space and let  $f : H \rightarrow (-\infty, \infty]$  be a lower semicontinuous proper convex function. Let  $x_1 = x \in H$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

$$J_{r_n} x_n = \arg \min \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 : z \in H \right\},$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy  $\alpha_n \in [0, k]$  for some  $k$  with  $0 < k < 1$  and  $\lim_{n \rightarrow \infty} r_n = \infty$ . If  $(\partial f)^{-1}0 \neq \phi$ , then  $\{x_n\}$  converges weakly to  $v \in H$ , which is a minimizer of  $f$ . Further

$$f(x_{n+1}) - f(v) \leq \alpha_n (f(x_n) - f(v)) + \frac{1 - \alpha_n}{r_n} \|J_{r_n} x_n - v\| \|J_{r_n} x_n - x_n\|.$$

Solodov and Svaiter [29] also proved the following strong convergence theorem.

**Theorem 14** ([29]). *Let  $H$  be a Hilbert space and let  $A \subset H \times H$  be a maximal monotone operator. Let  $x \in H$  and let  $\{x_n\}$  be a sequence defined by*

$$\begin{cases} x_1 = x \in H, \\ 0 = v_n + \frac{1}{r_n} (y_n - x_n), \quad v_n \in Ay_n, \\ H_n = \{z \in H : \langle z - y_n, v_n \rangle \leq 0\}, \\ W_n = \{z \in H : \langle z - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots, \end{cases}$$

where  $\{r_n\}$  is a sequence of positive numbers. If  $A^{-1}0 \neq \phi$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then  $\{x_n\}$  converges strongly to  $P_{A^{-1}0} x_1$ .

## 6. CONVERGENCE THEOREMS FOR ACCRETIVE OPERATORS

In this section, we study a strong convergence theorem of Halpern's type for accretive operators in a Banach space. We need the following lemma for the proof of our theorem.

**Lemma 15** ([35]). *Let  $E$  be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let  $A \subset E \times E$  be an accretive operator which satisfies the range condition. Suppose that every weakly compact convex subset of  $E$  has the fixed point property for nonexpansive mappings. Let  $C$  be a nonempty closed convex subset of  $E$  such that  $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ . If  $A^{-1}0 \neq \emptyset$ , then the strong  $\lim_{t \rightarrow \infty} J_t x$  exists and belongs to  $A^{-1}0$  for all  $x \in C$ .*

See also Reich [23]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [36] and Shioji and Takahashi [27].

**Theorem 16** ([10]). *Let  $E$  be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let  $A \subset E \times E$  be an accretive operator which satisfies the range condition, and let  $C$  be a nonempty closed convex subset of  $E$  such that  $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges strongly to an element of  $A^{-1}0$ .

As a direct consequence of Theorem 16, we have the following:

**Theorem 17.** *Let  $E$  be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let  $A \subset E \times E$  be an  $m$ -accretive operator. Let  $x_1 = x \in E$  and let  $\{x_n\}$  be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges strongly to an element of  $A^{-1}0$ .

Next, we prove a weak convergence theorem for Mann's type for accretive operators in a Banach space. Before proving the theorem, we need the following two lemmas.

**Lemma 18** ([3]). *Let  $C$  be a closed bounded convex subset of a uniformly convex Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself. If  $\{x_n\}$  converges weakly to  $z \in C$  and  $\{x_n - Tx_n\}$  converges strongly to 0, then  $Tz = z$ .*

**Lemma 19** ([22]). *Let  $E$  be a uniformly convex Banach space whose norm is Fréchet differentiable, let  $C$  be a nonempty closed convex subset of  $E$  and let  $\{T_0, T_1, T_2, \dots\}$  be a sequence of nonexpansive mappings of  $C$  into itself such that  $\bigcap_{n=0}^{\infty} F(T_n)$  is nonempty. Let  $x \in C$  and  $S_n = T_n T_{n-1} \cdots T_0$  for all  $n = 1, 2, \dots$ . Then the set  $\bigcap_{n=0}^{\infty} \overline{\text{co}}\{S_m x : m \geq n\} \cap U$  consists of at most one point, where  $U = \bigcap_{n=0}^{\infty} F(T_n)$ .*

For the proof of Lemma 19, see Takahashi and Kim [33]. Now we can prove the following weak convergence theorem.

**Theorem 20** ([10]). *Let  $E$  be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let  $A \subset E \times E$  be an accretive operator which satisfies the range condition, and let  $C$  be a nonempty closed convex subset of  $E$  such that  $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\limsup_{n \rightarrow \infty} \alpha_n < 1 \text{ and } \liminf_{n \rightarrow \infty} r_n > 0.$$

If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges weakly to an element of  $A^{-1}0$ .

As a direct consequence of Theorem 20, we have the following:

**Theorem 21.** *Let  $E$  be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let  $A \subset E \times E$  be an  $m$ -accretive operator. Let  $x_1 = x \in E$  and let  $\{x_n\}$  be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\limsup_{n \rightarrow \infty} \alpha_n < 1 \text{ and } \liminf_{n \rightarrow \infty} r_n > 0.$$

If  $A^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges weakly to an element of  $A^{-1}0$ .

## 7. CONVERGENCE THEOREMS FOR MAXIMAL MONOTONE OPERATORS

In this section, we study strong convergence theorems for resolvents of maximal monotone operators in a Banach space. Let  $E$  be a uniformly convex and smooth Banach space and let  $A$  be a maximal monotone operator from  $E$  into  $E^*$  such that  $A^{-1}0 \neq \emptyset$ . For  $x \in E$  and  $r > 0$ , we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

By Theorems 1 and 2, this equation has a unique solution  $x_r$ . We denote  $J_r$  by  $x_r = J_r x$  and such  $J_r$ ,  $r > 0$  are called resolvents of  $A$ . Now, we extend Solodov and Svaiter's result [29].

**Theorem 22** ([19]). *Let  $E$  be a uniformly convex and smooth Banach space and let  $A$  be a maximal monotone operator from  $E$  into  $E^*$  such that  $A^{-1}0 \neq \emptyset$ . Suppose  $\{x_n\}$  is the sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = J_{r_n} x_n, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots, \end{cases}$$

where  $\{r_n\}$  is a sequence of positive numbers. If  $A^{-1}0 \neq \emptyset$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then  $\{x_n\}$  converges strongly to  $P_{A^{-1}0} x_1$ .

Next, we establish another extension of Solodov and Svaiter's result [29]. Before establishing it, we give a definition. Let  $E$  be a reflexive, strictly convex and smooth Banach space. The function  $\phi: E \times E \rightarrow (-\infty, \infty)$  is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for  $x, y \in E$ . Let  $C$  be a nonempty closed convex subset of  $E$  and let  $x \in E$ . Then there exists a unique element  $x_0 \in C$  such that

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}. \quad (4)$$

So, if  $C$  is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space  $E$  and  $x \in E$ , we define the mapping  $Q_C$  of  $E$  onto  $C$  by  $Q_C x = x_0$ , where  $x_0$  is defined by (4). It is easy to see that in a Hilbert space, the mapping  $Q_C$  is coincident with the metric projection.

**Theorem 23** ([11]). *Let  $E$  be a uniformly convex and uniformly smooth Banach space and let  $A$  be a maximal monotone operator from  $E$  into  $E^*$  such that  $A^{-1}0 \neq \phi$ . Let  $Q_r = (J + rA)^{-1}J$  for all  $r > 0$  and let  $\{x_n\}$  be the sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = Q_{r_n} x_n, \\ H_n = \{z \in E : \langle z - y_n, Jx_n - Jy_n \rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots, \end{cases}$$

where  $\{r_n\}$  is a sequence of positive numbers such that  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then,  $\{x_n\}$  converges strongly to  $Q_{A^{-1}0} x_1$ .

Recently, Kohsaka and Takahashi [12] proved a strong convergence theorem of Halpen's type for maximal monotone operators in a Banach space.

**Theorem 24** ([12]). *Let  $E$  be a smooth and uniformly convex Banach space and let  $A \subset E \times E^*$  be a maximal monotone operator. Let  $Q_r = (J + rA)^{-1}J$  for all  $r > 0$  and let  $\{x_n\}$  be a sequence defined as follows:*

$$\begin{aligned} x_1 &= x \in E, \\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1 - \alpha_n) JQ_{r_n} x_n), \quad n = 1, 2, \dots, \end{aligned}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

If  $A^{-1}0 \neq \phi$ , then  $\{x_n\}$  converges strongly to  $Q_{A^{-1}0} x$ .

**Problem.** If  $E$  and  $E^*$  are uniformly convex Banach spaces, does Theorem 11 hold for maximal monotone operators  $A \subset E \times E^*$ ?

We can apply Theorems 22, 23 and 24 to find a minimizer of a convex function  $f$ . Let  $E$  be a real Banach space and let  $f: E \rightarrow (-\infty, \infty]$  be a proper convex lower semicontinuous function. Then the subdifferential  $\partial f$  of  $f$  is as follows:

$$\partial f(z) = \{v \in E^* : f(y) \geq f(z) + \langle y - z, v \rangle, \forall y \in E\}, \quad \forall z \in E.$$

**Theorem 25** ([19]). *Let  $E$  be a uniformly convex and smooth Banach space and let  $f : E \rightarrow (-\infty, \infty]$  be a proper convex lower semicontinuous function. Assume that  $\{r_n\} \subset (0, \infty)$  satisfies  $\liminf_{n \rightarrow \infty} r_n > 0$  and let  $\{x_n\}$  be the sequence generated by*

$$\begin{cases} x_1 \in E \\ y_n = \arg \min_{z \in E} \{f(z) + \frac{1}{2r_n} \|z - x_n\|^2\}, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots \end{cases}$$

*If  $(\partial f)^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges strongly to the minimizer of  $f$  nearest to  $x_1$ .*

*Proof.* Since  $f : E \rightarrow (-\infty, \infty]$  is a proper convex lower semicontinuous function, by Rockafellar [24], the subdifferential  $\partial f$  of  $f$  is a maximal monotone operator. We also know that

$$y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 \right\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n} J(y_n - x_n).$$

So, we have

$$0 \in J(y_n - x_n) + r_n \partial f(y_n).$$

Using Theorem 22, we get the conclusion.  $\square$

**Theorem 26** ([11]). *Let  $E$  be a uniformly convex and uniformly smooth Banach space and let  $f : E \rightarrow (-\infty, \infty]$  be a proper convex lower semicontinuous function. Assume that  $\{r_n\} \subset (0, \infty)$  satisfies  $\liminf_{n \rightarrow \infty} r_n > 0$  and let  $\{x_n\}$  be the sequence generated by*

$$\begin{cases} x_1 \in E \\ y_n = \arg \min_{z \in E} \{f(z) + \frac{1}{2r_n} \|z\|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle\}, \\ 0 = v_n + \frac{1}{r_n} (Jy_n - Jx_n), \quad v_n \in \partial f(y_n), \\ H_n = \{z \in E : \langle z - y_n, v_n \rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots \end{cases}$$

*If  $(\partial f)^{-1}0 \neq \emptyset$ , then  $\{x_n\}$  converges strongly to the minimizer of  $f$  nearest to  $x_1$ .*

*Proof.* We also know that

$$y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \|z\|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle \right\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n} Jy_n - \frac{1}{r_n} Jx_n.$$

So, we have  $v_n \in \partial f(y_n)$  such that  $0 = v_n + \frac{1}{r_n} (Jy_n - Jx_n)$ . Using Theorem 23, we get the conclusion.  $\square$

Using Theorem 24, we get the following theorem.

**Theorem 27** ([12]). *Let  $E$  be a smooth and uniformly convex Banach space and let  $f : E \rightarrow (-\infty, \infty]$  be a proper lower semicontinuous convex function such that  $(\partial f)^{-1}0$  is nonempty. Let  $\{x_n\}$  be a sequence defined as follows:*

$$\begin{aligned} x_1 &= x \in E, \\ y_n &= \arg \min_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n \rangle \right\}, \\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n), \quad n = 1, 2, \dots, \end{aligned}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} r_n = \infty.$$

Then,  $\{x_n\}$  converges strongly to  $Q_{(\partial f)^{-1}0}x$ .

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