

HERMITIAN OPERATORS ON BANACH ALGEBRAS OF VECTOR-VALUED LIPSCHITZ MAPS

(JOINT WORK WITH OSAMU HATORI)

SHIHO OI

NIIGATA PREFECTURAL NAGAOKA HIGH SCHOOL

ABSTRACT. Let H be a complex Hilbert space and $[\cdot, \cdot]$ an inner-product on H . A bounded linear operator T on H is a Hermitian operator if $[Tx, x] \in \mathbb{R}$ for each $x \in H$. In 1961, the Hermitian operator on a normed vector space was defined by means of the semi-inner product defined by Lumer [6]. Hermitian operators and their applications have been studied by many authors; a few of them are [1, 2, 5, 6, 7]. We exhibit forms of Hermitian operators on certain semisimple commutative Banach algebras.

1. INTRODUCTION

The notion of a Hermitian operator on a Banach space dates back to the seminal papers by Vidav [8] and Lumer [6]. Lumer considered a definition in terms of a semi-inner product.

Definition 1. Let V be a complex Banach space with the norm $\|\cdot\|_V$. A semi-inner product $[\cdot, \cdot]$ on V is a function from $V \times V$ into $\mathbb{C}$ with the following properties;

- (1) $[u + v, w] = [u, w] + [v, w]$,
 $[\lambda u, v] = \lambda[u, v]$ for $u, v, w \in V, \lambda \in \mathbb{C}$.
- (2) $[v, v] \geq 0$ for all $v \in V$ and $[v, v] \neq 0$ if $v \neq 0$.
- (3) $|[u, v]|^2 \leq [u, u][v, v]$ for $u, v \in V$.

In addition, if $[v, v] = \|v\|_V^2$ for every v in V , then $[\cdot, \cdot]$ is said to be a semi-inner product compatible with the norm of V .

In this note we abbreviate a semi-inner product compatible with the norm as a semi-inner product.

Definition 2. Let $[\cdot, \cdot]$ be a semi-inner product on a complex Banach space V . Then a bounded linear operator T on V is said to be a Hermitian operator if $[Tv, v] \in \mathbb{R}$ for all $v \in V$.

It is well-known that any Banach space has a semi-inner product, which needs not to be unique. We note that the above definition of a Hermitian operator is independent of the semi-inner product chosen.

2. KNOWN RESULTS FOR HERMITIAN OPERATORS AND THE MAIN THEOREM

2.1. Known results. Let B be a unital Banach algebra. For each $a \in B$, M_a denotes the multiplication operator on B , which is defined by $M_a = a \cdot I$ with the identity operator I on B . We introduce a Hermitian element.

Definition 3. Let B be a unital Banach algebra. The numerical range of $a \in B$ is

$$V(a) := \{f(a); \|f\| = f(1) = 1, f \in B^*\}.$$

Then $a \in B$ is said to be a Hermitian element if and only if $V(a) \subset \mathbb{R}$.

First proposition in this section summarizes some of the properties of Hermitian operators and Hermitian elements. In many situations this equivalent statements plays a pivotal role. The following is due to Theorem 5.2.6 in [3].

Proposition 2.1. Let T be a bounded linear operator on a Banach space V . Then the following are equivalent.

- (1) T is a Hermitian operator
- (2) $\|\exp(itT)\|_v = 1$ for any $t \in \mathbb{R}$
- (3) $\exp(itT)$ is an isometry for any $t \in \mathbb{R}$
- (4) T is a Hermitian element in $\mathfrak{B}(V)$, which stands for the space of all bounded linear operators on V equipped with the operator norm.

Proposition 2.2. Let B be a unital Banach algebra. If $a \in B$ is a Hermitian element, then the multiplication operator M_a is a Hermitian operator on B .

Proof. Let $a \in B$ be a Hermitian element. It is well-known that an element $a \in B$ is Hermitian if and only if $\|\exp(ita)\|_B = 1$ for any $t \in \mathbb{R}$. Thus, we deduce that

$$\|\exp(ita \cdot I)\| = 1$$

for all $t \in \mathbb{R}$. Applying Proposition 2.1, we conclude that M_a is a Hermitian operator on B . $\square$

We are interested in a problem that under which circumstances the converse statement of Proposition 2.2 holds; when is a Hermitian operator on a unital Banach algebra a multiplication operator? Our purpose of this note is to give a partial answer to the problem. Now we recall two observations about Hermitian operators.

Theorem 4. [2, Theorem 4] Let X be a compact Hausdorff space and E a complex Banach space. Suppose that $C(X, E)$ is the Banach space of all continuous functions on X with values in E with the supremum norm. A bounded linear operator T on $C(X, E)$ is a Hermitian operator if and only if for each $x \in X$ there is a Hermitian operator $A(x)$ on E such that for any $F \in C(X, E)$ we have

$$TF(x) = A(x)F(x) \quad x \in X.$$

Theorem 5. [1, Theorem 3.1] Let X be a compact metric space and $\text{Lip}(X)$ a complex Banach algebra of complex-valued Lipschitz functions with the norm $L(\cdot) + \|\cdot\|_\infty$. A bounded linear operator T on $\text{Lip}(X)$ is a Hermitian operator if and only if $T = \lambda \cdot I$ with $\lambda \in \mathbb{R}$.

2.2. The main theorem. The following is the main theorem in this note.

Theorem 6. Let B be a unital semisimple commutative Banach algebra. Suppose that every surjective unital isometry on B is multiplicative. If a bounded complex-linear operator T is a Hermitian operator, then

$$T = M_{T(1)}$$

3. A PROOF OF THE MAIN THEOREM

Proposition 3.1. *Let B be a unital Banach algebra. Suppose that T is a Hermitian operator on B . Then $T(\mathbf{1})$ is a Hermitian element in B .*

Proof. This proof is based on Lemma 3.2 in [1]. For any $f \in B^*$ with $\|f\| = f(\mathbf{1}) = 1$, we define $\Phi_f : \mathfrak{B}(B) \rightarrow \mathbb{C}$ by

$$\Phi_f(S) = f(S(\mathbf{1})) \quad (S \in \mathfrak{B}(B)).$$

We infer that Φ_f is a bounded linear functional on $\mathfrak{B}(B)$ and satisfies $\|\Phi_f\| = \Phi_f(I) = 1$. Since T is a Hermitian element, this implies

$$f(T(\mathbf{1})) = \Phi_f(T) \in \mathbb{R}$$

for any $f \in B^*$ with $\|f\| = f(\mathbf{1}) = 1$. Thus, we obtain that $T(\mathbf{1})$ is a Hermitian element of B . $\square$

Proposition 3.2. *Let T be a bounded complex linear operator on a unital semisimple commutative Banach algebra B . Then the following are equivalent.*

- (1) $T = M_{T(\mathbf{1})}$
- (2) $\exp(it(T - M_{T(\mathbf{1})}))$ is multiplicative for every $t \in \mathbb{R}$

Proof. Suppose that $T = M_{T(\mathbf{1})}$. Clearly we have

$$\exp(it(T - M_{T(\mathbf{1})})) = I$$

for every $t \in \mathbb{R}$.

In order to prove the converse, suppose that $\exp(it(T - M_{T(\mathbf{1})}))$ is multiplicative for every $t \in \mathbb{R}$. We define $H = T - M_{T(\mathbf{1})}$ and $U_t = \exp(itH)$ for every $t \in \mathbb{R}$. Differentiating U_t at $t = 0$, we get

$$U'_t|_{t=0}(ab) = iH(ab),$$

for any $a, b \in B$. As U_t is multiplicative, for any $a, b \in B$, we get

$$U'_t|_{t=0}(ab) = iaH(b) + iH(a)b.$$

It follows that H is a bounded derivation. By a theorem of Singer and Wermer, we observe that $H = 0$. $\square$

We now proceed with the details for the proof of our main theorem.

A proof of the main theorem. Let T be a Hermitian operator on B . Applying Proposition 3.1, $T(\mathbf{1})$ is a Hermitian element of B . According to Proposition 2.2, we see that $M_{T(\mathbf{1})}$ and $T - M_{T(\mathbf{1})}$ are Hermitian operators on B . Therefore, $\exp it(T - M_{T(\mathbf{1})})$ is a unital surjective isometry for any $t \in \mathbb{R}$. By the assumption, every surjective unital isometry on B is multiplicative, thus $\exp it(T - M_{T(\mathbf{1})})$ is multiplicative. Hence Proposition 3.2 provides that $T = M_{T(\mathbf{1})}$. $\square$

4. APPLICATIONS OF THE MAIN THEOREM

Let us begin with a definition of a vector-valued Lipschitz algebra. Let X be a compact metric space and A a uniform algebra on a compact Hausdorff space Y . A map F from X into A is said to be Lipschitz if it satisfies the following inequality

$$L(F) := \sup_{x \neq y \in X} \frac{\|f(x) - f(y)\|_\infty}{d(x, y)} < \infty.$$

The set of all Lipschitz maps is denoted by

$$\text{Lip}(X, A) := \{F : X \rightarrow A; L(F) < \infty\}$$

and this is a unital semisimple commutative Banach algebra with the norm of $\|\cdot\|_L = \|\cdot\|_\infty + L(\cdot)$.

We exhibit a theorem of Jarosz in [4]. Let E be a linear subspace of $C(X)$ which separates the points in X and contains constants. We denote by $\text{Ch}(E)$ the Choquet boundary for E . We call E a regular subspace of $C(X)$ if for any $\epsilon > 0$, $x_0 \in \text{Ch}(E)$, and open neighborhood U of x_0 , there exists an $f \in E$ with $\|f\|_\infty \leq 1 + \epsilon$, $f(x_0) = 1$, $|f(x)| < \epsilon$ for $x \in X \setminus U$.

Theorem 7. [4] *Let X and Y be compact Hausdorff spaces. Let A, B be complex linear subspaces of $C(X)$ and $C(Y)$ respectively. We assume that A and B satisfy the following;*

- (1) A and B contain constant functions.
- (2) A and B have $\|\cdot\|_A$ and $\|\cdot\|_B$ p -norm, q -norm.
- (3) A and B are regular subspaces.

If a bounded linear operator T from $(A, \|\cdot\|_A)$ onto $(B, \|\cdot\|_B)$ with $T(\mathbf{1}) = \mathbf{1}$ is a surjective linear isometry, then T is an isometry from $(A, \|\cdot\|_\infty)$ onto $(B, \|\cdot\|_\infty)$

Applying Theorem 7 by considering Lipschitz algebra $\text{Lip}(X, A)$ to be a subspace of $C(X \times Y)$, we get the following corollary.

Corollary 8. *If U is a linear isometry from $\text{Lip}(X, A)$ onto $\text{Lip}(Y, B)$ with $U(\mathbf{1}) = \mathbf{1}$ then U is also an isometry with the supremum norm.*

Now, we give a characterization of Hermitian operators on vector-valued Lipschitz algebras.

Theorem 9. *Let X be a compact metric space and A be a uniform algebra. A bounded linear operator $T : \text{Lip}(X, A) \rightarrow \text{Lip}(X, A)$ is a Hermitian operator if and only if there exists a real-valued function $a \in A$ with $T(\mathbf{1}) = \mathbf{1} \otimes a$ such that*

$$T = M_{T(\mathbf{1})}.$$

Proof. Actually real-valued function $a \in A$ is a Hermitian element of A . Therefore, for a real-valued function $a \in A$, we see that $T(\mathbf{1}) = \mathbf{1} \otimes a$ is a Hermitian element of $\text{Lip}(X, A)$. Applying Proposition 2.2, we get $T = M_{T(\mathbf{1})}$ is a Hermitian operator.

Now we consider the converse. Using Corollary 8, every surjective unital isometry on $\text{Lip}(X, A)$ with the norm of $\|\cdot\|_L$ is an isometry with the supremum norm. Moreover, Nagasawa's theorem shows that it is also multiplicative. Thus, Theorem 6 follows every Hermitian operator on $\text{Lip}(X, A)$ is a multiplication operator. $\square$

Remark 10. As corollaries of Theorem 6, we also have Theorem 4 in [2] and Theorem 3.1 in [1].

REFERENCES

- [1] F. Botelho and J. Jamison, A. Jiménez-Vargas, and M. Villegas-Vallecillos, *Hermitian operators on Banach algebras of Lipschitz functions*, Proc. Amer. Math. Soc. **142** (2014), 3469–3481.
- [2] R. J. Fleming and J. E. Jamison, *Hermitian Operators on $C(X, E)$ and the Banach-Stone Theorem*, Math. Z. **170** (1980), 77–84.
- [3] R. J. Fleming and J. E. Jamison, *Isometries on Banach spaces*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129. Chapman & Hall/CRC, Boca Raton, FL, 2003. x+197

- [4] K. Jarosz, *Isometries in semisimple, commutative Banach algebras*, Proc. Amer. Math. Soc. **94** (1985), 65–71.
- [5] D. Koehler and P. Rosenthal, *On isometries of normed linear spaces*, Studia Math. **34** (1970), 213–216.
- [6] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29–43.
- [7] G. Lumer, *On the isometries of reflexive Orlicz spaces*, Ann. Inst. Fourier(Grenoble) **13** (1963), 99–109.
- [8] I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z. **66** (1956), 121–128

E-mail address: shiho.oi.pmf20@gmail.com