

Difference approximation of evolution equations
and generation of nonlinear semigroups

by Yoshikazu Kobayashi

Department of Mathematics, Waseda University

We consider the following nonlinear evolution equation
(DE) $(d/dt) u(t) \in Au(t)$, $0 < t < T$,
where A is a (multi-valued) quasi-dissipative operator. In
this note, we construct the solution of the evolution equation
(DE) by the method of difference approximation. In addition,
we give a generation theorem of nonlinear semigroups through
the difference approximations.

1. Preliminaries. Let X be a real Banach space. For
the multi-valued operator A , we use the following notations:

$$D(A) = \{x \in X; Ax \neq \emptyset\} \quad , \quad R(A) = \bigcup_{x \in D(A)} \{y; y \in Ax\} \quad ,$$

$$\text{and } \|Ax\| = \inf\{\|y\|; y \in Ax\} \quad \text{for } x \in D(A).$$

We identify the multi-valued operator A with its graph, so that
we write $[x, y] \in A$ if $y \in Ax$.

Let F be the duality map in X . Then we set

$$\langle y, x \rangle_i = \inf\{\langle y, f \rangle; f \in F(x)\}$$

and $\langle y, x \rangle_s = -\langle -y, x \rangle_i = -\langle y, -x \rangle_i$ for $x, y \in X$.

Let $A \subset X \times X$. A is said to be dissipative if for any
 $[x_i, y_i] \in A$ ($i=1, 2$),

$$\langle y_1 - y_2, x_1 - x_2 \rangle_i \leq 0.$$

According to Takahashi [9], we introduce the following notion as a generalization of that of dissipative operator.

Definition 1. Let $A \subset X \times X$. A is said to be quasi-dissipative if for any $[x_i, y_i] \in A$ ($i=1,2$),

$$\langle y_1, x_1 - x_2 \rangle_i + \langle y_2, x_2 - x_1 \rangle_i \leq 0.$$

The following example shows that quasi-dissipative operators are not always dissipative.

Example (I. Miyadera). Let $X = \mathbb{R}^2$ with maximum norm. Let $x_1 = (1,1)$ and $x_2 = (0,0)$. We set $D(A) = \{x_1, x_2\}$, $Ax_1 = \{(\alpha, \beta); \alpha \leq 0 \text{ or } \beta \leq 0\}$ and $Ax_2 = \{(\alpha, \beta); \alpha \geq 0 \text{ or } \beta \geq 0\}$. Then A is quasi-dissipative in X but $A - \omega$ is not dissipative in X for any real ω . In addition, $R(I - \lambda A) \supset D(A)$ for any $\lambda > 0$.

For the quasi-dissipative operator, we have the following.

Lemma 1. Let $A \subset X \times X$. Then the followings are equivalent.

- (i) A is quasi-dissipative;
- (ii) for any $[x_i, y_i] \in A$ ($i=1,2$) and $\lambda, \mu > 0$,
 $(\lambda + \mu) \|x_1 - x_2\| \leq \lambda \|x_1 - x_2 - \mu y_1\| + \mu \|x_2 - x_1 - \lambda y_2\|;$
- (iii) for any $[x_i, y_i] \in A$ ($i=1,2$) and $\lambda > 0$,
 $2 \|x_1 - x_2\| \leq \|x_1 - x_2 - \lambda y_1\| + \|x_2 - x_1 - \lambda y_2\|.$

We can verify Lemma 1 similarly as the proof of Kato's lemma [4].

Let $X_0 \subset X$. A one parameter family $\{T(t); t \geq 0\}$ of operators from X_0 into itself is called (nonlinear) contraction semigroup on X_0 if it has the following properties:

- (i) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for $x, y \in X_0$ and $t \geq 0$;
- (ii) $T(0)x = x$ for $x \in X_0$ and $T(t+s) = T(t)T(s)$ for $t, s \geq 0$;
- (iii) for each $x \in X_0$, $T(t)x$ is strongly continuous in $t \geq 0$.

2. Cauchy problems and difference approximation.

Let A be a quasi-dissipative operator in X . Let $x_0 \in X$ and $T > 0$. Then we treat the following Cauchy problem for the evolution equation (DE):

$$(CP; x_0) \quad \begin{cases} (d/dt) u(t) \in Au(t) & \text{for } t \in (0, T), \\ u(0) = x_0. \end{cases}$$

For the Cauchy problem $(CP; x_0)$, we consider the following type of difference approximation:

$$(DS; x_0) \quad \begin{cases} \left\| \frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - y_k^n \right\| \leq \varepsilon_k^n, & k=1, 2, \dots, N_n; n \geq 1, \\ x_0^n = x_0, \end{cases}$$

where for each n , $[x_k^n, y_k^n] \in A$ ($k=1, 2, \dots, N_n$) and $\{t_k^n\}$ represents the partition of $[0, T]$ such that $0 = t_0^n < t_1^n < \dots < t_{N_n-1}^n < T \leq t_{N_n}^n$ and $\delta_n = \max_{1 \leq k \leq N_n} (t_k^n - t_{k-1}^n) \rightarrow 0$ as $n \rightarrow \infty$. The ε_k^n may be referred as an error bound which occurs at the k -th step of the n -th approximation of the difference approximation $(DS; x_0)$.

Definition 2. Let $u_n(t)$ be a sequence in $L^\infty(0, T; X)$.

We say that $u_n(t)$ is a (backward) DS-approximate solution of the Cauchy problem $(CP; x_0)$ if there exists a difference approximation $(DS; x_0)$ satisfying the following:

- (i) $u_n(0) = x_0^n = x_0$, $n \geq 1$;
- (ii) $u_n(t) = x_k^n$ for $t \in (t_{k-1}^n, t_k^n] \cap (0, T]$, $k=1, 2, \dots, N_n; n \geq 1$;
- (iii) $\sum_{k=1}^{N_n} \varepsilon_k^n (t_k^n - t_{k-1}^n) \rightarrow 0$ as $n \rightarrow \infty$.

Then we have

Theorem I. Let $x_0 \in \overline{D(A)}$ and $u_n(t)$ be a DS-approximate solution of $(CP; x_0)$ on $[0, T]$. Then there exists a $u(t) \in C([0, T]; X)$ satisfying the following:

$$(i) \quad u(t) = \lim_{n \rightarrow \infty} u_n(t) \quad \text{for } t \in [0, T],$$

and the convergence is uniform on $[0, T]$;

$$(ii) \quad u(t) \in \overline{D(A)} \quad \text{for } t \in [0, T] \text{ and } u(0) = x_0;$$

$$(iii) \quad \text{for any DS-approximate solution } \hat{u}_n(t) \text{ of } (CP; x_0),$$

$$u(t) = \lim_{n \rightarrow \infty} \hat{u}_n(t) \quad \text{for } t \in [0, T].$$

Remarks. 1) Kenmochi-Oharu [5] and Takahashi [9], [10] studied the convergence (i) under the additional condition, which is called the stability condition by them. Our result is an extension of their results.

2) By Benilan's method [2], we find that the limiting function $u(t)$ is the unique integral solution of the Cauchy problem $(CP; x_0)$.

The proof of Theorem I is based on the following.

Lemma 2. Let $(DS; x_0)$ and $(DS; \hat{x}_0)$ be two difference approximations as above of the Cauchy problems $(CP; x_0)$ and $(CP; \hat{x}_0)$ on $[0, T]$, respectively. Let the notations with $\hat{}$ represents the difference approximation $(DS; \hat{x}_0)$. Then

$$(1) \quad \|x_i^m - \hat{x}_j^n\| \leq \|x_0 - u\| + \|\hat{x}_0 - u\| \\ + \{(t_i^m - \hat{t}_j^n)^2 + \delta_m t_i^m + \delta_n \hat{t}_j^n\}^{1/2} \|Au\| \\ + \sum_{k=1}^i \varepsilon_k^m (t_k^m - t_{k-1}^m) + \sum_{k=1}^j \hat{\varepsilon}_k^n (\hat{t}_k^n - \hat{t}_{k-1}^n),$$

for $0 \leq i \leq N_m$, $0 \leq j \leq \hat{N}_n$ and $u \in D(A)$.

Proof. Let $u \in D(A)$ and $v \in Au$. We set $a_{i,j} = \|x_i^m - \hat{x}_j^n\|$, $h_i^m = t_i^m - t_{i-1}^m$ and $\hat{h}_j^n = \hat{t}_j^n - \hat{t}_{j-1}^n$ for $0 \leq i \leq N_m$ and $0 \leq j \leq \hat{N}_n$.

By (iii) of Lemma 1, we have

$$\|x_k^m - u\| \leq \|x_k^m - h_k^m y_k^m - u\| + h_k^m \|v\| \\ \leq \|x_{k-1}^m - u\| + h_k^m \varepsilon_k^m + h_k^m \|v\|$$

for $1 \leq k \leq N_m$. Therefore, inductively, we have

$$\|x_i^m - u\| \leq \|x_0^m - u\| + t_i^m \|v\| + \sum_{k=1}^i \epsilon_k^m h_k^m$$

or

$$a_{i,0} \leq \|x_0 - u\| + \|\hat{x}_0 - u\| + t_i^m \|Au\| + \sum_{k=1}^i \epsilon_k^m h_k^m.$$

This shows that (1) holds true for $(i,0)$ with $0 \leq i \leq N_m$.

Similarly we have (1) for $(0,j)$ with $0 \leq j \leq \hat{N}_n$. Furthermore,

by (ii) of Lemma 1, we have

$$\begin{aligned} (h_i^m + \hat{h}_j^n) a_{i,j} &\leq \hat{h}_j^n \|x_i^m - h_i^m y_i^m - \hat{x}_j^n\| + h_i^m \|\hat{x}_j^n - \hat{h}_j^n y_j^n - x_i^m\| \\ &\leq \hat{h}_j^n a_{i-1,j} + h_i^m a_{i,j-1} + h_i^m \hat{h}_j^n (\epsilon_i^m + \hat{\epsilon}_j^n) \end{aligned}$$

for $1 \leq i \leq N_m$ and $1 \leq j \leq \hat{N}_n$. Hence, using the Cauchy-Schwarz'inequality, we can verify (1) for every (i,j) by the induction for (i,j) . Q.E.D.

Remark. Let A be a dissipative operator in X such that $R(I - \lambda A) \supset D(A)$ for $\lambda > 0$. Then estimate (1) gives

$$\|(I - \lambda A)^{-n} x - (I - \mu A)^{-m} x\| \leq \{(n\lambda - m\mu)^2 + n\lambda^2 + m\mu^2\}^{1/2} \|Au\|$$

for $n, m \geq 1$, $\lambda, \mu > 0$ and $x \in D(A)$. This estimate is similar to but different from that of Crandall-Liggett [3].

Proof of Theorem I. Let $(DS; x_0)$ be the corresponding difference approximation to $u_n(t)$. Then by Lemma 2, we have

$$\begin{aligned} (2) \quad \|x_i^m - x_j^n\| &\leq 2\|x_0 - u_p\| \\ &\quad + \{(t_i^m - t_j^n)^2 + \delta_m t_i^m + \delta_n t_j^n\}^{1/2} \|Au_p\| \\ &\quad + \sum_{k=1}^{N_m} \epsilon_k^m (t_k^m - t_{k-1}^m) + \sum_{k=1}^{N_n} \epsilon_k^n (t_k^n - t_{k-1}^n) \end{aligned}$$

for $0 \leq i \leq N_m$ and $0 \leq j \leq N_n$, where $\{u_p\} \subset D(A)$ is a sequence such that $u_p \rightarrow x_0$ as $p \rightarrow \infty$. This estimate shows that there exists

$$\begin{aligned} u(t) &= \lim_{k \rightarrow \infty} x_k^n \quad \text{as } t_k^n \rightarrow t, n \rightarrow \infty, \\ &= \lim_{n \rightarrow \infty} u_n(t) \end{aligned}$$

for every $t \in [0, T]$. Furthermore, by (2), we have

$$\|u(t) - u(s)\| \leq 2\|x_0 - u_p\| + |t - s| \|Au_p\|$$

for $t, s \in [0, T]$. This shows that $u(t)$ is continuous on $[0, T]$.

The property (ii) is evident. Let $\hat{u}_n(t)$ be a DS-approximate solution of $(CP; \hat{x}_0)$ with $\hat{x}_0 \in \overline{D(A)}$. And let us set

$$\hat{u}(t) = \lim_{n \rightarrow \infty} \hat{u}_n(t) \quad \text{for } t \in [0, T].$$

Then by the estimate (1), we have

$$\|u(t) - \hat{u}(t)\| \leq \|x_0 - \hat{x}_0\| \quad \text{for } t \in [0, T].$$

Especially, we have (iii). Q.E.D.

By Theorem I, we define the following.

Definition 3. Let $u(t) \in C([0, T]; X)$ and $x_0 \in \overline{D(A)}$. We say that $u(t)$ is a (backward) DS-limit solution of the Cauchy problem $(CP; x_0)$ on $[0, T]$ if there exists a (backward) DS-approximate solution $u_n(t)$ of $(CP; x_0)$ on $[0, T]$, such that $u_n(t)$ converges to $u(t)$, uniformly for $t \in [0, T]$.

In the proof of Theorem I, we obtained the following.

Corollary. Let $u(t), \hat{u}(t)$ be two DS-limit solutions of (CP) on $[0, T]$. Then

$$\|u(t) - \hat{u}(t)\| \leq \|u(0) - \hat{u}(0)\| \quad \text{for } t \in [0, T].$$

3. Generation of semigroups

By Theorem I and the Corollary, we have a generation theorem of semigroups.

Definition 4. Let A be a quasi-dissipative operator in X . We say that A has the property $(\mathcal{Q})$ if for any $x \in \overline{D(A)}$ and $T > 0$, there exists a DS-approximate solution of the Cauchy problem $(CP; x)$ on $[0, T]$.

Theorem II. Let A be a quasi-dissipative operator in X , having the property $(\mathcal{Q})$. Then there exists a contraction

semigroup $\{T(t); t \geq 0\}$ on $\overline{D(A)}$ such that for any $x \in \overline{D(A)}$ and $T > 0$, $u(t) = T(t)x$ is the unique DS-limit solution of the Cauchy problem $(CP; x_0)$ on $[0, T]$.

Proof. Let $x \in \overline{D(A)}$ and $T > 0$. Then, by Theorem I, there exists the unique DS-limit solution $u(t)$ of $(CP; x)$ on $[0, T]$. By its corollary, we can extend the solution $u(t)$ onto $[0, \infty)$. Then we define $T(t): \overline{D(A)} \rightarrow \overline{D(A)}$ by

$$T(t)x = u(t) \quad \text{for } t \geq 0.$$

Using Theorem I and its corollary, we can verify that $\{T(t); t \geq 0\}$ is the desired contraction semigroup. Q.E.D.

4. Existence of difference approximation

In this section, we give a sufficient condition that a quasi-dissipative operator has the property $(\mathcal{L})$. Let A be a quasi-dissipative operator in X . We add the following condition on A :

$(R)_t$ for any $x \in \overline{D(A)}$, there exist sequences $\delta_n \downarrow 0$ and $[x_n, y_n] \in A$ ($n \geq 1$) such that

$$\lim_{n \rightarrow \infty} \delta_n^{-1} \|x_n - x - \delta_n y_n\| = 0.$$

Then we have

Theorem III. Let A be a quasi-dissipative operator in X , satisfying the condition $(R)_t$. Then A has the property $(\mathcal{L})$. Thus A generates a contraction semigroup on $\overline{D(A)}$, in the sense of Theorem II.

Remarks. 1) This theorem implies the fundamental result of Crandall-Liggett [3]: a part of the results of Martin [7] on ordinary differential equations; and the results of Webb [11] and Barbu [1] on the continuous perturbations of m -dissipative

operators. The details will be treated in [6].

2) Yorke announces in [12] that he obtained a similar result.

Proof of Theorem III. Let $x_0 \in \overline{D(A)}$ and $\varepsilon_n \downarrow 0$. Let n be fixed. Then for each $x \in \overline{D(A)}$, we set

$$\delta_n(x) = \sup\{ \delta ; 0 < \delta \leq \varepsilon_n \text{ and there exists } [x_\delta, y_\delta] \in A \text{ such that } \|x_\delta - x - \delta y_\delta\| \leq \delta \varepsilon_n \}.$$

Then $\delta_n(x)$ are positive by the assumption. Therefore, inductively, we can choose $h_k^n > 0$ and $[x_k^n, y_k^n] \in A$, for $k=1, 2, \dots$, so that they satisfy the following:

- (i) $x_0^n = x_0$;
- (ii) $(1/2) \delta_n(x_{k-1}^n) < h_k^n \leq \varepsilon_n$, for $k=1, 2, \dots$;
- (iii) $\|x_k^n - x_{k-1}^n - h_k^n y_k^n\| \leq h_k^n \varepsilon_n$, for $k=1, 2, \dots$.

Then we set $t_i^n = \sum_{k=1}^i h_k^n$. We may show that $t_i^n \rightarrow \infty$ as $i \rightarrow \infty$.

For the purpose, we use the following estimate:

$$(3) \quad \|x_i^n - x_j^n\| \leq (t_i^n - t_j^n) \|y_k^n\| + \varepsilon_n (t_i^n - t_k^n) + \varepsilon_n (t_j^n - t_k^n)$$

for any $i \geq j \geq k \geq 1$. This estimate may be verified by the induction for (i, j) with $i \geq j \geq k$ for each fixed $k \geq 1$, by using Lemma 1 as in the proof of Lemma 2.

Now, suppose that $t_i^n \rightarrow s_0 < +\infty$ as $i \rightarrow \infty$, for contradiction.

Then by (3), we see that there exists $u_0 \in \overline{D(A)}$ such that

$x_i^n \rightarrow u_0$ as $i \rightarrow \infty$. By the assumption, we can choose $\delta > 0$ and $[u_\delta, v_\delta] \in A$ such that $0 < \delta \leq \varepsilon_n$ and

$$\|u_\delta - u_0 - \delta v_\delta\| \leq \delta \varepsilon_n / 2.$$

Since $\delta_n(x_i^n) \rightarrow 0$ and $x_i^n \rightarrow u_0$ as $i \rightarrow \infty$, there exists i_0 such that $\delta_n(x_i^n) < \delta$ and $\|x_i^n - u_0\| \leq \delta \varepsilon_n / 2$ for $i \geq i_0$. Then we have

$$\|u_\delta - u_0 - \delta v_\delta\| \leq \delta \varepsilon_n \text{ for } i \geq i_0,$$

which is contrary to the definition of $\delta_n(x_i^n)$. Q.E.D.

Remark. The construction and properties of the DS-limit solution of evolution equations will be studied more systematically and generally in [6].

Acknowledgement. The author would like to express his hearty thanks to Prof. I. Miyadera and Mr. T. Takahashi for their valuable advices.

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