

On the diffeomorphism group of a smooth orbifold and its application

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§1. Introduction

Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of an n -dimensional smooth manifold M which are isotopic to the identity through compactly supported isotopies. In [TH], Thurston proved that the group $\mathcal{D}(M)$ is perfect, which means $\mathcal{D}(M)$ coincides with its commutator subgroup. There are many analogous results on the group of a smooth manifold M preserving a geometric structure of M .

In this note we shall study the case when M is a smooth orbifold. Since a smooth orbifold is locally diffeomorphic to the orbit space of a smooth G -manifold with finite group G , first we shall consider in the case of a representation space V of a finite group G . Let $\mathcal{D}_G(V)$ denote the group of equivariant smooth diffeomorphisms of V which are G -isotopic to the identity through compactly supported equivariant smooth isotopies. In general the group $\mathcal{D}_G(V)$ is not perfect. Then we calculate the first homology group $H_1(\mathcal{D}_G(V))$.

We shall prove that $\mathcal{D}_G(V)$ is perfect if $\dim V^G > 0$ and $H_1(\mathcal{D}_G(V))$ is isomorphic to $H_1(\text{Aut}_G(V)_0)$ if $\dim V^G = 0$. Here $\text{Aut}_G(V)_0$ is the identity component of the group of G -equivariant linear automorphisms of V , and V^G is the fixed point set of G on V ([AF5]).

Secondly we apply the above result to the case of smooth orbifold and also smooth G -manifold. Using the result by Biestone [BI1] and Schwarz [SC1], we see that $H_1(\mathcal{D}_G(V))$ is isomorphic to $H_1(\mathcal{D}(V/G))$. Combining those results and the fragmentation lemma we can determine the structure of $H_1(\mathcal{D}(N))$ of the diffeomorphism group $\mathcal{D}(N)$ for any smooth orbifold N . Then we see that $H_1(\mathcal{D}(N))$ describes a geometric structure around the isolated singularities.

Let M be a smooth G -manifold for a finite group G . Then $H_1(\mathcal{D}_G(M))$ is isomorphic to $H_1(\mathcal{D}(M/G))$, and we see that $H_1(\mathcal{D}_G(M))$ describes the properties of the isotropy representations at the isolated fixed points of M . We can also apply the above results to a smooth G -manifold when G is a compact Lie group. If M is a principal G -manifold with G a compact Lie group, then we proved that the group $\mathcal{D}_G(M)$ is perfect for $\dim(M/G) > 0$ (Banyaga [BA1] and Abe and Fukui [AF1]). In [AF2] we calculated $H_1(\mathcal{D}_G(M))$ when M is a smooth G -manifold with codimension

one orbit. We shall apply the above result to the case of a locally free $U(1)$ -action on the 3-sphere, and calculate $H_1(\mathcal{D}_{U(1)}(S^3))$ ([AF5]).

Thirdly we shall apply the results to the modular group. Let Γ be the modular group which acts on the upper half complex plane $\mathcal{H}$ by the Möbius transformations. Then the orbit space $\mathcal{H}/\Gamma$ is a smooth orbifold. Let $\mathcal{R}_\Gamma$ be the compactified space of $\mathcal{H}/\Gamma$ by adjoining the point $*$ which corresponds to the Γ -equivalence class of the parabolic cusps. With the canonical smooth coordinate around $*$, we shall calculate the group $H_1(\mathcal{D}(\mathcal{R}_\Gamma))$, which describes the elliptic points and the cusp point. We can also calculate the group for the case of the congruence subgroups of Γ .

We can apply the above results to the case of foliation preserving diffeomorphism groups. We studied for the similar problem in the Lipschitz category ([AF3], [AF4], [AF6], [AFM]).

§2. Recent results on the diffeomorphism groups on smooth orbifolds

Let G be a finite group and let M be a smooth connected G -manifold. Let $\mathcal{D}_G(M)$ denote the group of G -equivariant smooth diffeomorphisms of M which are G -isotopic to the identity through isotopies with compact support.

First we shall calculate $\mathcal{D}_G(V)$ for a finite dimensional G -module V . Let V^G be the subspace of the fixed point set of V . Let $A_G(V)$ denote the set of G -invariant automorphisms of V and let $A_G(V)_0$ be the identity component of $A_G(V)$. Then we have the following.

Theorem 1

- (1) If $\dim V^G > 0$, then $\mathcal{D}_G(V)$ is perfect.
- (2) If $\dim V^G = 0$, then $H_1(\mathcal{D}_G(V)) \cong H_1(A_G(V)_0)$.

We can decompose $V = \bigoplus_{i=1}^d k_i V_i$, where V_i runs over the inequivalent irreducible representation space of G and k_i is a positive integer. Let $\text{End}_G(V_i)$ denote the set of G -invariant endomorphisms of V_i . Then $\dim \text{End}_G(V_i) = 1, 2$ or 4 .

Corollary 2 If $\dim V^G = 0$, then

$$H_1(\mathcal{D}_G(V)) \cong \mathbf{R}^d \times \overbrace{U(1) \times \cdots \times U(1)}^{d_2},$$

where d_2 is the number of V_i with $\dim \text{End}_G(V_i) = 2$.

Definition 3 (smooth orbifold)

A paracompact Hausdorff space M is called a smooth orbifold if there exists an open covering $\{U_i \mid i \in \Lambda\}$ of M , closed under finite intersections, satisfying the following.

- (1) There exist an open subset $\tilde{U}_i$ in $\mathbf{R}^n$ such that a finite group Γ_i acts effectively on $\tilde{U}_i$ and a homeomorphism $\phi_i : \tilde{U}_i/\Gamma_i \rightarrow U_i$.
- (2) Whenever $U_i \subset U_j$, there exists a smooth embedding $\phi_{ij} : \tilde{U}_i \rightarrow \tilde{U}_j$ such that

$$\begin{array}{ccc}
 \tilde{U}_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j \\
 \downarrow \pi_i & & \downarrow \pi_j \\
 \tilde{U}_i/\Gamma_i & & \tilde{U}_j/\Gamma_j \\
 \downarrow \phi_i^{-1} & & \downarrow \phi_j^{-1} \\
 U_i & \xrightarrow{\subset} & U_j.
 \end{array}$$

(U_i, ϕ_i) is called a local chart of M .

Here we define the smooth maps between smooth orbifolds (c.f. [BI1]). $f : M \rightarrow \mathbf{R}$ is said to be smooth if for any local chart (U_i, ϕ_i) of M , $\tilde{U}_i \xrightarrow{\pi_i} \tilde{U}_i/\Gamma_i \xrightarrow{\phi_i} U_i \xrightarrow{f} \mathbf{R}$ is smooth. $h : M \rightarrow M$ is said to be smooth if for any smooth function $f : M \rightarrow \mathbf{R}$, $f \circ h$ is smooth. $h : M \rightarrow M$ is called a diffeomorphism if h and h^{-1} are smooth. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of M which are isotopic to the identity through isotopies with compact support.

$p \in M$ is said to be an isolated singular point of M if there exists a local chart (U_i, ϕ_i) around p such that $\tilde{p}$ is the isolated fixed point of $\tilde{U}_i$ with $\pi_i(\tilde{p}) = p$. Here $\phi_i : U_i \rightarrow \tilde{U}_i/\Gamma_i$ and $\pi_i : \tilde{U}_i \rightarrow U_i$ are the maps defined in Definition 3.

Let $(U_i, \phi_i), (U_j, \phi_j)$ be local charts of M around an isolated singular point p of M . Then we can assume that $\tilde{U}_i$ and $\tilde{U}_j$ are invariant open neighborhoods around the origin of linear representation spaces of Γ_i and Γ_j , respectively. By the result of Strub [ST], the groups Γ_i and Γ_j are isomorphic and the corresponding representations are equivalent. Then the isolated singular point p determines the equivalence class of the linear representation space V_p of a finite group Γ_p .

Theorem 4 If a smooth orbifold M has $\{p_1, \dots, p_k\}$ as the isolated singular point set, then

$$H_1(\mathcal{D}(M)) \cong H_1(A_{\Gamma_{p_1}}(V_{p_1})_0) \times \cdots \times H_1(A_{\Gamma_{p_k}}(V_{p_k})_0).$$

We can apply Theorem 4 to the case of smooth G -manifold with finite group G .

Theorem 5 *Let G be a finite group and M a smooth G -manifold. If the orbit space M/G has $\{G \cdot p_1, \dots, G \cdot p_k\}$ as the isolated singular points, then*

$$H_1(\mathcal{D}_G(M)) \cong H_1(A_{G_{p_1}}(T_{p_1}M)_0) \times \dots \times H_1(A_{G_{p_k}}(T_{p_k}M)_0).$$

Corollary 6 *Let $\tilde{\mathbf{R}}$ be the non-trivial one dimensional representation space of $\mathbf{Z}_2$. Then*

$$H_1(\mathcal{D}_{\mathbf{Z}_2}(\tilde{\mathbf{R}}^n)) \cong H_1(\mathcal{D}(\tilde{\mathbf{R}}^n/\mathbf{Z}_2)) \cong \mathbf{R}.$$

We can apply Corollary 6 to a smooth $U(1)$ -action on S^3 . Let

$$S^3 = \{(w_1, w_2) \in \mathbf{C}^2 \mid |w_1|^2 + |w_2|^2 = 1\}$$

with $U(1)$ -action given by

$$z \cdot (w_1, w_2) = (zw_1, z^2w_2), \quad z \in U(1).$$

Then it has two orbit types $\{(1), (\mathbf{Z}_2)\}$ and the orbit space $S^3/U(1)$ is homeomorphic to the space known as the tear drop which is the two dimensional sphere with one isolated singular point.

Theorem 7 $H_1(\mathcal{D}_{U(1)}(S^3)) \cong \mathbf{R} \times U(1)$.

Remark 8 *If we restrict the above action to $\mathbf{Z}_n$, then $\mathcal{D}_{\mathbf{Z}_n}(S^3)$ is perfect.*

§3. Application to the modular group

In this section we shall apply the results to the modular group. Let $\mathcal{H}$ be the upper half complex plane. Let $SL(2, \mathbf{R})$ be the group of real matrix with determinant 1. Then $SL(2, \mathbf{R})$ acts on $\mathcal{H}$ as follows.

$$g \cdot z = \frac{az + b}{cz + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad z \in \mathcal{H}.$$

Then $SL(2, \mathbf{R})$ acts transitively on $\mathcal{H}$ and the isotropy subgroup at $i = \sqrt{-1}$ is $SL(2, \mathbf{R})_i = SO(2)$. The kernel of the action is $\mathbf{Z}_2 = \{\pm 1\}$ and $PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm 1\}$ acts effectively on $\mathcal{H} \cong SL(2, \mathbf{R})/SO(2)$.

The action can be extended to the Riemannian sphere: $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \bar{\mathbf{C}},$$

$$g \cdot z = \begin{cases} \frac{az+b}{cz+d} & (z \neq -\frac{d}{c}, \infty) \\ \infty & (z = -\frac{d}{c}, z = d = 0) \\ \frac{a}{c} & (z = \infty) \end{cases}$$

Set

$$R_1 = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\},$$

$$R_2 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then each $g \in SL(2, \mathbf{R})$ is conjugate to one of the elements of $SO(2) \cup R_1 \cup R_2$, and $g \neq \pm 1$ is called *elliptic*, *hyperbolic* and *parabolic* if g is conjugate of an element in $SO(2)$, R_1 and R_2 , respectively.

Let $\Gamma = SL(2, \mathbf{Z})$ be the group of the integral matrices with determinant 1. Then $\bar{\Gamma} = \Gamma / \{\pm 1\}$ acts properly on $\mathcal{H}$ (i.e. for each $z \in \mathcal{H}$, there exists open neighborhood U of z such that $\bar{\Gamma}_U = \{g \in \bar{\Gamma} \mid g \cdot U = U\}$ is a finite group and if $\gamma \cdot U \cap U \neq \emptyset$ for $\gamma \in \bar{\Gamma}$, then $\gamma \in \bar{\Gamma}_U$).

$z \in \mathcal{H}$ is called *elliptic point* if there exists an elliptic element $g \in \Gamma$ such that $g \cdot z = z$. $x \in \mathbf{R} \cup \{\infty\}$ is called *cuspid point* if there exists a parabolic element $g \in \Gamma$ such that $g \cdot x = x$.

Proposition 9 (1) If z is a elliptic point, then Γ_z is a cyclic group which is conjugate to a cyclic subgroup of $SO(2)$.

(2) If x is a cuspid point, then Γ_x is isomorphic to $\mathbf{Z}$ which is conjugate to the group

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

(3) Γ acts transitively on the set of cuspid points which coincides with $Q \cup \{\infty\}$, where Q is the set of rational numbers.

Set

$$\mathcal{H}^* = \mathcal{H} \cup Q, \quad \mathcal{R}_\Gamma = \mathcal{H}^* / \Gamma = \mathcal{H} / \Gamma \cup \{*\}.$$

We give the set

$$\{*\} \cup \bigcup_{c>0} \{z \in \mathcal{H} \mid \Im z > c\}$$

as a fundamental system of open neighborhood of the point $*$. Then $\mathcal{R}_\Gamma$ is homeomorphic to S^2 .

Proposition 10 *There exists a Γ_∞ -invariant open neighborhood $\tilde{U}$ of $*$ satisfying the following.*

- (1) $\Gamma_\infty = \{g \in \Gamma \mid g \cdot \tilde{U} \cap \tilde{U} \neq \emptyset\}$.
- (2) Let $\varphi : \tilde{U}/\Gamma_\infty \rightarrow \mathbb{C}$ be the map given by $\varphi(\Gamma_\infty \cdot z) = \exp(2\pi\sqrt{-1}z)$ for $z \in \tilde{U}$. Then φ is a homeomorphism into an open set U of $\mathbb{C}$.

Let $\iota : \tilde{U}/\bar{\Gamma}_\infty \rightarrow \mathcal{R}_\Gamma$ be the natural map. Put $U = \iota(\tilde{U}/\bar{\Gamma}_\infty)$. By Proposition 10 U is an open neighborhood of $*$ and the homeomorphism $\phi = \varphi \circ \iota^{-1} : U \rightarrow \tilde{U}/\bar{\Gamma}_\infty$ is regarded as a local coordinate of $\mathcal{R}_\Gamma$.

We call $h : \mathcal{R}_\Gamma \rightarrow \mathcal{R}_\Gamma$ to be a diffeomorphism if the following conditions (1), (2), (3) is satisfied.

- (1) $h|(\mathcal{H}/\bar{\Gamma})$ is a diffeomorphism of $\mathcal{H}/\bar{\Gamma}$ as a smooth orbifold.
- (2) $\phi \circ h \circ \phi^{-1}$ is a diffeomorphism of U .
- (3) There exists $\bar{\Gamma}_\infty$ -equivariant diffeomorphism $\tilde{h}$ of $\mathcal{H}$ such that the induced diffeomorphism on $\mathcal{H}/\bar{\Gamma}$ coincides with h on $U \setminus \{*\}$.

Theorem 11

- (1) $H_1(\mathcal{D}_\Gamma(H^2)) \cong H_1(\mathcal{D}(H^2/\Gamma)) \cong \mathbb{R}^2 \times U(1)$.
- (2) $H_1(\mathcal{D}(\mathcal{R}_\Gamma)) \cong U(1) \times \mathbb{R}^3$.

The orbifold H^2/Γ has two isolated singular points which correspond to the elliptic subgroups of Γ with orders 2 and 3, which induces the isomorphism in Theorem 11, (1). In addition to those singular points, $\mathcal{R}_\Gamma$ has the singular point $*$ corresponding to the cusp point, which induces the isomorphism in Theorem 11, (2).

Let $\Gamma(N)$ denote the principal congruence subgroup of level N . Then

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N\mathbb{Z}} \right\}.$$

Similarly to the case of the modular group, we have the following.

Theorem 12 $H_1(\mathcal{D}(\mathcal{R}_\Gamma(N))) \cong \mathbb{R}^{t(N)}$, where $t(N)$ is the number of cusps of $\mathcal{H}/\Gamma(N)$.

The number $t(N)$ is known as:

$$t(1) = 1, \quad t(2) = 3,$$

$$t(N) = \frac{1}{2N}(N : \Gamma(N)) \quad (N \geq 3),$$

$$(N : \Gamma(N)) = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

We can also apply Theorem 1 to calculate the first homology group of the foliation preserving diffeomorphism group for a compact Hausdorff foliation.

§4. Outline of the proof of Theorem 1

First we prove Theorem 1 (1). Let G be a finite group and let V be a G -module with $\dim V^G > 0$. Then there exists a G -module W with $\dim W^G = 0$ such that $V = W \oplus \mathbf{R}^q$. We prove $\mathcal{D}_G(V)$ is perfect by induction of the order of G . If $G = \{1\}$, then $\mathcal{D}_G(V)$ is perfect by the result of Thurston [TH]. Assume that Theorem 1 (1) holds for any finite subgroup H with $|H| < |G|$.

To investigate the group structure of $\mathcal{D}_G(V)$, we give C^∞ -topology on $\mathcal{D}_G(V)$. For the proof we need the following fragmentation lemma.

Lemma 13 (fragmentation lemma)

Let M be a smooth G -manifold and let $\{U_i\}$ be a G -invariant open covering of M . Let N be a neighborhood of the identity in $\mathcal{D}_G(M)$. Then, for any $f \in \mathcal{D}_G(M)$, there exist $\{f_j \in N \mid 1 \leq j \leq k\}$ such that

- (1) f_j is equivariantly isotopic to the identity through G -diffeomorphisms with the support contained in U_j ,
- (2) $f = f_1 \circ \cdots \circ f_k$.

Let $f \in \mathcal{D}_G(V)$. In order to prove $f \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$, by the fragmentation lemma, we can assume f is sufficiently close to the identity. Then we can find $g_1, g_2 \in \mathcal{D}_G(V)$ satisfying

- (1) $g_1(x, y) = (x, \hat{g}_1(x)(y))$ with $\hat{g}_1(x) \in \mathcal{D}(\mathbf{R}^q)$,
- (2) $g_2(x, y) = (\hat{g}_2(y)(x), y)$ with $\hat{g}_2(y) \in \mathcal{D}_G(W)$ for $x \in W, y \in \mathbf{R}^q$,
- (3) $f = g_2 \circ g_1$.

By the result of Tsuboi [TS], we see that $g_1 \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$.

In the next we shall prove that $g_2 \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$. Let $\alpha_{g_2} : \mathbf{R}^q \rightarrow \text{Aut}_G(W)_0$ be a group homomorphism defined by $\alpha_{g_2}(y) = d\hat{g}_2(y)_0$, where $d\hat{g}_2(y)_0$ is the differential of $\hat{g}_2(y)$ at 0. Then α_{g_2} is a smooth map with compact support $\overline{\{p \in \mathbf{R}^q \mid \alpha_{g_2}(p) \neq e\}}$, where e is the unit element in $\text{Aut}_G(W)_0$.

If we take f close to the identity, then α_{g_2} is sufficiently close to the constant map e . Then applying [AF1], Lemma 4, we have

- (a) $\exists \varphi_i \in \mathcal{D}(\mathbf{R}^q)$, $\alpha_i \in C^\infty(\mathbf{R}^q, \text{Aut}_G(W)_0)$ ($i = 1, \dots, r = \dim \text{Aut}_G(W)_0$),
 (b) $\alpha_{g_2} = (\alpha_1^{-1} \cdot (\alpha_1 \circ \varphi_1)) \cdots (\alpha_r^{-1} \cdot (\alpha_r \circ \varphi_r))$.

Let $|\cdot|$ be a G -invariant norm of W . Let $\mu : W \rightarrow [0, 1]$ be a G -invariant smooth function satisfying

- (i) $\mu(x) = 1$ for $|x| \leq \frac{1}{2}$,
 (ii) $\mu(x) = 0$ for $|x| \geq 1$.

Define $h_i, F_i \in \mathcal{D}_G(V)$ ($i = 1, \dots, r$) by

$$\begin{aligned} h_i(x, y) &= (\mu(x)\alpha_i(y)(x) + (1 - \mu(x))x, y), \\ F_i(x, y) &= (x, \mu(x)\varphi_i(y) + (1 - \mu(x))y) \end{aligned}$$

for $x \in W$, $y \in \mathbf{R}^q$.

Lemma 14

$$(h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)(x, y) = ((\alpha_i^{-1} \cdot (\alpha_i \circ \varphi_i))(y)(x), y),$$

for $x \in W$, $y \in \mathbf{R}^q$ with $|x| \leq \frac{1}{2}$.

Set

$$g_3 = \prod_{i=1}^r (h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)^{-1} \circ g_2.$$

Then g_3 is written of the form $g_3(x, y) = (\hat{g}_3(x)(y), y)$ with $\hat{g}_3(x) \in \mathcal{D}_G(W)$ and $\alpha_{g_3} = e$.

For $0 < c < 1$, let $\psi_c \in \mathcal{D}_G(V)$ such that, for $x \in W$, $y \in \mathbf{R}^q$,

$$\psi_c(x, y) = \begin{cases} (cx, y) & (|x| \leq 1), \\ (x, y) & (|x| \geq 2). \end{cases}$$

Applying the result of Sternberg [S2], there exists $R \in \mathcal{D}(V)$ such that

- (1) R is of the form $R(x, y) = (\hat{R}(y)(x), y)$
 with $\hat{R}(y) \in \mathcal{D}(W, 0)$ and $\alpha_R = e$.

- (2) $R \circ (g_3 \circ \psi_c) \circ R^{-1} = \psi_c$ on a neighborhood U_0 of $\{0\} \times \mathbf{R}^q$.

Set

$$\tilde{R}(x, y) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x, y) \quad \text{for } x \in W, y \in \mathbf{R}^q.$$

Then

$$\psi_c \circ \tilde{R} = \tilde{R} \circ g_3 \circ \psi_c \quad \text{on } U_0.$$

Since $\tilde{R}$ is G -equivariant diffeomorphic on a neighborhood of $\{0\} \times \mathbf{R}^q$, we can find $\tilde{R}_1 \in \mathcal{D}_G(V)$ such that $\tilde{R}_1 = \tilde{R}$ on a neighborhood $U \subset U_0$ of $\{0\} \times \mathbf{R}^q$. Put

$$g_4 = g_3 \circ (\tilde{R}_1^{-1} \circ \psi_c \circ \tilde{R}_1 \circ \psi_c^{-1})^{-1}.$$

Then $g_4 = 1$ on U .

There exist a finite point $\{p_i \in V \setminus U \mid 1 \leq i \leq k\}$ and an open disk neighborhood $U(p_i)$ at p_i ($1 \leq i \leq k$) such that

- (1) $U(p_i)$ is a slice at p_i ,
- (2) $\text{supp}(g_4) \subset \bigcup_{i=1}^k G \cdot U(p_i)$.

By the fragmentation lemma there exist $h_j \in \mathcal{D}_G(V)$ ($1 \leq j \leq \ell$) such that

- (a) h_j is equivariantly isotopic to the identity through G -diffeomorphisms with the support contained in $G \cdot U(p_j)$,
- (b) $g_4 = h_1 \circ \cdots \circ h_\ell$.

Since $U(p_j)$ is a slice at p_j , the isotropy subgroup G_{p_j} acts on $U(p_j)$ and $G \cdot U(p_j)$ is a disjoint union of $|G/G_{p_j}|$ disks. Then from the above condition (a)

$$h_j(g \cdot U(p_j)) = g \cdot U(p_j) \quad \text{for } g \in G.$$

We assumed that $\mathcal{D}_H(V)$ is perfect when H is a finite group with $|H| < |G|$ and $\dim V^H > 0$. Therefore each h_j can be written as a commutator in $\mathcal{D}_G(V)$ and Theorem 1 (1) follows.

Secondary we prove Theorem 1 (2). Let V be a G -module with $\dim V^G = 0$. Let $\Phi : \mathcal{D}_G(V) \rightarrow \text{Aut}_G(V)_0$ be a group homomorphism defined by $\Phi(f) = (df)_0$. Since

$$1 \rightarrow \text{Ker}\Phi \xrightarrow{\iota} \mathcal{D}_G(V) \xrightarrow{\Phi} \text{Aut}_G(V)_0 \rightarrow 1$$

is a short exact sequence, we have the exact sequence.

$$\text{Ker}\Phi / [\text{Ker}\Phi, \mathcal{D}_G(V)] \xrightarrow{\iota_*} H_1(\mathcal{D}_G(V)) \xrightarrow{\Phi_*} H_1(\text{Aut}_G(V)_0) \rightarrow 1$$

Then Theorem 1 (2) follows from the following.

Proposition 15 $\text{Ker}\Phi = [\mathcal{D}_G(V), \mathcal{D}_G(V)]$

Proof. Let $f \in \text{Ker}\Phi$. For $0 < c < 1$, let $\psi_c \in \text{Aut}_G(V)_0$ as before. Applying the result by Sternberg [S2] there exists $R \in \mathcal{D}(V, 0)$ such that $(dR)_0 = 1_V$ and

$R \circ f \circ \psi_c \circ R^{-1} = \psi_c$ on a neighborhood of 0. Set

$$\tilde{R}(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x) \quad \text{for } x \in \mathbf{R}^n,$$

where $|G|$ is the order of G . Since $\tilde{R}$ is equivariant diffeomorphism on a neighborhood U of 0, we can find $\hat{R} \in \mathcal{D}_G(V)$ such that $\hat{R} = \tilde{R}$ on an open neighborhood $U_1 \subset U$ of 0. Then

$$f = \hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1} \quad \text{on } U_1.$$

Put

$$g = f \circ (\hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1})^{-1}.$$

Then $g = 1$ on U_1 . By the parallel way as in the proof of the case Theorem 1, (1), we can prove that g is written as a commutator in $\mathcal{D}_G(V)$.

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