

Propagation of microlocal solutions near a hyperbolic fixed point

東北大学大学院理学研究科 藤家雪朗 (Setsuro Fujiié)

Joint work with Jean-François Bony, Thierry Ramond
and Maher Zerzeri

1 Introduction

This is a partial report of the work in progress with Jean-François Bony, Thierry Ramond and Maher Zerzeri about the quantum monodromy operator associated to a homoclinic trajectory. A major part of the results here was already reported by one of the collaborators in [3].

The notion of monodromy operator was introduced by J. Sjöstrand and M. Zworski in [4] for a periodic trajectory. It consists in continuing microlocal solutions of the semiclassical Schrödinger equation

$$-\hbar^2 \Delta u + V(x)u = Eu \quad (1)$$

along a Hamilton flow H_p on $p^{-1}(E)$ of the corresponding classical mechanics:

$$H_p = \sum_{j=1}^d \left(\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \right), \quad p(x, \xi) = \xi^2 + V(x). \quad (2)$$

Recall briefly the notion of microlocal solution according to [4]. If $dp \neq 0$ at a point $(x^0, \xi^0) \in p^{-1}(E)$, there exists a local canonical transformation κ defined in a neighborhood of (x^0, ξ^0) with $\kappa(x^0, \xi^0) = (0, 0)$, and a semiclassical microlocal Fourier integral operator U associated to κ , such that $p = \kappa^* \xi_1$ and $UPU^{-1} = \hbar D_{x_1}$. We can then define the space of microlocal solution at (x^0, ξ^0) by

$$\ker_{(x^0, \xi^0)}(P) = U^{-1}(\ker(\hbar D_{x_1})), \quad \ker(\hbar D_{x_1}) = \{u \in \mathcal{D}'(\mathbb{R}^d) : \hbar D_{x_1} u = 0\}$$

Since $\ker(hD_{x_1})$ is identified with $\mathcal{D}'(\mathbb{R}^{d-1})$, so is $\ker_{(x^0, \xi^0)}(P)$. If $(x^1, \xi^1) = \exp tH_p(x^0, \xi^0)$ is another point on this flow, we can naturally define the propagator of microlocal solutions from $\ker_{(x^0, \xi^0)}(P)$ to $\ker_{(x^1, \xi^1)}(P)$ as operator on $\mathcal{D}'(\mathbb{R}^{d-1})$.

Here we study the case where $\exp tH_p(x^0, \xi^0)$ tends to a hyperbolic fixed point $(0, 0)$ as t tends to $+\infty$. To such a point associate the stable and unstable Lagrangian manifolds Λ_- and Λ_+ , on which Hamilton flows tend to $(0, 0)$ as t tends to $+\infty$ and $-\infty$ respectively. Moreover, any point close to Λ_+ comes from a point close to Λ_- . We expect, therefore, that a microlocal solution at a point on Λ_+ is determined by that on Λ_- .

The purpose of this report is to study this correspondence of microlocal solutions from Λ_- to Λ_+ . After preparing the geometrical setting in section 2, we state a uniqueness theorem in section 3, which says that if a solution to (1) is microlocally exponentially small on Λ_- , it is also microlocally exponentially small on Λ_+ for E away from a discrete subset $\Gamma(h)$. In section 4, based on an idea in [2], we construct a solution with a given microlocal data at a point (x^0, ξ_-^0) on Λ_- , as superposition of time-dependent WKB solutions via Fourier transform with respect to E , and formally calculate its microlocal output at the corresponding point (x^0, ξ_+^0) on Λ_+ . Section 5 is an appendix about the notion of *expandible symbol*, which is used repeatedly for the study of the large time behavior of both classical and quantum objects.

2 Symplectic geometry

Let $p(x, \xi) = \xi^2 + V(x)$ be the Hamiltonian associated to the semiclassical Schrödinger operator $-\hbar^2 \Delta + V(x)$ in $\mathbb{R}^d$. Here, we use the following notations:

$$x = (x_1, \dots, x_d), \quad \xi = (\xi_1, \dots, \xi_d), \quad \xi^2 = \sum_{j=1}^d \xi_j^2, \quad \Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

Suppose that the potential $V(x)$ is real and analytic in a neighborhood of $x = 0$, and that $x = 0$ is a non-degenerate minimum of $V(x)$, so that $(x, \xi) = (0, 0)$ is a saddle point of the Hamiltonian $p(x, \xi)$. After a change of variables, we can assume that $p(x, \xi)$ is of the form

$$p(x, \xi) = \xi^2 - \sum_{j=1}^d \frac{\lambda_j^2}{4} x_j^2 + O(|x|^3), \quad (x \rightarrow 0),$$

where $\{\lambda_j\}_{j=1}^d$ are positive numbers which we assume $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_d$.

Let H_p be the Hamilton vector field associated to p . In the (x, ξ) coordinates, the linearized vector field F_p of H_p at $(0, 0)$ is simply

$$F_p = d_{(0,0)}H_p = \begin{pmatrix} 0 & I \\ L^2 & 0 \end{pmatrix}, \quad (3)$$

where L is the $d \times d$ matrix defined as $L = \text{diag}(\lambda_1, \dots, \lambda_d)$. The eigenvalues of F_p are the λ_j 's and the $-\lambda_j$'s.

Associated to the hyperbolic fixed point, we have thus a natural decomposition of $T_{(0,0)}^*\mathbb{R}^d = \mathbb{R}^{2d}$ in a direct sum of two linear subspaces Λ_+^0 and Λ_-^0 , of dimension d , associated respectively to the positive and negative eigenvalues of F_p . These spaces $\Lambda_{\pm}^0$ are given by

$$\Lambda_{\pm}^0 = \{(x, \xi); \xi_j = \pm \frac{\lambda_j}{2} x_j, j = 1, \dots, d\}. \quad (4)$$

The stable/unstable manifold theorem gives us the existence of two Lagrangian manifolds Λ_+ and Λ_- , defined in a vicinity Ω of $(0, 0)$, which are stable under the H_p flow and whose tangent space at $(0, 0)$ are precisely Λ_+^0 and Λ_-^0 . In particular, we see that these manifolds can be written as

$$\Lambda_{\pm} = \{(x, \xi); \xi = \nabla \phi_{\pm}(x)\}, \quad (5)$$

for some smooth functions ϕ_+ and ϕ_- , which can be chosen so that

$$\phi_{\pm}(x) = \pm \sum_{j=1}^d \frac{\lambda_j}{4} x_j^2 + o(x^2). \quad (6)$$

We shall say that Λ_+ is the outgoing Lagrangian manifold and Λ_- the incoming Lagrangian manifold associated to the hyperbolic fixed point. Indeed Λ_+ (resp. Λ_-) can be characterized as the set of points $(x, \xi) \in \Omega$ such that $\exp tH_p(x, \xi) \rightarrow (0, 0)$ as $t \rightarrow -\infty$ (resp. as $t \rightarrow +\infty$): Take a point $x^0 \in \mathbb{R}^d$ near 0. Then there exist unique $\xi_+^0 \in \mathbb{R}^d$ and $\xi_-^0 \in \mathbb{R}^d$ such that $(x^0, \xi_{\pm}^0) \in \Lambda_{\pm}$. Let $\gamma_{\pm}(t) = \exp tH_p(x^0, \xi_{\pm}^0)$ be the Hamilton flow emanating from $(x^0, \xi_{\pm}^0)$. Then, we know from Proposition 10 in Appendix that $\gamma_{\pm}(t)$ are expandible, i.e.

$$\gamma_{\pm}(t) \sim \sum_{k=1}^{\infty} e^{\pm \mu_k t} \gamma_{\pm,k}(t), \quad t \rightarrow \mp \infty, \quad (7)$$

where $\gamma_{\pm,k}(t)$ are vectors whose elements are polynomials in t ($\gamma_{\pm,1}$ is constant) and $0 < \mu_1 < \mu_2 < \dots$ are the various non-vanishing linear combinations over $\mathbb{N}$ of the λ_j 's. In particular, $\mu_1 = \lambda_1$. If we assume

(A1) $\lambda_1 < \lambda_2$,

then there exists a constant $\gamma_1 = \gamma_1(x^0)$ such that

$$\gamma_{\pm}(t) = \gamma_1 e^{\pm \lambda_1 t} \times {}^t(1, 0, \dots, 0, \pm \lambda_1/2, 0, \dots, 0) + O(e^{\pm \mu_2 t}), \quad (t \rightarrow \mp \infty). \quad (8)$$

We see that $\gamma_{\pm}(t)$ is tangential to the (x_1, ξ_1) -plane if $c \neq 0$.

3 Uniqueness

We begin this section by introducing the notion of *microsupport* of solutions.

For $u \in L^2(\mathbb{R}^n)$, the Bargman transform (or global FBI transform) is defined by

$$Tu(x, \xi; h) = c_d(h) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi / h - (x-y)^2 / 2h} u(y; h) dy.$$

$Tu(x, \xi; h)$ belongs to $L^2(\mathbb{R}_{x,\xi}^{2d})$ and $c_d(h)$ is taken so that T be an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$. It is seen that by this transform, the function u is localized in x by a Gaussian up to $O(\sqrt{h})$ when h is small. Moreover, it is localized also in ξ up to $O(\sqrt{h})$. Indeed we have an identity

$$Tu(x, \xi; h) = e^{ix \cdot \xi / h} T\hat{u}(\xi, -x; h),$$

where $\hat{u}$ is the semiclassical Fourier transform

$$\hat{u}(\xi) = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} e^{-x \cdot \xi / h} u(x) dx. \quad (9)$$

A (h -dependent) function $u \in L^2$ is said to be *zero* at a point (x^0, ξ^0) in the phase space iff there exists a neighborhood U of (x^0, ξ^0) and a positive number ϵ such that

$$Tu(x, \xi; h) = O(e^{-\epsilon/h})$$

as $h \rightarrow 0$ uniformly in U . The complement of such points is called *microsupport* of u and denoted by $MS(u)$. Microsupport is a closed set. Two functions u and v are identified near (x^0, ξ^0) if $(x^0, \xi^0) \notin MS(u - v)$.

Microsupport has the following properties: Let u be a solution of $Pu = E(h)u$ in a domain $\Omega \subset \mathbb{R}^n$, where $E(h) = O(h)$, and assume that $\|u\|_{L^2(\Omega)} \leq 1$.

- The microsupport of u is included in the energy surface $p^{-1}(0)$.
- The microsupport of u propagates along a simple Hamilton flow in $p^{-1}(0)$.
- The microsupport of a WKB solution $u = e^{i\psi(x)/h}b(x, h)$, $b(x, h) = O(h^{-N})$ for some $N \in \mathbb{R}$ as h tends to 0, is included in the Lagrangian submanifold $\{(x, \xi); \xi = \partial_x \psi(x)\}$.

Now we come back to our problem near the hyperbolic fixed point. Let $\Gamma(h)$ be the discrete subset of $\mathbb{C}$ defined by

$$\Gamma(h) = \{E_\alpha = -ih \sum_{j=1}^d \lambda_j (\alpha_j + \frac{1}{2}); \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d\}. \quad (10)$$

Notice that for $E = E_\alpha$, the functions

$$u_\alpha = \prod_{j=1}^d H_{\alpha_j} \left(e^{-\pi i/4} \frac{\sqrt{\lambda_j}}{\sqrt{2h}} x_j \right) \exp \left(i \sum_{j=1}^m \frac{\lambda_j}{4h} x_j^2 \right),$$

where H_n is the Hermite polynomial, satisfy the equation

$$-h^2 \Delta u_\alpha - \sum_{j=1}^m \frac{\lambda_j^2}{4} x_j^2 u_\alpha = E_\alpha u_\alpha.$$

These functions are of WKB form and, by the above third property, the microsupport of u_α is Λ_+^0 .

Let us assume

(A2) $|E(h)| \leq Ch$ in $\mathbb{C}$ with $C > 0$, and there exists $\delta > 0$ such that $d(E(h), \Gamma(h)) > \delta h$ for all small h .

The following theorem says that the solution of (1) is uniquely determined microlocally in a neighborhood of $(0, 0)$, modulo microlocally small functions, by its data on $\Lambda_- \setminus (0, 0)$ if $E(h)$ is away from the exceptional set $\Gamma(h)$.

Theorem 1 Assume (A2). If an h -dependent function $u \in L^2(\mathbb{R}^d)$ with $\|u\|_{L^2} \leq 1$ satisfies

$$MS((P - E(h))u) = \emptyset, \quad MS(u) \cap \{\Lambda_- \setminus (0, 0)\} = \emptyset,$$

in a neighborhood of $(0, 0)$, then $(0, 0) \notin MS(u)$.

4 Integral representation of the solution

In order to study the correspondence of microlocal solutions from Λ_- to Λ_+ , we fix a point (x^0, ξ_-^0) on Λ_- sufficiently close to the origin and consider solutions of (1) whose microsupport on Λ_- is included in a neighborhood of $\exp tH_p(x^0, \xi_-^0)$ (recall that the microsupport is invariant by the Hamilton flow). Then, under the assumption (A2), the solution u is uniquely determined in a full neighborhood of the origin, in particular on Λ_+ , if a microlocal data u_0 is given at (x^0, ξ_-^0) . We study in this section the map $\mathcal{I}_S$ which associates u_0 to the microlocal solution u at (x^0, ξ_+^0) , which we call here *propagator* (it is called *singular part of the monodromy operator* in [3]).

The symbol p is of principal type at $(x^0, \xi_-^0) \in \Lambda_-$ and the space of microlocal solutions $\ker_{(x^0, \xi_-^0)}(P)$ is identified with $\mathcal{D}'(\mathbb{R}^{d-1})$. If we assume

(A3) $\gamma_1(x^0) \neq 0$,

where $\gamma_1(x^0)$ is defined in (8), a microlocal solution $u_0 \in \ker_{(x^0, \xi_-^0)}(P)$ can be considered as distribution on

$$H_0 = \{x \in \mathbb{R}^d; x_1 = x_1^0\},$$

since (the projection of) the Hamilton flows are tangential to the x_1 axis at the origin.

Let $u_0(x') \in \mathcal{D}'(\mathbb{R}^{d-1})$ be such that $\hat{u}_0(\eta)$, the semiclassical Fourier transform of u_0 (see (9)), is supported in a small neighborhood of ξ_0' .

Following an idea of Helffer and Sjöstrand [2], we write the solution u in the form

$$u(x, h) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} \int_0^{+\infty} e^{i\phi(t, x, \eta)/h} a(t, x, \eta, h) \hat{u}_0(\eta) dt d\eta, \quad (11)$$

with

$$\left\{ \frac{h}{i} \frac{\partial}{\partial t} + P(x, hD) - E(h) \right\} (e^{i\phi/h} a) = O(h^\infty).$$

If a and the energy $E(h)$ have classical asymptotic expansions with respect to h :

$$a(t, x, \eta, h) \sim \sum_{l=0}^{\infty} a_l(t, x, \eta) h^l, \quad E(h) \sim \sum_{l=0}^{\infty} E_l h^{l+1},$$

(recall here that $E(h)$ is assumed to be of $O(h)$ in (A2)) then ϕ and a should satisfy the eikonal and transport equations respectively:

$$\partial_t \phi + p(x, \nabla_x \phi) = 0, \quad (12)$$

$$\partial_t a_0 + 2\nabla_x \phi \cdot \nabla_x a_0 + (\Delta \phi - iE_0)a_0 = 0, \quad (13)$$

$$\partial_t a_l + 2\nabla_x \phi \cdot \nabla_x a_l + (\Delta \phi - iE_0)a_l = i\Delta a_{l-1} + i \sum_{m=1}^l E_m a_{l-m} \quad (l \geq 1). \quad (14)$$

The phase function ϕ will be constructed as generating function of the evolution $\Lambda_t^\eta = \exp tH_p(\Lambda_0^\eta)$ of a suitably chosen Lagrangian manifold Λ_0^η transverse to Λ_- at (x^0, ξ_-^0) . Let us fix η sufficiently close to $\xi_-^{0'}$, and look at the integral in (11) with respect to t . It will be shown that, for x close to x^0 , there exists a unique critical point $t = t(x, \eta)$. On the other hand, the Lagrangian manifold Λ_t^η tends to Λ_+ as $t \rightarrow +\infty$, which means that $\partial_t \phi$ tends to ϕ_+ . Thus we will have microlocally

$$\int_0^{+\infty} e^{i\phi(t, x, \eta)/h} a(t, x, \eta, h) dt \sim \begin{cases} e^{i\psi(x, \eta)} b(x, \eta, h) & \text{near } (x, \xi) = (x^0, \xi_-^0), \\ e^{i\theta(x, \eta)} c(x, \eta, h) & \text{near } (x, \xi) = (x^0, \xi_+^0), \end{cases}$$

with $\psi(x, \eta) = \phi(t(x, \eta), x, \eta)$ and $\theta(x, \eta) = \phi_+(x) + \tilde{\psi}(\eta)$ for some $\tilde{\psi}$.

We require that u is equal to u_0 on H_0 microlocally near (x^0, ξ^0) , which is satisfied if

$$\psi(x, \eta) = x' \cdot \eta, \quad b(x, \eta, h) = 1 \quad \text{on } H_0. \quad (15)$$

We will see in the following that it is possible to construct ϕ and a so that ψ and b satisfy the condition (15) and to calculate θ and c . Then we will write $\mathcal{I}_S$ as Fourier integral operator.

4.1 The phase function

Since γ_- is a simple characteristic for the operator p , by the usual Hamilton-Jacobi theory we have first the

Lemma 2 *For all $\eta \in \mathbb{R}^{d-1}$ close enough to $\xi^{0'}$, there is a unique function $\psi_\eta = \psi(x, \eta)$, defined in a neighborhood ω_0 of x_0 such that*

$$\begin{cases} p(x, \nabla \psi_\eta(x)) = 0 & \text{in } \omega_0, \\ \psi_\eta(x) = x' \cdot \eta & \text{on } H_0 \cap \omega_0. \end{cases}$$

We denote by Λ_ψ^η the corresponding Lagrangian manifold

$$\Lambda_\psi^\eta = \{(x, \xi) \in T^*\mathbb{R}^d, x \in \omega_0, \xi = \nabla\psi_\eta(x)\}. \quad (16)$$

Lemma 3 *The Lagrangian manifolds Λ_- and Λ_ψ^η intersect along an integral curve γ^η for H_p , and the intersection is clean. In particular, $\gamma^{\xi^{0'}} = \gamma_-$.*

Let $(x^0(\eta), \xi^0(\eta))$ be the intersection of γ^η and $H_0 \times \mathbb{R}_\xi^d$. The curve γ^η is parametrized as $\gamma^\eta(t) = \exp tH_p(x^0(\eta), \xi^0(\eta))$, and it has the asymptotic property like (8);

$$\gamma^\eta(t) \sim \gamma_1(\eta)e^{-\lambda_1 t} \times {}^t(1, 0, \dots, 0, -\lambda_1/2, 0, \dots, 0) \quad (t \rightarrow +\infty), \quad (17)$$

with a non vanishing constant $\gamma_1(\eta)$ for η close to $\xi^{0'}$.

Let Γ_0^η be the level set of ψ_η passing by $x^0(\eta)$:

$$\Gamma_0^\eta = \{(x, \xi) \in \Lambda_\psi^\eta, \psi_\eta(x) = \psi_\eta(x^0(\eta))\}. \quad (18)$$

Lemma 4 *For any η close enough to $\xi^{0'}$, one can find a Lagrangian manifold Λ_0^η such that*

1. Λ_0^η intersects cleanly with Λ_ψ^η along Γ_0^η ,
2. for any $t \geq 0$, the projection $\Pi : \Lambda_t^\eta = \exp(tH_p)(\Lambda_0^\eta) \rightarrow \mathbb{R}_x^d$ is a diffeomorphism in a neighborhood of $\gamma^\eta(t) \in \Lambda_t^\eta$.

The Lagrangian manifold $\Lambda_t^\eta = \exp(tH_p)(\Lambda_0^\eta)$ is then represented by a generating function $\phi(t, x, \eta)$:

$$\Lambda_t^\eta = \{(x, \xi); \xi = \nabla_x \phi(t, x, \eta)\}. \quad (19)$$

and $\phi(t, x, \eta)$ satisfies the eikonal equation (12) for every η .

Now we fix η and define

$$\Gamma_t^\eta = \Lambda_t^\eta \cap \Lambda_\psi^\eta \quad (= \exp(tH_p)\Gamma_0^\eta). \quad (20)$$

If $(x, \xi) \in \Gamma_t^\eta$, then $\xi = \nabla_x \phi(t, x, \eta)$ and $p(x, \xi) = 0$ ($\Lambda_\psi^\eta \subset p^{-1}(0)$). Together with (12), we get that t is a critical point for the function $t \mapsto \phi(t, x, \eta)$ if and only if $x \in \Pi_x \Gamma_t^\eta$. More precisely, we have

Proposition 5 For each x close enough to γ^η , there is a unique time $t = t(x, \eta)$ such that $x \in \Pi_x \Gamma_t^\eta$. Moreover, it is the only critical point for the function $t \mapsto \phi(t, x, \eta)$ and it is non-degenerate, $\partial_t^2 \phi(t(x, \eta), x, \eta) > 0$.

As a consequence, we obtain

$$\nabla_x \psi_\eta(x) = \nabla_x (\phi(t(x, \eta), x)), \quad (21)$$

so that $x \mapsto \psi_\eta(x)$ and $x \mapsto \phi(t(x), x)$ are equal up to constant. We choose ϕ so that

$$\phi(t(x, \eta), x, \eta) = \psi_\eta(x). \quad (22)$$

Finally we observe the asymptotic behavior of the phase function $\phi(t, x, \eta)$ when t tends to $+\infty$.

Proposition 6 The phase function $(t, x) \mapsto \phi(t, x, \eta)$ is expandible uniformly with respect to η :

$$\phi(t, x, \eta) - (\phi_+(x) + \tilde{\psi}(\eta)) \sim \sum_{j \geq 1} e^{-\mu_j t} \phi_j(t, x, \eta). \quad (23)$$

Here $\tilde{\psi}$ is a generating function of the $d-1$ dimensional Lagrangian submanifold $\Lambda_- \cap (H_0 \times \mathbb{R}_\xi^d)$, i.e.

$$\{(y', \eta) \in T^*\mathbb{R}^{d-1}; \eta = \nabla_{y'} \phi_-(x_1^0, y')\} = \{(y', \eta) \in T^*\mathbb{R}^{d-1}; y' = \nabla_\eta \tilde{\psi}(\eta)\},$$

and so

$$\tilde{\psi}(\eta) \sim - \sum_{j=2}^d \frac{1}{\lambda_j} \eta_j^2, \quad (\eta \rightarrow 0).$$

Moreover, the function ϕ_1 does not depend on t , and

$$\phi_1(x, \eta) = -2\lambda_1 \gamma_1(\eta) x_1 + O(x^2), \quad (24)$$

where $\gamma_1(\eta)$ is defined in (17).

4.2 Transport equations

We study the transport equations (13), (14), using the informations about the phase function $\phi(t, x, \eta)$ obtained in the previous subsection. We want to solve these equations under the condition

$$a(t(x, \eta), x, \eta, h)|_{H_0} = e^{-\pi i/4} \sqrt{\partial_t^2 \phi(t(x, \eta), x, \eta)}, \quad (25)$$

so that the right hand side of (11), after the stationary phase method applied to the integration with respect to t at the critical point $t = t(x, \eta)$, reduces to u_0 on H_0 . Notice that the initial condition (25) determines uniquely the solutions of (13), (14) on the hypersurface $\{(t, x); t = t(x, \eta)\}$, since this hypersurface is invariant under the flow of the vector field $\partial_t + 2\nabla_x \phi \cdot \nabla_x$.

As for the asymptotic behavior as $t \rightarrow +\infty$, we recall that ϕ is expandible and

$$\nabla_x \phi \cdot \nabla_x = \sum_{j=1}^d \left(\frac{\lambda_j}{2} x_j + O(x^2) \right) \frac{\partial}{\partial x_j}, \quad \Delta \phi = \sum_{j=1}^d \frac{\lambda_j}{2} + O(x) \quad (x \rightarrow 0).$$

Then again by Proposition 10 applied to $e^{St} a_j$, where

$$S = \frac{1}{2} \sum_{j=1}^d \lambda_j - iE_0,$$

we have the following asymptotic expansion.

Proposition 7 *For each l , $a_l(t, x, \eta)$ is expandible and has an asymptotic expansion as $t \rightarrow \infty$*

$$a_l(t, x, \eta) \sim e^{-St} \sum_{k=0}^{\infty} a_{l,k}(t, x, \eta) e^{-\mu_k t}, \quad (26)$$

which is uniform with respect to η . Here μ_0 is defined to be 0, and $a_{0,0}$ is independent of t .

4.3 Asymptotics of the propagator

Let us fix η close to $\xi^{0'}$ and x close to γ_η . Then there are two t 's which contribute in the semiclassical limit to the integration with respect to t of

the expression (11). One is $t = t(x, \eta)$, which is the unique critical point, and the other is $t = +\infty$. They correspond to the Lagrangian manifolds $\Lambda_{t(x, \eta)}^\eta$ and Λ_+ respectively.

Since the contribution from $t = t(x, \eta)$ reproduces the given data $u_0(x')$ on H_0 after integration with respect to η , we will obtain the propagator $\mathcal{I}_S$ in the form of Fourier integral operator after calculating the contribution from $t = +\infty$.

Lemma 8 Suppose $b \in \mathbb{R}$, $\lambda > 0$ and $\rho > 0$. Then as $h \rightarrow 0$, we have

$$\begin{aligned} & \int_0^\infty \exp\{ibe^{-\lambda t}/h - \rho t\} dt - \frac{1}{\lambda} \left(\frac{ih}{b}\right)^{\rho/\lambda} \Gamma\left(\frac{\rho}{\lambda}\right) \\ & \sim \frac{e^{ib/h}}{\lambda} \sum_{n=0}^\infty \binom{\rho/\lambda - 1}{n} n! \left(\frac{ih}{b}\right)^{n+1} \end{aligned}$$

Let us compute the contribution from $t = \infty$ of the integral

$$\int_0^\infty e^{i\phi(t, x, \eta)/h} a(t, x, \eta, h) dt.$$

If we substitute $\phi_+(x) + \tilde{\psi}(\eta) + e^{-\lambda_1 t} \phi_1(x, \eta)$ to $\phi(t, x, \eta)$ and $a_{0,0}(x, \eta) e^{-St}$ to $a(t, x, \eta, h)$ according to (23), (26), we get

$$\int_0^\infty e^{i\phi/h} a dt = e^{i(\phi_+ + \tilde{\psi})/h} a_{0,0} \int_0^\infty \exp\{i\phi_1 e^{-\lambda_1 t}/h - St\} dt$$

Applying Lemma 8 with $b = \phi_1$, $\lambda = \lambda_1$ and $\rho = S$, we get

$$\begin{aligned} & \int_0^\infty e^{i\phi/h} a dt \sim e^{i(\phi_+ + \tilde{\psi})/h} a_{0,0} \\ & \times \left\{ \frac{1}{\lambda_1} \Gamma\left(\frac{S}{\lambda_1}\right) \left(\frac{ih}{\phi_1}\right)^{S/\lambda_1} + \frac{e^{i\phi_1/h}}{\lambda_1} \frac{ih}{\phi_1} + O(h^2) \right\} \quad (h \rightarrow 0). \end{aligned}$$

The leading term of the left hand side changes according to the real part of S/λ_1 :

$$\operatorname{Re} S/\lambda_1 > 1 \Leftrightarrow \operatorname{Im} E_0 > \left(\lambda_1 - \sum_{j=2}^d \lambda_j \right) / 2.$$

Theorem 9 *The propagator $\mathcal{I}_S$ can be written in the form*

$$\frac{1}{\sqrt{2\pi h}^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\theta(x,\eta)} c(x, \eta, h) \hat{u}_0(\eta) d\eta,$$

microlocally near (x^0, ξ_+^0) with

$$\theta(x, \eta) = \phi_+(x) + \tilde{\psi}(\eta),$$

and if $\text{Im } E_0 < (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x, \eta, h) \sim \frac{1}{\sqrt{2\pi h} \lambda_1} \Gamma\left(\frac{S}{\lambda_1}\right) \left(\frac{ih}{\phi_1(x)}\right)^{S/\lambda_1} a_{0,0}(x, \eta),$$

if $\text{Im } E_0 > (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x, \eta, h) \sim \frac{1}{\sqrt{2\pi h} \lambda_1} e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} a_{0,0}(x, \eta),$$

and if $\text{Im } E_0 = (\lambda_1 - \sum_2^d \lambda_j)/2$

$$c(x, \eta, h) \sim \frac{1}{\sqrt{2\pi h} \lambda_1} \left(\Gamma\left(\frac{S}{\lambda_1}\right) \left(\frac{ih}{\phi_1(x)}\right)^{S/\lambda_1} + e^{i\phi_1(x)/h} \frac{ih}{\phi_1(x)} \right) a_{0,0}(x, \eta),$$

where $\tilde{\psi}(\eta)$ and $\phi_1(x)$ are given in Proposition 6 and $a_{0,0}$ is given in Proposition 7.

5 Appendix - Expandible symbols

Here we recall from [2] the notion of expandible symbol.

We denote by $(\mu_j)_{j \geq 0}$ the strictly growing sequence of linear combinations over $\mathbb{N}$ of the λ_j 's. We have for example $\mu_0 = 0$, $\mu_1 = \lambda_1$ and $\mu_2 = 2\lambda_1$ or $\mu_2 = \lambda_2$, whether $2\lambda_1 < \lambda_2$ or not.

First we introduce a convenient notation for error terms. We shall write, with $\mu \in \mathbb{R}^+$, $M \in \mathbb{N}$,

$$w(t, x) = \tilde{O}(e^{-\mu t} |x|^M) \quad (27)$$

if, for every $\epsilon > 0$, we have

$$w(t, x) = O(e^{-(\mu-\epsilon)t} |x|^M). \quad (28)$$

Definition 1 ([2], Definition 3.1) Let ω be a neighborhood of 0 in $\mathbb{R}^d$. A smooth function $u : [0, +\infty[\times \omega \rightarrow \mathbb{R}$ is *expandible* if there exists a sequence (u_k) of smooth functions on $[0, +\infty[\times \omega$, which are polynomials in t , such that for any $n, N \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$

$$\partial_t^n \partial_x^\alpha \left(u(t, x) - \sum_{j=0}^N u_k(t, x) e^{-\mu_k t} \right) = \tilde{O}(e^{-\mu_{N+1} t}) \quad (29)$$

If (29) holds, we write simply

$$u(t, x) \sim \sum_{k \geq 0} u_k(t, x) e^{-\mu_k t}. \quad (30)$$

Proposition 10 ([2], Theorem 3.8) Let $A(t, x)$ be a real smooth expandible matrix with $A(0, 0) = \text{diag}(\lambda_1, \dots, \lambda_d)$. Then, if $v(t, x)$ is expandible, the solution $u(t, x)$ to the problem

$$\begin{cases} \partial_t u + A(t, x) x \cdot \partial_x u = v, & t \geq 0, x \in \omega, \\ u|_{t=0} = 0, \end{cases} \quad (31)$$

is expandible.

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